TSUKUBA J. MATH.
Vol. 8 No. 2 (1984). 255–259

POLYNOMIAL INVARIANTS OF EUCLIDEAN LIE ALGEBRAS

By

Zenji KOBAYASHI

1. Introduction.

Let Δ be a root system in the sense of Bourbaki [1] and W be its Weyl group. Δ spans an l dimensional real vector space V on which W acts as a finite linear group. By extension of the transpose action, W acts on the symmetric algebra $S(V)$ of the dual space V^*. There is a well known theorem of Chevalley [2], that is, the ring of W-invariant elements of $S(V)$ is generated by l algebraically independent homogeneous polynomials.

How will the situation change when Δ is an infinite root system and W is an infinite group acting in V defined by a generalized Cartan matrix of non-finite type? With regard to this, there is a work of Moody [4], that the indefinite quadratic form by itself generates the entire ring of invariants, when Δ is defined by a generalized Cartan matrix of hyperbolic type.

In this paper, we study the ring of polynomial invariants of the Weyl group of a Euclidean Lie algebra. In this case, V is not isomorphic to V^* as W-module. So we have to consider both the ring of invariants of $S(V)$ and of $S(V^*)$, the symmetric algebra of V^*.

It becomes clear that the unique invariant vector, called null root, by itself generates the entire ring of invariants of $S(V^*)$ (*Theorem* 1) while the ring of invariants of $S(V)$ is isomorphic to that of corresponding finite type, which is generated by algebraically independent homogeneous elements (*Theorem* 2). This indicates that the polynomial invariants of Euclidean Lie algebras are critically situated between those of finite types and those of hyperbolic types.

When Δ is a root system in the sense of Bourbaki [1], this subject has some relation with classical Harish-Chandra's theorem, which states that the center of the universal enveloping algebra of corresponding finite dimensional complex simple Lie algebra isomorphic to the ring of W-invariants of $S(V^*)$. This theorem cannot be extended when Δ is an infinite root system. For example, we cannot use a number of propositions with respect to even the Casimir operator in this case. We hope to discuss this case in near feature.

Received October 26, 1983.

The author expresses his hearty thanks to Professor R. V. Moody for his valuable advice and to Professor E. Abe for his helpful suggestion.

2. Preliminary.

A generalized Cartan matrix $A=(a_{ij})$ is a square matrix of integers satisfying $a_{ii}=2$, $a_{ij}\leqq 0$ if $i\neq j$, $a_{ij}=0$ if and only if $a_{ji}=0$. For any generalized Cartan matrix $A=(a_{ij})$ of size $l\times l$ and for any field $\boldsymbol{F}$ of characteristic zero, $\mathfrak{G}=\mathfrak{G}_F(A)$ denotes the Lie algebra over $\boldsymbol{F}$ generated by $3l$ generators $e_1, \cdots, e_l, h_1, \cdots, h_l, f_1, \cdots, f_l$ with the defining relations $[h_i, h_j]=0$, $[e_i, f_j]=\delta_{ij}h_i$, $[h, e_j]=a_{ij}e_j$, $[h_i, f_j]=-a_{ij}f_j$ for all i, j, and $(ad\ e_i)^{-a_{ij}+1}e_j=0$, $(ad\ f_i)^{-a_{ij}+1}f_j=0$ for distinct i, j. We call this algebra $\mathfrak{G}$ the Kac-Moody Lie algebra over $\boldsymbol{F}$ associated with A. Let Γ be a free $\boldsymbol{Z}$-module of rank l, and choose a free basis $\{\alpha_1, \cdots, \alpha_l\}$ of Γ. By defining $\deg(e_i)=\alpha_i$, $\deg(h_i)=0$, $\deg(f_i)=-\alpha_i$ for all i, we can view $\mathfrak{G}$ as a Γ-graded Lie algebra $\mathfrak{G}=\bigoplus_{\alpha\in\Gamma}\mathfrak{G}_\alpha$, where $\mathfrak{G}_\alpha$ is the subspace of $\mathfrak{G}$ corresponding to α. Put $\Delta=\}\alpha\in\Gamma|\mathfrak{G}_\alpha\neq 0\}$, called *the root system* of $\mathfrak{G}$. We call $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ *a fundamental root system* of Δ. Let w_i be a $\boldsymbol{Z}$-module automorphism of Γ defined by $w_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$, and let W be the subgroup of $GL(\Gamma)$ generated by w_i for all i. We call W the *Weyl groop* of A. Then Δ is W-stable. Set $\Delta^{re}=\{w(\alpha_i)|w\in W$, for all $i\}$ *real roots*, and $\Delta^{im}=\Delta-\Delta^{re}$, *imaginary roots*.

A generalized Cartan matrix A is called *of finite type* if $\mathfrak{G}_F(A)$ is of finite dimension. A generalized Cartan matrix A is called *of Euclidean type* if A is singular and possesses the property that removal of any row and the corresponding column leaves a Cartan matrix of finite type, in which case $\mathfrak{G}_F(A)$ is called *a Euclidean Lie algebra*. Generalized Cartan matrices of Euclidean type are classified (cf. [5]).

A generalized Cartan matrix A is called *symmetrizable* if there is a positive rational diagonal matrix D such that $B=(b_{ij})=DA$ is a symmetric matrix. Generalized Cartan matrices of finite type and of Euclidean type are symmetrizable. Usually D is normalized by the properties that b_{ij} is a half-integer for distinct i, j, and that b_{ii} is an integer and the greatest common divisor of $\{b_{ii}|$ for all $i\}$ is 1 (cf. [3]). Then we can define a symmetric bilinear form on Γ by $(\alpha_i, \alpha_j)=b_{ij}$, and we can see easily that this form is W-invariant and $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)=a_{ij}$.

3. Ring of invariants of $S(V^*)$.

From now on we assume that $A=(a_{ij})$ is a generalized Cartan matrix of

Euclidean type of size $(l+1)\times(l+1)$. According to the classification (cf. [3]), A is expressed as $\begin{pmatrix} A_0 & * \\ * & 2 \end{pmatrix}$, where A_0 is of type X_l (resp. B_l, C_l, F_4, G_2, B_l) where A is of type $X_l^{(1)}$ (resp. $D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}, D_4^{(3)}, A_2^{(2)}$). We call A_0 *the corresponding finite type of* A. The set Δ^{im} is a free $\boldsymbol{Z}$-module of rank 1 and a nonzero element of Δ^{im} is called *a null root*. Let ν be the generator of Δ^{im} with the coefficient of $\alpha_{l+1}=1$ and V be the $\boldsymbol{R}$-span of Δ, then we obtain:

THEOREM 1. *The ring of W-invariants of the symmetric algebra $S(V^*)$ of V is generated by ν.*

PRROF. Let V_0 be the subspace of V generated by $\alpha_1, \cdots, \alpha_l$. Then $V=V_0+\boldsymbol{R}\nu$ and $\Delta_0=\Delta\cap V_0$ is the root system of the corresponding finite type of A. The root ν is expressed as $\alpha_{l+1}+\phi$, where ϕ is the highest long root of Δ_0 when A is of type $X_l^{(1)}$, highest short root when A is of type $D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}$ and $D_4^{(3)}$ and $2\times$highest short root when A is of type $A_2^{(2)}$. The restriction of $(\ ,\ )$ to V_0 is equal to the Killing form on $\mathfrak{G}_C(A_0)$. Considering that ν is W-invariant and $2(\alpha_{l+1}, \alpha_j)/(\alpha_{l+1}, \alpha_{l+1})=a_{l+1\,j}$, we deduce $w_{l+1}(\alpha_j)=w_\phi(\alpha_j)-a_{l+1\,j}\nu$, $w_{l+1}(\nu)=\nu$, where w_ϕ is the reflection with respect to ϕ in V_0 and $j=1, \cdots, l$. We can see easily that there exists an element w of W such that $w(\alpha_j)=\alpha_j-a_{l+1\,j}$, $w(\nu)=\nu$ $(j=1, \cdots, l)$.

Let f be a W-invariant of $S(V^*)$, expressed by a polynomial of $\alpha_1, \cdots \alpha_l, \nu$, then $f(\alpha_1, \cdots, \alpha_l, \nu)=f(\alpha_1-a_{l+1\,1}\nu, \cdots, \alpha_l-a_{l+1}\nu, \nu)$. Differentiating the both sides with respect to ν, we deduce $\sum_{j=1}^{l} a_{l+1\,j}(\partial f/\partial\alpha_j)(\alpha_1, \cdots, \alpha_l, 0)=0$. If $A\neq A_l^{(1)}$, using the classification of Euclidean type, we can see easily that there exists a unique j $(1\leqq j\leqq l)$ such that $a_{l+1\,j}\neq 0$. This indicates $(\partial f/\alpha x_j)(\alpha_1, \cdots, \alpha_l, 0)=0$. As f is W-invariant, we can prove $(\partial f/\partial\alpha_j)(\alpha_1, \cdots, \alpha_l, 0)=0$ for all j $(1\leqq j\leqq l)$. This indicates $f(\alpha_1, \cdots, \alpha_l, 0)$ is constant and $f\in\boldsymbol{R}[\nu]$.

If $A=A_l^{(1)}$ $(l\geqq 3)$, we can easily see

$$(\partial f/\partial\alpha_1)(\alpha_1, \cdots, \alpha_l, 0)+(\partial f/\partial\alpha_l)(\alpha_1, \cdots, \alpha_l, 0)=0$$

when A is expressed as

$$\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

As f is W-invariant, we can prove

$$-(\partial f/\partial \alpha_1)(\alpha_1, \cdots, \alpha_l, 0)+(\partial f/\partial \alpha_l)(\alpha_1, \cdots, \alpha_l, 0)=0 .$$

This indicates $(\partial f/\partial \alpha_1)(\alpha_1, \cdots, \alpha_l, 0)=0$ and $f(\alpha_1, \cdots, \alpha_l, 0)$ is constant.

If $A=A_2^{(1)}$, we can deduce

$$(\partial f/\partial \alpha_1)(\alpha_1, \alpha_2, 0)+(\partial f/\partial \alpha_2)(\alpha_1, \alpha_2, 0)=0 ,$$

$$-(\partial f/\partial \alpha_1((\alpha_1, \alpha_2, 0)+(\partial f/\partial(\alpha_1+\alpha_2))(\alpha_1, \alpha_2, 0)=0$$

and

$$(\partial f/\partial(\alpha_1+\alpha_2))(\alpha_1, \alpha_2, 0)-(\partial f/\partial \alpha_2)(\alpha_1, \alpha_2, 0)=0 .$$

This indicates $(\partial f/\partial \alpha_1)(\alpha_1, \alpha_2, 0)=(\partial f/\partial \alpha_2)(\alpha_1, \alpha_2, 0)=0$. G. E. D.

4. Ring of invariants of $S(V)$.

Let W_0 be the subgroup of W generated by $w_1, \cdots, w_l$, then W_0 acts on V_0 as the Weyl group of Δ_0. This induces the decomposition of V as W_0-module, $V=V_0+\boldsymbol{R}\nu$ and $S(V_0)$ can be considered as a subalgebra of $S(V)$ by using the inclusion from V_0^* to V^* with respect to this decomposition. Then we obtain:

THEOREM 2. *The ring of W-invariants of $S(V)$ is equal to the ring of W_0-invariants of $S(V_0)$. Consequently the ring of W_0-invariants of $S(V)$ is generated by l algebraically independent homogeneous polynomials.*

PROOF. Let $\{\alpha_1^*, \cdots, \alpha_l^*, \nu^*\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_l, \nu\}$. Easy calculations show that $w_{l+1}(\alpha_j^*)=w_\phi(\alpha_j^*)$ $(j=1, \cdots, l)$, $w_{l+1}(\nu^*)=\nu^*-\sum_{j=1}^{l} a_{l+1\,j}\alpha_j^*$. It is clear that $S(V)^{W_0}$, the ring of W_0-invariants of $S(V_0)$, is a subalgebra of the ring of W-invariants of $S(V)$ and the ring of W_0-invariants of $S(V)$ is $S(V_0)^{W_0}[\nu^*]$, the polynomial ring of the commutative algebra $S(V_0)^{W_0}$ with the indeterminate ν^*.

Clearly the ring of W-invariants of $S(V)$ is a subalgebra of the ring of W_0-invariants of $S(V)$. Let f be any element of the ring of W-invariants of $S(V)$. Then f can be expressed by a polynomial of ν^* with coefficients in $S(V_0)^{W_0}$ and satisfying the equation $f(\nu^*)=f\left(\nu^*-\sum_{j=1}^{l} a_{l+1\,j}\alpha_j^*\right)$, where $\sum_{j=1}^{l} a_{l+1\,j}\alpha_j^*$ is non-zero from the classification. Then the following lemma completes the proof of the theorem. Q. E. D.

LEMMA. *Let R be a commutative algebra over a field of characteristic zero and $f(X)$ be any element of $R[X]$. If $f(X)$ satisfies the equation $f(X)=f(X+c$ where c is not a zero divisor of R, then deg $f=0$.*

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Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki, 305 Japan