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REMARKS ON HYPERBOLIC POLYNOMIALS

By

Seiichiro WAKABAYASHI

1. Introduction.

In the study of hyperbolic partial differential operators, it is important to investigate properties of the characteristic roots. Bronshtein [2] proved the Lipschitz continuity of the characteristic roots of hyperbolic operators with variable coefficients, and he studied the hyperbolic Cauchy problem in Gevrey classes (see [3]). Ohya and Tarama [7] extended the results in [2] and, also, studied the Cauchy problem.

In this paper we shall give an alternative proof of Bronshtein's results, which seems to be simpler. Also, we shall prove the inner semi-continuity of the cones defined for the localization polynomials of hyperbolic operators (see Theorem 3 below). In studying singularities of solutions the inner semi-continuity of the cones plays a key role (see [8], [9], [10]). We note that our method can be applicable to the mixed problem.

Let $p(t, x, y)=t^m+\sum_{j=1}^m a_j(x, y)t^{m-j}$ be a polynomial in t, where the $a_j(x, y)$ are defined for $x=(x_1, \cdots, x_n)\in X$ and $y\in Y$, X is an open convex subset of $\boldsymbol{R}^n$ and Y is a compact Hausdorff topological space. We assume that

(A-1) $p(t, x, y)\neq 0$ *if* $\operatorname{Im} t\neq 0$ *and* $(x, y)\in X\times Y$,

(A-2) $\partial_x^\alpha a_j(x, y)$ $(|\alpha|\leqq k, 1\leqq j\leqq m)$ *are continuous and there are* $C>0$ *and* δ *with* $0<\delta\leqq 1$ *such that*

$$|\partial_x^\alpha a_j(x, y)-\partial_x^\alpha a_j(x', y)|\leqq C|x-x'|^\delta$$

if $|\alpha|=k$, $x, x'\in X$ *and* $y\in Y$, *where* k *is a nonnegative integer and* $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$.

THEOREM 1. *Assume that* (A-1) *and* (A-2) *are satisfied. Then, for any open subset* U *of* X *with* $U\Subset X$ *there is* $C=C(U)>0$ *such that*

$$|\lambda_j(x, y)-\lambda_j(x', y)|\leqq C|x-x'|^r \qquad \text{for } 1\leqq j\leqq m,\ x, x'\in U \text{ and } y\in Y,$$

where $p(t, x, y)=\prod_{j=1}^m (t-\lambda_j(x, y))$, $\lambda_1(x, y)\leqq\lambda_2(x, y)\leqq\cdots\leqq\lambda_m(x, y)$, *and* $r=\min(1, (k+\delta)$

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$/m$). Here $U \Subset X$ means that $\bar{U}$ is a compact subset of X.

REMARK. Under the assumptions (A-1) and (A-2)$'$ below Bronshtein [2] proved the above theorem. Theorem 1 was announced by Ohya and Tarama [7] who proved it by the same argument as in [2].

THEOREM 2. *Assume that*
(A-1)$'$ *$p(t, x, y) \neq 0$ if $\operatorname{Im} t < 0$ and $(x, y) \in X \times Y$*
and (A-2) *are satisfied. Then, for any open subset U of X with $U \Subset X$ there is $C = C(U) > 0$ such that*

$$|\partial_t^j \partial_x^\alpha p(t, x, y)| / |p(t, x, y)| \leqq C |\operatorname{Im} t|^{-j-|\alpha|/r_j} \tag{1.1}$$

if $0 \leqq j \leqq m-1$, $|\alpha| \leqq k$, $-1 \leqq \operatorname{Im} t < 0$ and $(x, y) \in X \times Y$, where $r_j = \min(1, (k+\delta)/(m-j))$. Moreover, if $\delta = 1$ and $\partial_x^\alpha a_j(x, y)$ ($|\alpha| = k+1$) are continuous, then (1.1) *holds for $|\alpha| \leqq k+1$.*

REMARK. The above theorem was announced by Ohya and Tarama [7] under the assumptions (A-1) and (A-2).

Let us assume that (A-1)$'$ is valid and that

(A-2)$'$ *$\partial_x^\alpha a_j(x, y)$ ($|\alpha| \leqq m$) are continuous.*

Define the localization polynomial $p_{(t,x;y)}(\tau, \xi)$ of p at $(t, x, y) \in \boldsymbol{R} \times X \times Y$ as

$$p(t+s\tau, x+s\xi, y) = s^\mu (p_{(t,x;y)}(\tau, \xi) + o(1)) \qquad \text{as } s \to 0,$$

where $p_{(t,x;y)}(\tau, \xi) \not\equiv 0$ in $(\tau, \xi) \in \boldsymbol{R}^{n+1}$. Then $p_{(t,x;y)}(\tau, \xi)$ is a homogeneous polynomial of degree μ. Moreover, it follows from Rouché's theorem and Lemma 2.4 below that

$$p_{(t,x;y)}(\tau, \xi) \neq 0 \qquad \text{if } \operatorname{Im} \tau \neq 0 \text{ and } \xi \in \boldsymbol{R}^n$$

(see, e.g., Hörmander [5]). We denote by $\Gamma(p_{(t,x;y)}, \vartheta)$ the connected component of the set $\{(\tau, \xi) \in \boldsymbol{R}^{n+1}; p_{(t,x;y)}(\tau, \xi) \neq 0\}$ which contains $\vartheta = (1, 0) \in \boldsymbol{R}^{n+1}$. For some properties of hyperbolic polynomials and $\Gamma(p_{(t,x;y)}, \vartheta)$ we refer to Atiyah, Bott and Gårding [1].

THEOREM 3. *Assume that* (A-1)$'$ *and* (A-2)$'$ *are sartisfied, and let $(t_0, x^0, y_0) \in \boldsymbol{R} \times X \times Y$. Then, for any compact subset M of $\Gamma(p_{(t_0,x^0;y_0)}, \vartheta)$ there is a neighborhood $\mathcal{U}$ of (t_0, x^0, y_0) in $\boldsymbol{R} \times X \times Y$ such that $M \subset \Gamma(p_{(t,x;y)}, \vartheta)$ for $(t, x, y) \in \mathcal{U}$.*

REMARK. In [9] we proved the above theorem when the $a_j(x, y)$ are sufficiently smooth.

In the rest of this paper we shall prove the above theorems.

2. Preliminaries.

Let $p(t)=t^m+\sum_{j=1}^m a_j t^{m-j}$ be a polynomial in t, where $a_j\in \boldsymbol{C}$.

LEMMA 2.1. *Let* $q(t)=\sum_{j=1}^m b_j t^{m-j}$, *and write* $p(t)+q(t)=\prod_{j=1}^m (t-\alpha_j(b_1, \cdots, b_m))$, *where the* $\alpha_j(b_1, \cdots, b_m)$ *are continuous functions of* $(b_1, \cdots, b_m)\in \boldsymbol{C}^m$. *Then there is a positive constant* $C(m)$, *depending only on* m, *such that*

(2.1) $|\alpha_j(b_1, \cdots, b_m)-\alpha_j^0|\leqq C(m)\max_{1\leqq k\leqq m}(|b_k|^{1/k}+|b_k|^{1/m}|\alpha_j^0|^{1-k/m}), \quad 1\leqq j\leqq m,$

where $\alpha_j^0=\alpha_j(0, \cdots, 0)$.

PROOF. There is an integer k_0 with $1\leqq k_0\leqq m$ such that $\alpha_j^0\notin\{z\in \boldsymbol{C}; (k_0-1)A\leqq |z-\alpha_1^0|<k_0A\}$ for $2\leqq j\leqq m$, where $A>0$ is determined latter. Therefore, we have

$$|p(z)|-|q(z)|\geqq (A/2)^m-\sum_{j=1}^m |b_j||z|^{m-j} \qquad \text{if } |z-\alpha_1^0|=(k_0-2^{-1})A\,.$$

It is easy to see that there is $C'(m)>0$, depending only on m, such that

$$(A/2)^m>m|b_j|(|\alpha_1^0|+(k_0-2^{-1})A)^{m-j}$$

if $1\leqq j\leqq m$, $A\geqq C'(m)(|b_j|^{1/j}+|b_j|^{1/m}|\alpha_1^0|^{1-j/m})$ and $b_j\neq 0$. Thus, Rouché's theorem shows that (2.1) with $C(m)=(m-2^{-1})C'(m)$ holds for $j=1$. Q.E.D.

In the proofs of theorems, we shall use Nuij's approximations (see [6]) and need the following

LEMMA 2.2. *Let* $p(t)=\prod_{j=1}^m (t-\alpha_j^0)$, *where* $\alpha_1^0\leqq\alpha_2^0\leqq\cdots\leqq\alpha_m^0$. *Then one can write* $(1+s(d/dt))^{m-1}p(t)=\prod_{j=1}^m(t-\alpha_j(s))$ *for* $s\in\boldsymbol{R}$, *where* $\alpha_1(s)\leqq\alpha_2(s)\leqq\cdots\leqq\alpha_m(s)$ *and* $\alpha_j(0)=\alpha_j^0$. *Moreover, there are positive constants* $c_1(m)$ *and* $c_2(m)$ *such that*

(2.2) $\qquad \alpha_j(s)-\alpha_{j-1}(s)\geqq c_1(m)|s| \qquad$ *for* $s\in\boldsymbol{R}$ *and* $2\leqq j\leqq m$,

(2.3) $\qquad 0<\pm(\alpha_j^0-\alpha_j(s))\leqq c_2(m)|s| \qquad$ *for* $\pm s>0$ *and* $1\leqq j\leqq m$.

PROOF. The first part of the lemma is obvious. Consider the case where $s>0$. Similarly, one can prove the lemma in the case where $s<0$. Assume that for a fixed l with $1\leqq l\leqq m-1$ there is $c_1(l)>0$ such that

(2.4) $\qquad \alpha_j^l(s)-\alpha_{j-1}^l(s)\geqq c_1(l)s \qquad$ *for* $s>0$ *and* $2\leqq j\leqq l$,

where $(1+s(d/dt))^{l-1}p(t)=\prod_{j=1}^m (t-\alpha_j^l(s))$ and $\alpha_1^l(s)\leqq\alpha_2^l(s)\leqq\cdots\leqq\alpha_m^l(s)$. Put

$$f(t, s)=(1+s(d/dt))^l p(t)/(1+s(d/dt))^{l-1}p(t)$$
$$(=1+s\sum_{j=1}^m (t-\alpha_j^l(s))^{-1})\,.$$

If $s>0$, $1\leqq h\leqq m$ and $\alpha_{h-1}^l(s)<t<\alpha_h^l(s)$, then

$$1+ms(t-\alpha_1^l(s))^{-1}<f(t, s)<1+s(t-\alpha_1^l(s))^{-1} \qquad \text{when } h=1,$$

$$1+(m-h+1)s(t-\alpha_h^l(s))^{-1}+s(t-\alpha_{h-1}^l(s))^{-1}<f(t,s)$$
$$<A_h+s(t-\alpha_h^l(s))^{-1}+s(t-\alpha_{h-1}^l(s))^{-1} \qquad when\ 2\leqq h\leqq m,$$

where $\alpha_0^l(s)=-\infty$ and $A_h=1$ if $h=2$ and $A_h=1+(h-2)s(\alpha_{h-1}^l(s)-\alpha_{h-2}^l(s))^{-1}$ if $3\leqq h\leqq m$. Therefore, we have

$$\begin{cases} \alpha_{h-1}^l(s)<\alpha_h^{l+1}(s)<\alpha_h^l(s), \\ \alpha_1^l(s)-ms<\alpha_1^{l+1}(s)<\alpha_1^l(s)-s \qquad when\ h=1, \\ \alpha_h^l(s)-2^{-1}(X_h+(m-h+2)s-[(X_h-(m-h+2)s)^2+4sX_h]^{1/2}) \\ \quad <\alpha_h^{l+1}(s)<\alpha_h^l(s)-F(X_h, 2s/A_h)/2 \qquad when\ 2\leqq h\leqq m, \end{cases} \tag{2.5}$$

where $X_h=\alpha_h^l(s)-\alpha_{h-1}^l(s)$ and $F(u,v)=u+v-(u^2+v^2)^{1/2}$, if $s>0$ and $\alpha_{h-1}^l(s)<\alpha_h^l(s)$. It is obvious that $\alpha_1^{l+1}(s)\leqq\alpha_1^l(s)\leqq\alpha_2^{l+1}(s)\leqq\cdots\leqq\alpha_m^{l+1}(s)\leqq\alpha_m^l(s)$ for $s\geqq 0$. Since $(X_h-(m-h+2)s)^2+4sX_h=(X_h-(m-h)s)^2+4(m-h+1)s^2\geqq(X_h-(m-h)s)^2$, (2.5) gives

$$0\leqq\alpha_h^l(s)-\alpha_h^{l+1}(s)\leqq(m-h+1)s \qquad for\ s\geqq 0\ and\ 1\leqq h\leqq m. \tag{2.6}$$

Moreover, it follows from (2.4) and (2.5) that

$$\alpha_{h+1}^{l+1}(s)-\alpha_h^{l+1}(s)\geqq\begin{cases} s \quad (h=1), \\ sF(c_1(l), 2c_1(l)/(h-2+c_1(l)))/2 \quad (2\leqq h\leqq l), \end{cases} \tag{2.7}$$

since $F(u_1,v_1)\geqq F(u_2,v_2)$ for $u_1\geqq u_2\geqq 0$ and $v_1\geqq v_2\geqq 0$. (2.7) shows that (2.4) is valid, replacing l with $l+1$, where $c_1(l+1)=\min\{1, F(c_1(l), 2c_1(l)/(l-2+c_1(l)))/2\}$ (>0). This proves (2.2). With $c_2(m)=m(m-1)$ (2.3) follows from (2.6). Q.E.D.

LEMMA 2.3. *If* $p(t)\neq 0$ *for* $\operatorname{Im} t<0$, *then*

$$(1+s(d/dt))p(t)\neq 0 \qquad for\ \operatorname{Im} t<0\ and\ \operatorname{Im} s\leqq 0.$$

PROOF. Let $p(t)=\prod_{j=1}^m(t-\alpha_j)$, where $\operatorname{Im}\alpha_j\geqq 0$. Then we have

$$(1+s(d/dt))p(t)=p(t)(1+s\textstyle\sum_{j=1}^m(t-\alpha_j)^{-1}).$$

It is obvious that $\operatorname{Im}(t-\alpha_j)^{-1}>0$ and $\operatorname{Im} s^{-1}\geqq 0$ if $\operatorname{Im} t<0$, $s\neq 0$ and $\operatorname{Im} s\leqq 0$. This proves the lemma (see [6]). Q.E.D.

LEMMA 2.4. *Let* $(t_0, x^0, y_0)\in \boldsymbol{R}\times X\times Y$, *and assume that* (A-1)′ *and* (A-2) *are satisfied. If* $\partial_t^h p(t_0,x^0,y_0)=0$ *for* $0\leqq h<l$ *and* $\partial_t^l p(t_0,x^0,y_0)\neq 0$, *then*

$$\partial_t^j\partial_x^\alpha p(t_0,x^0,y_0)=0 \qquad when\ j<l\ and\ |\alpha|<(l-j)r',$$

where $r'=\min(1,(k+\delta)/l)$.

PROOF. The lemma is well-known if (A-2)′ is satisfied (see, e.g., [5]). And we can prove the lemma similarly. Assume that there are j_0 and α^0 such that $j_0<l$, $|\alpha^0|<(l-j_0)r'$ and $\partial_t^{j_0}\partial_x^{\alpha^0}p(t_0,x^0,y_0)\neq 0$. Then we have $r''\equiv\min\{|\alpha|/(l-j);$

$\partial_t^j\partial_x^\alpha p(t_0, x^0, y_0)\neq 0$, $j<l$ and $|\alpha|\leqq k\}<r'$ $(\leqq 1)$. Write $r''=b/a$, where a and b are positive integers and mutually prime. Note that $1\leqq b<a$. It is easy to see that $p(t_0+s^{r''}\tau, x^0+s\xi, y_0)=q(\tau, \xi)s^{lr''}+o(s^{lr''})$ as $s\downarrow 0$, where $q(\tau, \xi)=c\tau^l+\sum_{0<j\leqq l/a} c_j(\xi)\tau^{l-aj}$, $c=\partial_t^l p(t_0, x^0, y_0)/l!\neq 0$ and $c_j(\xi)=\sum_{|\alpha|=bj}\partial_t^{l-aj}\partial_x^\alpha p(t_0, x^0, y_0)\cdot\xi^\alpha/((l-aj)!\alpha!)$. By assumption there is $\xi^0\in\boldsymbol{R}^n$ such that all $c_j(\xi^0)$ do not vanish. So there is $\tau_0\in\boldsymbol{C}\setminus\{0\}$ such that $q(\tau_0, \xi^0)=0$. Then we have $q(\tau, \pm\xi^0)=0$ if $\tau^a=(\pm 1)^b\tau_0^a$. On the other hand, (A-1)$'$ implies that $q(\tau, \pm\xi^0)\neq 0$ if $\operatorname{Im}\tau<0$. This gives $a=b=1$, which contradicts $a>1$.

Q. E. D.

LEMMA 2.5. *Let M be an arcwise connected subset of $\boldsymbol{R}^{n'}$, U a Hausdorff topological space and $S=\{s\in\boldsymbol{C}; |s|\leqq s_0$ and $\operatorname{Im} s\leqq 0\}$. Let $f(s, w, u)$ be a continuous function on $S\times M\times U$ which satisfies the following conditions;* (i) *$f(s, w, u)$ is analytic in s if* $\operatorname{Im} s<0$, (ii) *there is a dense subset U' of U such that $f(s, w, u)\neq 0$ for $s\in S\cap\boldsymbol{R}$, $w\in M$ and $u\in U'$,* (iii) *$f(s, w, u)\neq 0$ if $|s|=s_0$, and* (iv) *there is $w^0\in M$ such that $f(s, w^0, u)\neq 0$ if* $\operatorname{Im} s<0$. *Then*

$$f(s, w, u)\neq 0 \qquad \text{if } \operatorname{Im} s<0\,.$$

PROOF. Assume that there are $(s_1, w^1, u_1)\in S\times M\times U$ such that $\operatorname{Im} s_1<0$ and $f(s_1, w^1, u_1)=0$. Since $f(s, w^1, u_1)\not\equiv 0$ in s, applying Rouché's theorem (or a variant of the Weierstrass preparation theorem), we may assume that $u_1\in U'$. Let $\{w(\theta)\}_{0\leqq\theta\leqq 1}$ be a continuous curve in M satisfying $w(0)=w^1$ and $w(1)=w^0$. Then it follows from the conditions (i)–(iii) that there is a continuous function $s(\theta)$ defined on $[0, 1]$ such that $s(0)=s_1$ and $f(s(\theta), w(\theta), u_1)=0$ for $\theta\in[0, 1]$. Observe that $\operatorname{Im} s(\theta)<0$ and $|s(\theta)|<s_0$ for $\theta\in[0, 1]$. Therefore we have $f(s(1), w^0, u_1)=0$, which contradicts the condition (iv). This proves the lemma. Q. E. D.

The following lemma is elementary (see, e. g., [10]).

LEMMA 2.6. *Let V_l be the vector space of all homogeneous polynomials with real coefficients in ξ of degree l. Then there are $p_1(\xi), \cdots, p_\nu(\xi)\in V_1$ such that $\{p_1(\xi)^l, \cdots, p_\nu(\xi)^l\}$ is a basis of V_l, where $\nu=\dim V_l$.*

3. Proof of Theorem 1.

Put

$$\tilde{p}(t, x, y, z)=(1+z^r\partial_t)^{m-1}p(t, x, y) \qquad \text{for } z\in\boldsymbol{C} \text{ with } \operatorname{Im} z\leqq 0\,,$$

where $1^r=1$. By Lemma 2.2 the equation $\tilde{p}(t, x, y, z)=0$ has only real roots for $(x, y)\in X\times Y$, if $z\geqq 0$ or $z\in\boldsymbol{R}$ and $r=1$. Moreover, if $z\geqq 0$ or $z\in\boldsymbol{R}$ and $r=1$, then we have

$$(3.1)\qquad \tilde{\lambda}_j(x, y, z)-\tilde{\lambda}_{j-1}(x, y, z)\geqq c_1(m)|z|^r, \qquad 2\leqq j\leqq m,$$

$$(3.2)\qquad |\lambda_j(x, y)-\tilde{\lambda}_j(x, y, z)|\leqq c_2(m)|z|^r, \qquad 1\leqq j\leqq m,$$

for $(x, y)\in X\times Y$, where $\tilde{p}(t, x, y, z)=\prod_{j=1}^m(t-\tilde{\lambda}_j(x, y, z))$ and $\tilde{\lambda}_1(x, y, z)\leqq\tilde{\lambda}_2(x, y, z)\leqq\cdots\leqq\tilde{\lambda}_m(x, y, z)$. If $z\leqq 0$ and $r<1$, then Lemma 2.3 gives

$$(3.3)\qquad \tilde{p}(t+z^r, x, y, z)\neq 0 \qquad \textit{when } \operatorname{Im} t<|z|^r\sin r\pi.$$

Write

$$a_j(x+z\xi, y)=\sum_{|\alpha|\leqq k} z^{|\alpha|}\xi^\alpha\partial_x^\alpha a_j(x, y)/\alpha!+\tilde{a}_j(x, \xi, y, z),$$

where $z\in\boldsymbol{R}$, $\xi\in\boldsymbol{R}^n$, $x\in X$, $x+z\xi\in X$ and $y\in Y$. Then the condition (A-2) implies that there is $A>0$ such that

$$(3.4)\qquad |\tilde{a}_j(x, \xi, y, z)|\leqq A|z|^{mr}|\xi|^{mr}$$

if $z\in\boldsymbol{R}$, $\xi\in\boldsymbol{R}^n$, $x\in X$, $x+z\xi\in X$ and $y\in Y$. Let U be an open subset of X such that $U\Subset X$, and put

$$P(t, x, \xi, y, z)=(1+z^r\partial_t)^{m-1}(t^m+\sum_{j=1}^m t^{m-j}\sum_{|\alpha|\leqq k} z^{|\alpha|}\xi^\alpha\partial_x^\alpha a_j(x, y)/\alpha!).$$

From Lemma 2.1 and (3.1) it follows that there are $\delta_0>0$ and $\delta_1>0$ such that $P(t, x, \xi, y, z)=0$ has only simple roots for $(x, \xi, y)\in\Omega(U;\delta_1)$ if $0<z\leqq\delta_0$ or $z\in[-\delta_0, \delta_0]\backslash\{0\}$ and $r=1$, where $\Omega(U;\delta_1)=\{(x, \xi, y)\in U\times\boldsymbol{R}^n\times Y;\ |\xi|\leqq\delta_1\}$. Since the $\tilde{a}_j(x, \xi, y, z)$ are real-valued, $P(t, x, \xi, y, z)=0$ has only real roots for $(x, \xi, y)\in\Omega(U;\delta_1)$ if $0\leqq z\leqq\delta_0$ or $-\delta_0\leqq z\leqq\delta_0$ and $r=1$. Therefore, we can write

$$P(t+z^r, x, \xi, y, z)=\prod_{j=1}^m(t-\Lambda_j(x, \xi, y, z))$$

for $(x, \xi, y)\in\Omega(U;\delta_1)$ and $0\leqq z\leqq\delta_0$, where $\Lambda_1(x, \xi, y, z)\leqq\Lambda_2(x, \xi, y, z)\leqq\cdots\leqq\Lambda_m(x, \xi, y, z)$. It follows from Lemma 2.1 that there is $c>0$ such that

$$(3.5)\qquad |\Lambda_j(x, \xi, y, z)-\tilde{\lambda}_j(x+z\xi, y, z)|\leqq cz^r$$

if $(x, \xi, y)\in\Omega(U;\delta_1)$ and $0\leqq z\leqq\delta_0$. Moreover, by Lemma 2.1, (3.3) and (3.4) we have $P(t+z^r, x, \xi, y, z)\neq 0$ for $(x, \xi, y)\in\Omega(U;\delta_1)$, $\operatorname{Im} t<0$ and $z\in[-\delta_0, \delta_0]$, if necessary, modifying δ_0 and δ_1. Let $t\in\boldsymbol{R}$, $z\in(0, \delta_0/2]$ and $(x, \xi, y)\in\Omega(U;\delta_1)$, and write

$$P(t+(z+s\zeta)^r+z^{r-1}s\tau, x, \xi, y, z+s\zeta)=s^\mu(P_{(t,z;x,\xi,y)}(\tau, \zeta)+o(1)) \qquad \textit{as } s\downarrow 0,$$

where $P_{(t,z;x,\xi,y)}(\tau, \zeta)\not\equiv 0$ in (τ, ζ). Then $P_{(t,z;x,\xi,y)}(\tau, \zeta)$ is a homogeneous polynomial in (τ, ζ) of degree μ and satisfies

$$(3.6)\qquad P_{(t,z;x,\xi,y)}(\tau, \zeta)\neq 0 \qquad \textit{if } \operatorname{Im}\tau<0 \textit{ and } \zeta\in\boldsymbol{R}.$$

In fact, $P(z^{r-1}\tilde{t}+\tilde{z}^r, x, \xi, y, \tilde{z})$ is analytic in $(\tilde{t}, \tilde{z})$ and microhyperbolic with respect to $(-1, 0)\in\boldsymbol{R}^2$ near $(\tilde{t}, \tilde{z})=(z^{1-r}t, z)$. This verifies (3.6) (see, e.g., Lemma 8.7.2 in [5]), which easily follows from Lemma 2.4 and Rouché's theorem. Note that

$P_{(t,z;x,\xi,y)}(\tau,\zeta)$ can be defined and satisfies (3.6) when $r=1$ and $z=0$. Put $f(s,\zeta,(t,x,\tau,\xi,y,z))=P(t+(z+s\zeta)^r+z^{r-1}s\tau,x,\xi,y,z+s\zeta)$ for $s\in\boldsymbol{C}$ with $\operatorname{Im}s\leqq 0$ and $|s|\leqq s_0$, $\tau\in[1/2,2]$, $\zeta\in[0,1]$, $t\in\boldsymbol{C}$ with $\operatorname{Im}t\leqq 0$, $(x,\xi,y)\in\Omega(U;\delta_1)$ and $z\in(0,\varepsilon]$, where $s_0\leqq\delta_0/2$ and $\varepsilon\leqq\delta_0/2$. If $r<1$, then it is easy to see that (i) f is analytic in s for $\operatorname{Im}s<0$, (ii) $f(s,\zeta,(t,x,\tau,\xi,y,z))\neq 0$ when $\operatorname{Im}t<0$ and $s\in\boldsymbol{R}$, (iii) for any $T>0$ there is $\varepsilon>0$ such that $f(s,\zeta,(t,x,\tau,\xi,y,z))\neq 0$ when $|s|=s_0$, $|t|\leqq T$ and $z\in(0,\varepsilon]$, and (iv) $f(s,0,(t,x,\tau,\xi,y,z))\neq 0$ when $\operatorname{Im}s<0$. In fact, the assertions (i), (ii) and (iv) are obvious. Since $\lim_{|t|\to\infty}t^{-m}P(t,x,\xi,y,z)=1$, the assertion (iii) is also obvious. Therefore, it follows from Lemma 2.5 that

$$P(t+(z+s\zeta)^r+z^{r-1}s\tau,x,\xi,y,z+s\zeta)\neq 0 \tag{3.7}$$

if $r<1$, $\operatorname{Im}s<0$, $|s|\leqq s_0$, $\tau\in[1/2,2]$, $\zeta\in[0,1]$, $\operatorname{Im}t\leqq 0$, $|t|\leqq T$, $(x,\xi,y)\in\Omega(U;\delta_1)$ and $z\in(0,\varepsilon]$. Next let us consider the case where $r=1$. From (3.6), for any $(t_0,x^0,\xi^0,y_0)\in\boldsymbol{R}\times U\times\boldsymbol{R}^n\times Y$ with $|\xi^0|\leqq\delta_1/2$ there is $c>0$ such that

$$P_{(t_0,0;x^0,\xi^0,y_0)}(1,\zeta)\neq 0 \qquad \textit{if } \zeta\in[0,c].$$

Therefore, there are $s_0>0$, $\varepsilon>0$ and a neighborhood V of y_0 in Y such that

$$P(t+(z+s\zeta)+s\tau,x,\xi,y,z+s\zeta)\neq 0 \tag{3.8}$$

if $|s|=s_0$, $\tau\in[1-\varepsilon,1+\varepsilon]$, $\zeta\in[0,c]$, $|t-t_0|<\varepsilon$, $(x,\xi,y)\in X\times\boldsymbol{R}^n\times V$, $|x-x^0|<\varepsilon$, $|\xi-\xi^0|<\varepsilon$ and $z\in[0,\varepsilon]$. For we can write

$$P(t+(z+s\zeta)+s\tau,x,\xi,y,z+s\zeta)=\textstyle\sum_{j=0}^{\mu_0}s^jP_j(t,x,\xi,y,z,\tau,\zeta)+o(s^{\mu_0}) \qquad \textit{as } s\to 0,$$

where $P_{\mu_0}(t_0,x^0,\xi^0,y_0,0,\tau,\zeta)=P_{(t_0,0;x^0,\xi^0,y_0)}(\tau,\zeta)$. Since $P_j(t_0,x^0,\xi^0,y_0,0,\tau,\zeta)\equiv 0$ for $j<\mu_0$, we have (3.8). Similarly, it follows from Lemma 2.5 that (3.7) is valid if $r=1$ and $\operatorname{Im}s<0$, $|s|\leqq s_0$, $\tau\in[1-\varepsilon,1+\varepsilon]$, $\zeta\in[0,c]$, $\operatorname{Im}t\leqq 0$, $|t-t_0|<\varepsilon$, $(x,\xi,y)\in X\times\boldsymbol{R}^n\times V$, $|x-x^0|<\varepsilon$, $|\xi-\xi^0|<\varepsilon$ and $z\in[0,\varepsilon]$. Since $\bar{U}$ and Y are compact, for any $T>0$ there are positive constant c, s_0, ε and δ_1 such that (3.7) holds if $r\leqq 1$, $\operatorname{Im}s<0$, $|s|\leqq s_0$, $\tau\in[1-\varepsilon,1+\varepsilon]$, $\zeta\in[0,c]$, $\operatorname{Im}t\leqq 0$, $|t|\leqq T$, $(x,\xi,y)\in\Omega(U;\delta_1)$ and $z\in(0,\varepsilon]$. This implies that $P_{(t,z;x,\xi,y)}(1,\zeta)\neq 0$ if $t\in\boldsymbol{R}$, $|t|<T$, $z\in(0,\varepsilon)$, $(x,\xi,y)\in\Omega(U;\delta_1)$ and $\zeta\in[0,c]$. In fact, if there are $t_0\in\boldsymbol{R}$, $z_0\in(0,\varepsilon)$, $(x^0,\xi^0,y_0)\in\Omega(U;\delta_1)$ and $\zeta_0\in[0,c]$ such that $|t_0|<T$ and $P_{(t_0,z_0;x^0,\xi^0,y_0)}(1,\zeta_0)=0$, then Rouché's theorem gives a contradiction to the fact that (3.7) is valid when $r\leqq 1$, $\operatorname{Im}s<0$, $|s|\leqq s_0$, $\tau\in[1-\varepsilon,1+\varepsilon]$, $\zeta\in[0,c]$, $\operatorname{Im}t\leqq 0$, $|t|\leqq T$, $(x,\xi,y)\in\Omega(U;\delta_1)$ and $z\in(0,\varepsilon]$. This proves the assertion.

Now we can prove Theorem 1. It is obvious that

$$\begin{aligned}0&=P(\Lambda_j(x,\xi,y,z+s\zeta)+(z+s\zeta)^r,x,\xi,y,z+s\zeta)\\&=s^{\mu}(P_{(\Lambda_j(x,\xi,y,z),z;x,\xi,y)}(z^{1-r}s^{-1}(\Lambda_j(x,\xi,y,z+s\zeta)\\&\quad-\Lambda_j(x,\xi,y,z)),\zeta)+o(1)) \qquad \textit{as } s\downarrow 0,\end{aligned}$$

where μ depends on (x, ξ, y, z) and j, if $(x, \xi, y)\in\Omega(U;\delta_1)$, $\zeta\in[0, c]$ and $z\in(0, \varepsilon)$. Therefore, we have

$$\partial_s \Lambda_j(x, \xi, y, z+s\zeta)|_{s=0}<z^{r-1} \tag{3.9}$$

when $(x, \xi, y)\in\Omega(U;\delta_1)$, $\zeta\in[0, c]$ and $z\in(0, \varepsilon)$. It follows from (3.2), (3.5) and (3.9) that there is $C>0$ such that

$$\lambda_j(x+z\xi, y)-\lambda_j(x, y)\leqq Cz^r \quad \textit{if } (x, \xi, y)\in\Omega(U;\delta_1) \textit{ and } z\in[0, \varepsilon]. \tag{3.10}$$

Replacing $x+z\xi$ and x with x and $x+z\xi$ in (3.10), respectively, we have, with some constant $C'>0$,

$$|\lambda_j(x^1, y)-\lambda_j(x^2, y)|\leqq C'|x^1-x^2|^r \quad \textit{if } x^1, x^2\in U \textit{ and } y\in Y.$$

This proves Theorem 1.

4. Proof of Theorem 2.

From Lemma 4.1.1 in [4] it follows that there is $C>0$ such that

$$|\partial_t^j p(t, x, y)|/|p(t, x, y)|\leqq C|\mathrm{Im}\, t|^{-j}$$

if $\mathrm{Im}\, t<0$, $x\in X$ and $y\in Y$. Therefore, it suffices to prove (1.1) for $j=0$. In fact, the Gauss theorem implies that $\partial_t^j p(t, x, y)$ satisfies (A-1)$'$. First let us consider the case where $r=1$. Write

$$(1+i)p(t, x, y)=p_1(t, x, y)+ip_2(t, x, y),$$

where p_h $(h=1, 2)$ are polynomials in t with real coefficients for $(x, y)\in X\times Y$. Then the Hermite theorem implies that $p_h(t, x, y)\neq 0$ if $\mathrm{Im}\, t\neq 0$, $x\in X$ and $y\in Y$. From (A-1)$'$ it follows that $|p_h(t, x, y)|\leqq 2^{1/2}|p(t, x, y)|$ if $\mathrm{Im}\, t<0$ and $(x, y)\in X\times Y$. In fact, it is obvious that $|t-\bar{\alpha}|/|t-\alpha|\leqq 1$ if $\mathrm{Im}\, t<0$ and $\mathrm{Im}\, \alpha\geqq 0$. Therefore, it suffices to prove Theorem 2 in the case where p satisfies (A-1) and (A-2). Assume that p satisfies (A-1) and (A-2). Then, with the notations in § 3, similarly we have $P(t+s, x, \xi, y, s\zeta)\neq 0$ if $\mathrm{Im}\, s<0$, $|s|\leqq s_0$, $\zeta\in[-c, c]$, $\mathrm{Im}\, t\leqq 0$ and $(x, \xi, y)\in\Omega(U;\delta_1)$. So there is $c>0$ such that $P(t, x, \xi, y, z)\neq 0$ if $-2\leqq\mathrm{Im}\, t<0$, $(x, \xi, y)\in\Omega(U;\delta_1)$, $z\in\boldsymbol{C}$ and $|z|\leqq c|\mathrm{Im}\, t|$. Since $P(t, x, \xi, y, z)$ is a polynomial in (t, z), it follows from Lemma 4.1.1 in [4] that there is $C>0$ such that

$$|\partial_t^j\partial_z^h P(t, x, \xi, y, z)|_{z=0}/|P(t, x, \xi, y, 0)|\leqq C|\mathrm{Im}\, t|^{-j-h}$$

if $-1\leqq\mathrm{Im}\, t<0$ and $(x, \xi, y)\in\Omega(U;\delta_1)$. It is obvious that

$$\partial_t^j\partial_z^h((1+z\partial_t)^{m-1}p(t, x+z\xi, y)-P(t, x, \xi, y, z))|_{z=0}=0$$

for $0\leqq h\leqq m-1$. So we have, inductively, (4.1) $$|\partial_t^j\partial_z^h p(t, x+z\xi, y)|_{z=0}/|p(t, x, y)|\leqq C|\mathrm{Im}\, t|^{-j-h}$$

if $-1\leqq \mathrm{Im}\, t<0$, $(x, \xi, y)\in\Omega(U;\delta_1)$ and $0\leqq h\leqq m-1$. It is obvious that (4.1) holds for $j=0$ and $h=m$ if $\partial_x^\alpha a_j(x, y)$ $(|\alpha|=m)$ are continuous. Therefore, Theorem 2 immediately follows from Lemma 2.6 if $r=1$. Next consider the case $r<1$. Put

$$P(t, x, \xi, y)=t^m+\textstyle\sum_{j=1}^m t^{m-j}\sum_{|\alpha|\leqq k}\xi^\alpha\partial_x^\alpha a_j(x, y)/\alpha!\,,$$
$$f(s, \xi, (t, x, y, \nu))=P(t+s^r+\nu\omega s^r|\xi|^r, x, s\xi, y)$$

for $\mathrm{Im}\, t\leqq 0$, $\mathrm{Im}\, s\leqq 0$, $(x, \xi, y)\in X\times \boldsymbol{R}^n\times Y$ and $\nu>0$, where $\omega=\exp[i(r-1)\pi/2]$ and $1^r=1$. Let $(t_0, x^0, y_0)\in\boldsymbol{R}\times X\times Y$. Then we have the following: (i) $f(s, \xi, (t, x, y, \nu))$ is analytic in s if $\mathrm{Im}\, s<0$. (ii) For any open subset U of X with $U\Subset X$, there are positive constants ν_U, δ_0 and δ_1 such that $f(s, \xi, (t, x, y, \nu_U))\neq 0$ if $s\in[-\delta_0, \delta_0]$, $\mathrm{Im}\, t<0$ and $(x, \xi, y)\in\Omega(\bar{U};\delta_1)$. (iii) There are positive constants c, s_0 and ε and a neighborhood V of y_0 in Y such that $s_0\leqq\delta_0$, $c\leqq\delta_1$ and $f(s, \xi, (t, x, y, \nu_0))\neq 0$ if $|s|=s_0$, $(t, x, \xi, y)\in\boldsymbol{C}\times X\times\boldsymbol{R}^n\times V$, $|t-t_0|\leqq\varepsilon$, $|x-x^0|\leqq\varepsilon$ and $|\xi|\leqq c$, where $\nu_0=\nu_U$ with $U=\{x\in X;|x-x^0|<\varepsilon\}$. (iv) $f(s, 0, (t, x, y, \nu))\neq 0$ if $\mathrm{Im}\, s<0$. In fact, we have

$$p(t+s^r+\nu\omega s^r|\xi|^r, x+s\xi, y)\neq 0$$

if $\mathrm{Im}\, t<\nu|s|^r|\xi|^r\sin(1-r)\pi/2$, $s\in\boldsymbol{R}$ and $x+s\xi\in X$. Since

$$|\tilde{a}_j(x, s\xi, y)|\leqq A|s|^{mr}|\xi|^{mr}$$

for $(x, \xi, y)\in X\times\boldsymbol{R}^n\times Y$, $s\in\boldsymbol{R}$ and $x+s\xi\in X$, where $p(t, x+\xi, y)-P(t, x, \xi, y)=\sum_{j=1}^m\tilde{a}_j(x, \xi, y)t^{m-j}$, the assertion (ii) easily follows from Lemma 2.1. Write

$$P(t_0+s^r\tau, x^0, s\xi, y_0)=s^{\mu_0}(P_{(t_0,x^0,y_0)}(\tau, \xi)+o(1))\qquad as\ s\to 0\,,$$

where $P_{(t_0,x^0,y_0)}(\tau, \xi)\not\equiv 0$ in (τ, ξ). Then we have $\mu_0\leqq mr$ and

$$P_{(t_0,x^0,y_0)}(\tau, \xi)=\textstyle\sum_{jr+|\alpha|=\mu_0}\tau^j\xi^\alpha\partial_t^j\partial_x^\alpha p(t_0, x^0, y_0)/(j!\alpha!)$$

if $\mu_0<mr$. Therefore, it follows from Lemma 2.4 that $P_{(t_0,x^0,y_0)}(1, 0)\neq 0$. One can also prove that $P_{(t_0,x^0,y_0)}(\tau, \xi)=P_{(t_0,x^0,y_0)}(\tau, 0)$ if $\mu_0<mr$. We can write

$$P(t+s^r\tau, x, s\xi, y)=\textstyle\sum_{\mu\leqq\mu_0}s^\mu f_\mu(t, x, y;\tau, \xi)+o(s^{\mu_0})\qquad as\ s\to 0\,,$$
$$f_\mu(t_0, x^0, y_0;\tau, \xi)\equiv 0\qquad for\ \mu<\mu_0\,,$$
$$f_{\mu_0}(t_0, x^0, y_0;\tau, \xi)=P_{(t_0,x^0,y_0)}(\tau, \xi)\,.$$

This verifies the assertion (iii). From Lemma 2.5 it follows that $f(s, \xi, (t, x, y, \nu_0))\neq 0$ if $\mathrm{Im}\, s<0$, $|s|\leqq s_0$, $(t, x, \xi, y)\in\boldsymbol{C}\times X\times\boldsymbol{R}^n\times V$, $\mathrm{Im}\, t\leqq 0$, $|t-t_0|\leqq\varepsilon$, $|x-x^0|\leqq\varepsilon$ and $|\xi|\leqq c$. Therefore, there are positive constants ε' and δ' such that $P(t, x, s\xi, y)\neq 0$ if $(t, x, \xi, y)\in\boldsymbol{C}\times X\times\boldsymbol{R}^n\times V$, $|\mathrm{Re}\, t-t_0|\leqq\varepsilon'$, $-\varepsilon'\leqq\mathrm{Im}\, t<0$, $|x-x^0|\leqq\varepsilon$, $|\xi|\leqq 1$, $s\in\boldsymbol{C}$ and $|s|^r\leqq\delta'|\mathrm{Im}\, t|$. In fact, we have $\{(t, s\xi);|\mathrm{Re}\, t-t_0|\leqq\varepsilon', -\varepsilon'\leqq\mathrm{Im}\, t<0, s\in\boldsymbol{C}$ and $|s|^r\leqq\delta'|\mathrm{Im}\, t|\}\subset\{(t+s^r(1+\nu_0\omega c^r|\hat{\xi}|^r), cs\hat{\xi});|t-t_0|\leqq\varepsilon, \mathrm{Im}\, t\leqq 0, \mathrm{Im}\, s\leqq 0, \mathrm{Im}\, t+\mathrm{Im}\, s<0, |s|\leqq s_0$

and $\hat{\xi}=\pm\xi\}$ if $\xi\in\boldsymbol{R}^n$, $|\xi|\leqq 1$, $3\varepsilon'\leqq\varepsilon$, $\delta'<(c^{-r}+\nu_0)^{-1}$ and $\delta'\varepsilon'\leqq c^r s_0$. Applying Lemma 4.1.1 in [4] to the polynomial $P(t, x, s\xi, y)$ in s, we have

$$(4.2)\qquad |\partial_s^j P(t, x, s\xi, y)|_{s=0}/|P(t, x, 0, y)|\leqq C|\mathrm{Im}\, t|^{-j/r}$$

if $(t, x, \xi, y)\in\boldsymbol{C}\times X\times\boldsymbol{R}^n\times V$, $|\mathrm{Re}\, t-t_0|\leqq\varepsilon'$, $-\varepsilon'\leqq\mathrm{Im}\, t<0$, $|x-x^0|\leqq\varepsilon$, $|\xi|\leqq 1$. Since $\partial_s^j P(t, x, s\xi, y)|_{s=0}=\partial_s^j p(t, x+s\xi, y)|_{s=0}$ for $j\leqq k$, (4.2) and Lemma 2.6 prove the first part of theorem 2. Then the second part of Theorem 2 is obvious.

5. Proof of Theorem 3.

Write

$$(5.1)\qquad p(t_0+s\tau, x^0+s\xi, y_0)=s^{\mu}(p_{(t_0,x^0;y_0)}(\tau, \xi)+o(1))\qquad as\ s\to 0\,,$$

and put $\alpha=p_{(t_0,x^0;y_0)}(1, 0)$ $(\in\boldsymbol{C}\backslash\{0\})$ and

$$p_1(t, x, y)+ip_2(t, x, y)=\bar{\alpha}(1+i)p(t, x, y)\,,$$

where $p_j(t, x, y)$ $(j=1, 2)$ are polynomials in t with real coefficients. Then it follows from the Hermite theorem that $p_j(t, x, y)$ $(j=1, 2)$ satisfy (A-1), and that

$$p_j(t_0+s\tau, x^0+s\xi, y_0)=s^{\mu}(\bar{\alpha}p_{(t_0,x^0;y_0)}(\tau, \xi)+o(1))\qquad as\ s\to 0\,.$$

Thus we have $\Gamma(p_{(t_0,x^0;y_0)}, \vartheta)=\Gamma(p_{j(t_0,x^0;y_0)}, \vartheta)$ $(j=1, 2)$. On the other hand, $\Gamma(p_{(t_0,x^0;y_0)}, \vartheta)$ is equal to at least one of $\Gamma(p_{j(t_0,x^0;y_0)}, \vartheta)$ $(j=1, 2)$. Therefore, it suffices to prove the theorem under the assumptions (A-1) and (A-2)$'$. Assume that p satisfies (A-1) and (A-2)$'$. Put

$$P(t, x, \xi, y, s, \nu)=(1+s\nu|\xi|\partial_t)^{m-1}(t^m+\textstyle\sum_{j=1}^m t^{m-j}\sum_{|\alpha|\leq m} s^{|\alpha|}\xi^{\alpha}\partial_x^{\alpha}a_j(x, y)/\alpha!)\,.$$

Then, for any $U\Subset X$ and any $\nu>0$ there is $\delta_0\equiv\delta_0(U, \nu)>0$ such that

$$(5.2)\qquad P(t, x, \xi, y, s, \nu)\neq 0$$

if $\mathrm{Im}\, t\neq 0$, $(x, \xi, y)\in U\times\boldsymbol{R}^n\times Y$, $|\xi|\leqq 2$ and $s\in[-\delta_0, \delta_0]$. In fact, we have

$$(1+s\nu|\xi|\partial_t)^{m-1}p(t, x+s\xi, y)-P(t, x, \xi, y, s, \nu)=\textstyle\sum_{j=1}^m \tilde{a}_j(x, \xi, y, s, \nu)t^{m-j}\,,$$
$$\tilde{a}_j(x, \xi, y, s, \nu)=o(s^m|\xi|^m)$$

if $(x, \xi, y)\in U\times\boldsymbol{R}^n\times Y$, $|\xi|\leqq 2$, $s\in[-1, 1]$ and $x+s\xi\in X$. Thus Lemmas 2.1 and 2.2 give (5.2), applying the same argument as in § 3. Since $\mu\leqq m$ in (5.1), we have

$$P(t_0+s\tau, x^0, \xi, y_0, s, \nu)=s^{\mu}\{(1+\nu|\xi|\partial_\tau)^{m-1}p_{(t_0,x^0;y_0)}(\tau, \xi)+o(1)\}\qquad as\ s\to 0\,.$$

From Lemma 2.2 or its proof, it follows that

$$(5.3)\qquad \{(\tau, \xi)\in\boldsymbol{R}^{n+1}\,;(\tau-c_2(m)\nu|\xi|, \xi)\in\Gamma_\nu\}\subset\Gamma(p_{(t,x;y)}, \vartheta)\subset\Gamma_\nu\,,$$

where $\Gamma_\nu\equiv\Gamma_{(t,x,y,\nu)}=\Gamma((1+\nu|\xi|\partial_\tau)^{m-1}p_{(t,x;y)}(\tau, \xi), \vartheta)$. For a compact subset M of

$\Gamma(p_{(t_0,x^0;y_0)}, \vartheta) \cap \{(\tau,\xi) \in \boldsymbol{R}^{n+1}; |\xi| \leqq 1\}$ there are $\nu_0 > 0$ and a compact subset $\tilde{M}$ of $\Gamma_{(t_0,x^0,y_0,\nu_0)}$ such that $\{(\tau,\xi); (\tau + c_2(m)\nu_0|\xi|, \xi) \in M\} \subset \mathring{\tilde{M}}$, where $\mathring{\tilde{M}}$ denotes the interior of $\tilde{M}$. It is easy to see that there are $s_0 > 0$, $\varepsilon > 0$ and a neighborhood V of y_0 in Y such that

$$P(t+s\tau, x, \xi, y, s, \nu_0) \neq 0$$

if $|t - t_0| \leqq \varepsilon$, $|x - x^0| \leqq \varepsilon$, $y \in V$, $|s| = s_0$ and $(\tau, \xi) \in \tilde{M}$. We may assume that $\tilde{M}$ is convex and $\vartheta \in \tilde{M}$. So we can apply Lemma 2.5 and obtain

$$P(t+s\tau, x, \xi, y, s, \nu_0) \neq 0 \tag{5.4}$$

If $\operatorname{Im} t \leqq 0$, $|t - t_0| \leqq \varepsilon$, $|x - x^0| \leqq \varepsilon$, $y \in V$, $\operatorname{Im} s < 0$, $|s| \leqq s_0$ and $(\tau, \xi) \in \tilde{M}$. Assume that there are $t_1 \in \boldsymbol{R}$, $x^1 \in X$, $y_1 \in V$ and $(\tau_1, \xi^1) \in \mathring{\tilde{M}}$ such that $|t_1 - t_0| < \varepsilon$, $|x^1 - x^0| < \varepsilon$ and $(1 + \nu_0|\xi^1|\partial_\tau)^{m-1} p_{(t_1,x^1;y_1)}(\tau_1, \xi^1) = 0$. Then there is $\delta' > 0$ such that $(\tau_1 \pm \delta', \xi^1) \in \tilde{M}$ and $|(1 + \nu_0|\xi^1|\partial_\tau)^{m-1} p_{(t_1,x^1;y_1)}(\tau_1 + \lambda, \xi^1)| > c$ for $\lambda \in \boldsymbol{C}$ with $|\lambda| = \delta'$, where $c > 0$. Rouché's theorem implies that there are $s_1 > 0$ and a function $\lambda(s)$ defined on $[0, s_1]$ such that $|\lambda(s)| < \delta'$ and

$$P(t_1 + s \operatorname{Im} \lambda(s) - is(\tau_1 + \operatorname{Re} \lambda(s)), x^1, \xi^1, y_1, -is, \nu_0) = 0$$

for $0 < s \leqq s_1$. This contradicts (5.4). Therefore, we have $\mathring{\tilde{M}} \subset \Gamma_{(t,x,y,\nu_0)}$ if $(t, x, y) \in \boldsymbol{R} \times X \times V$, $|t - t_0| < \varepsilon$ and $|x - x^0| < \varepsilon$. From (5.3) it follows that $M \subset \Gamma(p_{(t,x;y)}, \vartheta)$ if $(t, x, y) \in \boldsymbol{R} \times X \times V$, $|t - t_0| < \varepsilon$ and $|x - x^0| < \varepsilon$. This proves the theorem.

We remark that one can easily prove Theorem 3 and, therefore, Theorems 1 and 2 if the coefficients $a_j(x, y)$ satisfy the condition (A-2) with $k = m$. In fact, one has only to apply the above argument to $P(t, x, \xi, y, s) = (t - \omega s^a)^m + \sum_{j=1}^m (t - \omega s^a)^{m-j} \times \sum_{|\alpha| \leq m} s^{|\alpha|} \xi^\alpha \partial_x^\alpha a_j(x, y)/\alpha!$, where $1 < a < 1 + \delta/m$ $(\leqq 2)$, $\omega = \exp[i(a-1)\pi/2]$ and $(-1)^a = \exp[-ia\pi]$.

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Institute of Mathematics
University of Tsukuba
Ibaraki 305, Japan