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HALF CONFORMALLY FLAT STRUCTURES AND THE DEFORMATION OBSTRUCTION SPACE

Dedicated to Professor H. Nakagawa on his sixtieth birthday

By

Mitsuhiro ITOH

1. A compact connected oriented Riemannian 4-manifold (M, g) is called half conformally flat, or a Riemannian metric g on M is called self-dual or anti-self-dual, when $W^{-}=0$ or $W^{+}=0$ where $W^{\pm}$ is the self-dual (anti-self-dual) part of the Weyl conformal curvature tensor W of g.

We denote for an arbitrary Riemannian metric g by $R=(R_{ijkl})$, $Ric=(R_{ij})$ and ρ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. Then the Weyl conformal curvature tensor $W=(W_{ijkl})$, considered as a section of the symmetric product bundle $S^{2}(\Omega^{2})$, is defined by

$$R=W+L\owedge g$$

($L=1/2\ (Ric-(\rho/6)g)$ is the Schouten tensor and $\owedge$ is the Kulkarni-Nomizu product).

In terms of the Hodge star operator the bundles Ω^{2} and $S^{2}(\Omega^{2})$ decompose as $\Omega^{2}=\Omega^{+}\oplus\Omega^{-}$ and $S^{2}(\Omega^{2})=S^{2}(\Omega^{+})\oplus(\Omega^{+}\otimes\Omega^{-})\oplus(\Omega^{-}\otimes\Omega^{+})\oplus S^{2}(\Omega^{-})$, respectively and then the tensors R and W split as $R=\begin{pmatrix} R^{+} & R^{+-} \\ R^{-+} & R^{-} \end{pmatrix}$ and $W=\begin{pmatrix} W^{+} & 0 \\ 0 & W^{-} \end{pmatrix}$ in such a way that $R^{\pm}=W^{\pm}+(\rho/12)I$.

The notion "half conformal flatness" depends only on a conformal structure $[g]$, the conformal equivalence class represented by a Riemannian metric g, because W and the Hodge star operator are conformal invariants.

The significance of the half conformally flat structure is that it ensures the integrability of the almost complex structure which is naturally defined on the twistor space $Z_{M}\rightarrow M$, the unit sphere bundle of Ω^{+} such that Z_{M} becomes a complex 3-fold admitting a real structure ([1]).

Like Yang-Mills instantons on 4-manifolds, every half conformally flat structure $[g]$ enjoys an elliptic complex at any representative within $[g]$ provided $W=W^{-}$ i.e., $W^{+}=0$

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$$C^\infty(M, T_M) \xrightarrow{L_g} C^\infty(M, S_0^2(T_M^*)) \xrightarrow{D_g} C^\infty(M, S_0^2(\Omega^+)),$$

where L_g is the tracefree Lie derivative of g and the second order operator D_g is the linearization of the self-dual part W^+ of W ([10]).

The index of this complex is from the Atiyah-Singer index theorem 1/2 $(29\tau(M)+15\chi(M))$ ([10]).

This elliptic complex gives rise to a local description of the moduli space $\mathcal{M}_M$ of half conformally flat structures on M, the space of all diffeomorphism-equivalence classes of half conformally flat structures on M. The moduli space has a structure of real analytic variety. In fact it is written as Zero$(K)/C_{[g]}$, the zero locus of the map K between cohomologies $\boldsymbol{H}^1$, $\boldsymbol{H}^2$ of the complex divided by the $[g]$-conformal diffeomorphism group $C_{[g]}$ at each $[g]$ ([Theorem 2, 10]). In the sense of this local description $\boldsymbol{H}^1$ and $\boldsymbol{H}^2$ represent the space of infinitesimal deformations of half conformally flat structures and the space of obstruction for local deformations, respectively.

We restrict ourself in this article to 4-manifolds of certain type, namely, Kähler surfaces of zero scalar curvature and investigate how the second cohomology group $\boldsymbol{H}^2$ relates with certain cohomology groups of other elliptic complexes which are holomorphically defined.

The following are several examples of half conformally flat 4-manifolds for which $\boldsymbol{H}^2$ is computed: (i) the 4-sphere S^4 with the standard metric; $\boldsymbol{H}^2=0$, (ii) the complex projective plane $\boldsymbol{C}P^2$ with Fubini-Study metric; $\boldsymbol{H}^2=0$, (iii) a complex 2-torus with a flat metric; $\boldsymbol{H}^2\cong\boldsymbol{R}^5$, (iv) a K3 surface with Ricci flat metric; $\boldsymbol{H}^2\cong\boldsymbol{R}^5$.

The latter two 4-manifolds are examples of Kähler surface of zero scalar curvature. Another example of those Kähler surfaces is a ruled surface M_k of genus $k(\geqq 2)$ (a $\boldsymbol{C}P^1$ bundle over a compact Riemann surface Σ_k of genus $k\geqq 2$) with a Kähler metric induced from the product metric on $D^1\times\boldsymbol{C}P^1$ (D^1: the unit disk in $\boldsymbol{C}$) of metrics of curvature ± 1.

Any compact Kähler surface of scalar curvature $\rho=0$ is necessarily one of the following ([9], [4])

i) a Kähler surface covered by a complex 2-torus with a flat metric (a complex 2-torus and a hyperelliptic surface)

ii) a Kähler surface covered by a K 3 surface with a Ricci flat Kähler metric (a K 3 surface and an Enriques surface)

iii) a ruled surface M_k ($k\geqq 2$) with a Kähler metric of zero scalar curvature

iv) a Kähler surface, obtained by blowing up l times either $\boldsymbol{C}P^2$ ($l\geqq 10$), a ruled surface of genus 0 ($l\geqq 9$) or a ruled surface of genus k ($l\geqq 1$).

We remark that LeBrun showed recently by using the generalized Hawking Ansatz the existence of Kähler metric of zero scalar curvature on a compact complex surface of type (iv) above, namely a surface derived by blowing up l points of a ruled surface M_k of genus $k\geqq 2$, $l\geqq 2$ ([15]).

When $\boldsymbol{H}^2$ vanishes, the moduli space $\mathcal{M}_M$ admits at worst conformal symmetry singularities. On the other hand the vanishing of $\boldsymbol{H}^2$ might give like Yang-Mills instantons guarantee to the grafting procedure for concentrating half conformally flat structure (see [7], where $\boldsymbol{CP}^2$ with the Fubini-Study metric behaves as "half conformally flat 1-instanton"), which corresponds in terms of twistor spaces to the connected sum procedure of half conformally flat 4-manifolds (see [6] in which the isomorphism $\boldsymbol{H}^2\cong H^2(Z_M, \mathcal{O}(T_Z^{1,0}))$ is asserted).

As for half conformally flat structures of positive scalar curvature on a simply connected 4-manifold, the positivity might ensure vanishing of $\boldsymbol{H}^2$ as is conjectured and is proved partially under the condition $\dim|K_Z^{-1/2}|>0$ ([14]).

This conjecture corresponds to the vanishing theorem in the Atiyah-Hitchin-Singer complex for Yang-Mills instantons ([1]) and other vanishing theorems of half spinorially defined operators ([p. 178, 8]).

For Kähler surfaces of $\rho=0$ the bundles $S_0^2(T_M^*)$, $S_0^2(\Omega^+)$ appeared in the elliptic complex have natural decompositions and the kernel of the adjoint D^* can be described in terms of cohomologies $H^0(M, \mathcal{O}(K_M^i))$, $i=1, 2$ and $H^2(M, \mathcal{O}(T_M^{1,0}))$.

In fact we have the following for $\operatorname{Ker} D_g^*=\boldsymbol{H}_g^2$

THEOREM 1. *Let* (M, g) *be a compact Kähler surface of zero scalar curvature. Let* $S_0^2(\Omega^+)=V^0\oplus V^1\oplus V^2$, $V^0=1_{\boldsymbol{R}}$ *be the real subbundle decomposition corresponding to the identification* $S_0^2(\Omega^+)\cong 1_{\boldsymbol{R}}\oplus K_M\oplus K_M^2$. *Then* (i) $\operatorname{Ker}(D^*|_{C^\infty(M,V^0)})\cong\{f\in C^\infty(M)\,;\operatorname{Hes}(f)=-1/4\Delta f g-1/2 f\,Ric\}$, (ii) $\operatorname{Ker}(D^*|_{C^\infty(M,V^1)})\cong H^0M, \mathcal{O}(K_M))$ *and,* (iii) *if* $H^2(M, \mathcal{O}(T_M^{1,0}))=0$, *then*

$$\operatorname{Ker}(D^*|_{C^\infty(M,V^2)})\cong H^0(M, \mathcal{O}(K_M^2))\,.$$

Moreover (iv) *under the condition* $H^2(M, \mathcal{O}(T_M^{1,0}))=0$

$$\operatorname{Ker}(D^*|_{C^\infty(M,V^1\oplus V^2)})\cong H^0(M, \mathcal{O}(K_M))\oplus H^0(M, \mathcal{O}(K_M^2))\,.$$

By applying this theorem we get the

THEOREM 2. *Let* (M, g) *be a complex* 2-*torus or a* $K\,3$ *surface with a flat Kähler metric or a Ricci flat Kähler metric. Then* $\boldsymbol{H}_g^2\cong\boldsymbol{R}^5$.

2. Proof of Theorem 1.

(i) To verify our theorem we begin with a sufficient but small amount of Kähler geometry needed for our study.

Assume that (M, g) is a Kähler surface with a complex structure J. The Kähler form is $\theta=\sqrt{-1}g_{\alpha\bar{\beta}}dz^{\alpha}\wedge dz^{\bar{\beta}}$.

We omit the symbol of summation by Einstein convention.

The bundles of self-dual (anti-self-dual) 2-forms Ω^{+}, Ω^{-} on M are identified by the aid of (p, q)-form bundles $\Omega^{p,q}$ as

$$\Omega^{+}\cong \boldsymbol{R}\theta\oplus\Omega^{2,0}$$

$$\Omega^{-}=\Omega_{0}^{1,1}=\{\xi\in\Omega^{1,1},\ \bar{\xi}=\xi,\ \theta\wedge\xi=0\}. \tag{2.1}$$

Here $\Omega^{2,0}=K_{M}$, the canonical line bundle of M and the identification for Ω^{+} is $a\theta+\varphi\in\boldsymbol{R}\theta\oplus\Omega^{2,0}\leftrightarrow a\theta+(\varphi+\bar{\varphi})\in\Omega^{+}$.

Fix a point $p\in M$ and take at p complex coordinates $\{z^{1}=x^{1}+\sqrt{-1}x^{2}, z^{2}=x^{3}+\sqrt{-1}x^{4}\}$ in such a way that $\{\partial/\partial x^{a}\}_{a=1,\cdots,4}$ is an orthonormal frame at p with dual frame $\{dx^{a}\}_{a=1,\cdots,4}$.

So $\{\omega=dx^{1}\wedge dx^{2}+dx^{3}\wedge dx^{4},\ \eta=dx^{1}\wedge dx^{3}+dx^{4}\wedge dx^{2},\ \zeta=dx^{1}\wedge dx^{4}+dx^{2}\wedge dx^{3}\}$ forms a frame of Ω^{+} at p.

Notice that $2\omega=\theta$ and $\phi=\eta+i\zeta$ is a (2, 0)-form.

The bundle $S_{0}^{2}(\Omega^{+})$ has then the following canonical basis at p:

$$\Phi=\omega^{2}-1/2(\eta^{2}+\zeta^{2})=1/4\theta^{2}-1/2\phi\cdot\bar{\phi},$$

$$\omega\cdot\eta=1/4\theta(\phi+\bar{\phi}),\quad \omega\cdot\zeta=1/(4\sqrt{-1})\theta(\phi-\bar{\phi}), \tag{2.2}$$

$$\eta^{2}-\zeta^{2}=1/2(\phi^{2}+\bar{\phi}^{2}),\quad 2\eta\cdot\zeta=1/(2\sqrt{-1})(\phi^{2}-\bar{\phi}^{2}).$$

From this we have the identification $S_{0}^{2}(\Omega^{+})\cong 1_{\boldsymbol{R}}\oplus K_{M}\oplus K_{M}^{2}$ in such a way that

$$a\Phi+\varphi+\psi\longleftrightarrow a\Phi+(\varphi+\bar{\varphi})\wedge\theta+(\psi+\bar{\psi}),$$

$a\in\boldsymbol{R}$, $\varphi\in K_{M}$, $\psi\in K_{M}^{2}$. Here $\eta^{2}+\zeta^{2}$ and hence Φ does not depend on a choice of η, ζ so that Φ is globally defined.

LEMMA 2. 1 ([5]). *The globally defined tensor Φ is g-parallel.*

PROOF. Since $\nabla\omega=1/2\nabla\theta=0$, it suffices to show $\nabla(\eta^{2}+\zeta^{2})=2(\eta\cdot\nabla\eta+\zeta\cdot\nabla\zeta)=0$.

For any point $p\in M$ choose an orthonormal frame field $\{e_{a}\}_{a=1,\cdots,4}$ defined around p satisfying $e_{a}=\partial/\partial x^{a}$, $a=1,\cdots,4$ at p and $e_{2}=Je_{1}$, $e_{4}=Je_{3}$.

Since $\nabla J=0$, we have the connection forms $\{\omega_{a}^{b}\}$ associated to $\{e_{a}\}$ in the following form

$$(\omega_a^b)=\begin{pmatrix} 0 & \omega_1^2 & \omega_1^3 & \omega_1^4 \\ -\omega_1^2 & 0 & -\omega_1^4 & \omega_1^3 \\ -\omega_1^3 & \omega_1^4 & 0 & \omega_3^4 \\ -\omega_1^4 & -\omega_1^3 & -\omega_3^4 & 0 \end{pmatrix}.$$

For the dual frame $\{e^a\}_{a=1,\dots,4}$ of $\{e_a\}$ the connection forms $\{\tilde{\omega}_b^a\}$, defined by $\nabla e^a=\Sigma_b\tilde{\omega}_b^a e^b$, satisfy $\tilde{\omega}_b^a=-\omega_a^b$.

By the aid of these connection forms we have for the frame field $\{\omega=e^1\wedge e^2+e^3\wedge e^4,\ \eta=e^1\wedge e^3+e^4\wedge e^2,\ \zeta=e^1\wedge e^4+e^2\wedge e^3\}$ of Ω^+ the following connection forms

$$\nabla\omega=(\tilde{\omega}_1^3+\tilde{\omega}_4^2)\zeta+(\tilde{\omega}_3^2-\tilde{\omega}_1^4)\eta=0\,,$$

$$\Delta\eta=(\tilde{\omega}_1^4+\tilde{\omega}_2^3)\omega+(\tilde{\omega}_2^1+\tilde{\omega}_4^3)\zeta=(\tilde{\omega}_2^1+\tilde{\omega}_4^3)\zeta$$

$$\nabla\zeta=(-\tilde{\omega}_1^3+\tilde{\omega}_2^4)\omega+(\tilde{\omega}_1^2+\tilde{\omega}_3^4)\eta=(\tilde{\omega}_1^2+\tilde{\omega}_3^4)\eta\,.$$

Hence $\eta\cdot\nabla\eta+\zeta\cdot\nabla\zeta=0$.

We remark that for an arbitrary Kähler surface the self-dual part W^+ of W is given by $W^+=c\rho\Phi$ for a constant $c>0$ ([5]). A Kähler surface is then anti-self-dual if and only if $\rho=0$.

We would like now to decompose the bundle $S_0^2(T_M^*)$, the tracefree symmetric product of the real cotangent bundle, whose sections give the space of infinitesimal deformations of metrics of a fixed volume form.

As is shown in [9], the bundle $S_0^2(T_M^*)$ is in general isomorphic to $\mathrm{Hom}_R(\Omega^+, \Omega^-)\cong\Omega^+\otimes\Omega^-$ in such a way that from the identification (2.1) we have

$$S_0^2(T_M^*)=\mathrm{Her}_0(T^*)\oplus\mathrm{Sk}_0(T^*)\,, \tag{2.3}$$

where $\mathrm{Her}_0(T^*)=\{h\in S_0^2(T_M^*)\,;\ h(JX, JY)=h(X, Y)\}$, isomorphic to $\Omega_0^{1,1}$ and $\mathrm{Sk}_0(T^*)=\{h\in S_0^2(T^*)\,;\ h(JX, JY)=-h(X, Y)\}\cong S^2(\Omega^{1,0})$.

By making use of these identifications one can represent $D_g: C^\infty(M, S_0^2(T^*))\to C^\infty(M, S_0^2(\Omega^+))$ and its formal adjoint D_g^* in terms of naturally defined operators.

(ii) Let g be an arbitrary anti-self-dual metric. The operator D_g is represented as

$$D(h)=(U(h))^+ +(V(h))^+\,, \tag{2.4}$$

$h\in C^\infty(M, S_0^2(T^*))$ (see [Appendix, 10]). Here $(U(h))^+$ and $(V(h))^+$ are the $S_0^2(\Omega^+)$-components of $U(h)$ and $V(h)$, which are defined as

$$U,\,V: C^\infty(M, S_0^2(T))\longrightarrow C^\infty(M, \Omega^+\otimes\Omega^+)\,,$$

(2.5) $$U(h)_{ijkl}=1/2\,(\nabla_k\nabla_j h_{il}-\nabla_l\nabla_j h_{ik}-\nabla_k\nabla_i h_{jl}+\nabla_l\nabla_i h_{jk})$$

(2.6) $$V(h)=1/4\,(B)\bigcirc\!\!\!\!\wedge\, h), \qquad B=Ric-\rho/4\,g\,.$$

Here the Kulkarni-Nomizu product is

$$(h\bigcirc\!\!\!\!\wedge\, k)_{ijkl}=h_{kj}k_{il}-h_{lj}k_{ik}-h_{ki}k_{jl}+h_{li}k_{jk}\,.$$

Assume that (M, g) is a Kähler surface of zero scalar curvature.

LEMMA 2.2. *In the decomposition* (2.3) $(h\bigcirc\!\!\!\!\wedge\, k)^+=0$ *for* $h\in \mathrm{Her}_0(T^*)$, $k\in \mathrm{Sk}_0(T^*)$ *and* $(h\bigcirc\!\!\!\!\wedge\, k)^+=-1/3\,(h, k)\Phi$ *for* $h, k\in \mathrm{Her}_0(T^*)$ ((h, k) *is the inner product induced from the metric* g).

REMARK. The tracefree Ricci tensor B is in $C^\infty(M, \mathrm{Her}_0(T^*))$ and then $(V(h+k))^+=(V(h))^+=-1/3\,(Ric, h)\Phi$ for $h\in \mathrm{Her}_0(T^*)$, $k\in \mathrm{Sk}_0(T^*)$.

The formulae in Lemma 2.2 follow from simple computation.

We extend the forth order covariant tensors $U(h)$ and $V(h)$ over $\boldsymbol{C}$ as $U(h)=(U_{ABCD})$, $\overline{U_{ABCD}}=U_{\bar{A}\bar{B}\bar{C}\bar{D}}$,

$$U_{ABCD}=1/2\,(\nabla_C\nabla_B h_{AD}-\nabla_D\nabla_B h_{AC}-\nabla_C\nabla_A h_{BD}+\nabla_D\nabla_A h_{BC*}),$$

with respect to complex coordinates $\{z^A,\ A=1, 2, \bar{1}, \bar{2}\}$.

We symmetrize $U(h)\in C^\infty(M, \Omega^2\otimes\Omega^2)$ as $\tilde{U}(h)=1/4\,\tilde{U}_{ABCD}(dz^A\wedge dz^B)\cdot(dz^C\wedge dz^D)$, $\tilde{U}_{ABCD}=1/2\,(U_{ABCD}+U_{CDAB})$. Then $(U(h))^+$ is the $S^2_0(\Omega^+)$-component of $\tilde{U}(h)$.

By using the canonical basis (2.2) for $S^2_0(\Omega^+)$ we write the $S^2_0(\Omega^+)$-component of $U(h)$ or more generally of $Z\in C^\infty(M, S^2(\Omega^+))$.

Then the $S^2_0(\Omega^+)$-component Z^+ of $Z=1/4\,Z_{ABCD}(dz^A\wedge dz^B)\cdot(dz^C\wedge dz^D)\in S^2(\Omega^2\otimes \boldsymbol{C})$, $Z_{ABCD}=Z_{CDAB}$ is given as

$$\begin{aligned}Z^+=&-1/6(Z_{1\bar{1}1\bar{1}}+2Z_{1\bar{1}2\bar{2}}+Z_{2\bar{2}2\bar{2}}+8Z_{12\bar{1}\bar{2}})\Phi\\&+(Z_{1212}+Z_{\bar{1}\bar{2}\bar{1}\bar{2}})(\eta^2-\zeta^2)+(Z_{1212}-Z_{\bar{1}\bar{2}\bar{1}\bar{2}})(2\sqrt{-1}\eta\cdot\zeta)\\&+(Z_{121\bar{1}}+Z_{122\bar{2}}+Z_{1\bar{1}\bar{1}\bar{2}}+Z_{2\bar{2}\bar{1}\bar{2}})(-\sqrt{-1}\omega\cdot\eta)\\&+(Z_{121\bar{1}}+Z_{122\bar{2}}-Z_{1\bar{1}\bar{1}\bar{2}}-Z_{2\bar{2}\bar{1}\bar{2}})(\omega\cdot\zeta)\,.\end{aligned}$$

We have then

LEMMA 2.3. *With respect to* $\{\Phi,\ \theta\cdot\phi,\ \theta\cdot\bar{\phi},\ \phi^2,\ \bar{\phi}^2\}$ Z^+ *is*

(2.7) $$\begin{aligned}Z^+=&-1/6\,(Z_{1\bar{1}1\bar{1}}+2Z_{1\bar{1}2\bar{2}}+Z_{2\bar{2}2\bar{2}}+8Z_{12\bar{1}\bar{2}})\Phi\\&+\frac{1}{2\sqrt{-1}}(Z_{121\bar{1}}+Z_{122\bar{2}})\theta\cdot\phi+\frac{1}{2\sqrt{-1}}(Z_{\bar{1}\bar{2}1\bar{1}}+Z_{\bar{1}\bar{2}2\bar{2}})\theta\cdot\bar{\phi}+Z_{1212}\phi^2+Z_{\bar{1}\bar{2}\bar{1}\bar{2}}\bar{\phi}^2.\end{aligned}$$

Next we consider the adjoint operator

$$D^*: C^\infty(M, S_0^2(\Omega^+)) \longrightarrow C^\infty(M, S_0^2(T^*)).$$

In real local coordinates $\{x^a\}$ it has the form

$$(D^*Z)_{ab}=(\nabla^c\nabla^d+\nabla^d\nabla^c)Z_{acdb}+R^{cd}Z_{acdb}, \tag{2.8}$$

$Z=(Z_{abcd})\in C^\infty(M, S_0^2(\Omega^+))$ where $R^{cd}=g^{ac}g^{bd}R_{ab}$ is the Ricci tensor of g ([Appendix, 10]).

By using complex coordinates $\{z^A\}$ we rewrite this formula as

$$(D^*Z)_{AB}=(\nabla^C\nabla^D+\nabla^D\nabla^C)Z_{ACDB}+R^{CD}Z_{ACDB} \tag{2.9}$$

for $Z=1/2\, Z_{ABCD}(dz^A\wedge dz^B)\cdot(dz^C\wedge dz^D)$, $\overline{Z_{ABCD}}=Z_{\bar{A}\bar{B}\bar{C}\bar{D}}$ so that for the adjoints $(U^+)^*$, $(V^+)^*$

$$\begin{aligned}((U^+)^*Z)_{AB}&=(\nabla^C\nabla^D+\nabla^D\nabla^C)Z_{ACDB},\\ ((V^+)^*Z)_{AB}&=R^{CD}Z_{ACDB}.\end{aligned} \tag{2.10}$$

iii) We deduce first the Hessian equation from $\operatorname{Ker}(D^*|_{C^\infty(M,V^0)})$.

From the decomposition $S_0^2(T^*)=\mathrm{Her}_0(T^*)\oplus\mathrm{Sk}_0(T^*)$ we have

LEMMA 2.4. *The* $\mathrm{Her}_0(T^*)$*-part of* $D^*(f\Phi)$, $f\Phi\in C^\infty(M, V^0)$ *is*

$$D^*(f\Phi)_{\alpha\bar{\beta}}=-\nabla_\alpha\nabla_{\bar{\beta}}f-1/4\,\Delta f\cdot g_{\alpha\bar{\beta}}-1/2\,fR_{\alpha\bar{\beta}}, \tag{2.11}$$

($\Delta f=2\Box f=-2g^{\alpha\bar{\beta}}\nabla_\alpha\nabla_{\bar{\beta}}f$ *is the real Laplacian) and the* $\mathrm{Sk}_0(T^*)$*-part is*

$$D^*(f\Phi)_{\alpha\beta}=1/2\,\nabla_\alpha\nabla_\beta f. \tag{2.12}$$

PROOF. From (2.10) and the definition of ϕ and Φ the $\mathrm{Her}_0(T^*)$-part of $(U^+)^*(f\Phi)$ is computed as

$$(U^+)^*(f\Phi)_{\alpha\bar{\beta}}=-\nabla^\delta\nabla^{\bar{\gamma}}f\phi_{\alpha\delta}\bar{\phi}_{\gamma\beta}-\nabla_\alpha\nabla_{\bar{\beta}}f.$$

Substitute the value of ϕ at a point p. Then $(U^+)^*(f\Phi)_{\alpha\bar{\beta}}=-\nabla_\alpha\nabla_{\bar{\beta}}f-1/2\Box f g_{\alpha\bar{\beta}}$. On the other hand $(V^+)^*(f\Phi)_{\alpha\bar{\beta}}$ is

$$(V^+)^*(f\Phi)_{\alpha\bar{\beta}}=-1/4\,fR^{\delta\bar{\gamma}}\phi_{\alpha\delta}\bar{\phi}_{\gamma\beta}-1/4\,fR_{\alpha\bar{\beta}}$$

reducing to $-1/2\,fR_{\alpha\bar{\beta}}$ so that (2.11) is derived.

The Sk_0-part of $D^*(f\Phi)$ is from Lemma 2.2 equal to that of $(U^+)^*(f\Phi)$ which is

$$(U^+)^*(f\Phi)_{\alpha\beta}=\nabla^C\nabla^D f\Phi_{\alpha CD\beta}+\nabla^D\nabla^C f\Phi_{\alpha CD\beta}=-2\nabla^{\bar{\delta}}\nabla^{\bar{\gamma}}f\Phi_{\alpha\bar{\gamma}\beta\bar{\delta}}.$$

Then (2.12) follows since $\Phi_{\alpha\bar{\gamma}\beta\bar{\delta}}=-1/4\,g_{\alpha\bar{\gamma}}g_{\beta\bar{\delta}}$.

REMARK. From this lemma $f\Phi\in C^{\infty}(M, V^{0})$ is in Ker D_{g}^{*} if and only if

$$\mathrm{Hes}(f)=-1/4\Delta f\cdot g-1/2\, fB\,, \tag{2.13}$$

since $Ric=B$ for a Riemannian metric of zero scalar curvature. Similar Hessian equations with a positive smooth function solution were dealt with in [11].

Next we consider the kernel of D^{*} restricted to $C^{\infty}(M, V^{1})$, namely, to those $Z\in C^{\infty}(M, S_{0}^{2}(\Omega^{+}))$ of the form $Z=\theta(\varphi+\bar{\varphi})$, $\varphi\in C^{\infty}(M, K_{M})$.

$Z=(Z_{ABCD})$, $Z_{ABCD}=(\theta\varphi)_{ABCD}+(\theta\bar{\varphi})_{ABCD}$ has nontrivial components $Z_{\alpha\bar{\beta}\gamma\delta}=1/2\sqrt{-1}g_{\alpha\bar{\beta}}\varphi_{\gamma\delta}$ and $Z_{\alpha\bar{\beta}\bar{\gamma}\bar{\delta}}=1/2\sqrt{-1}g_{\alpha\bar{\beta}}\bar{\varphi}_{\gamma\delta}$ and other components are zero.

The Hermitian part $h_{\alpha\bar{\delta}}$ of $h=D^{*}(Z)$ is from (2.9)

$$\begin{aligned}h_{\alpha\bar{\delta}}&=\nabla^{B}\nabla^{C}Z_{\alpha BC\bar{\delta}}+\nabla^{C}\nabla^{B}Z_{\alpha BC\bar{\delta}}\\&=(\nabla^{\beta}\nabla^{\gamma}+\nabla^{\gamma}\nabla^{\beta})Z_{\alpha\beta\gamma\bar{\delta}}+(\nabla^{\beta}\nabla^{\bar{\gamma}}+\nabla^{\bar{\gamma}}\nabla^{\beta})Z_{\alpha\beta\bar{\gamma}\bar{\delta}}\\&\quad+(\nabla^{\bar{\beta}}\nabla^{\gamma}+\nabla^{\gamma}\nabla^{\bar{\beta}})Z_{\alpha\bar{\beta}\gamma\bar{\delta}}+(\nabla^{\bar{\beta}}\nabla^{\bar{\gamma}}+\nabla^{\bar{\gamma}}\nabla^{\bar{\beta}})Z_{\alpha\bar{\beta}\bar{\gamma}\bar{\delta}}\,.\end{aligned}$$

Since $Z_{\alpha\beta\bar{\gamma}\bar{\delta}}=Z_{\alpha\bar{\beta}\gamma\bar{\delta}}=0$, $h_{\alpha\bar{\delta}}=2(\nabla^{\beta}\nabla^{\gamma}Z_{\alpha\beta\gamma\bar{\delta}}+\nabla^{\bar{\beta}}\nabla^{\bar{\gamma}}Z_{\alpha\bar{\beta}\bar{\gamma}\bar{\delta}})$ reduces to $\sqrt{-1}\{\nabla_{\bar{\delta}}(\nabla^{\beta}\varphi_{\alpha\beta})+\nabla_{\alpha}\nabla^{\bar{\gamma}}\bar{\varphi}_{\gamma\delta}\}$.

Thus we have

LEMMA 2.5. *The Hermitian part of* $D^{*}Z$, $Z=\theta(\varphi+\bar{\varphi})$, $\varphi\in C^{\infty}(M, K_{M})$ *is*

$$(D^{*}Z)_{\alpha\bar{\delta}}=\sqrt{-1}(\nabla_{\bar{\delta}}\nabla^{\beta}\varphi_{\alpha\beta}+\nabla_{\alpha}\nabla^{\bar{\gamma}}\bar{\varphi}_{\gamma\delta})\,. \tag{2.14}$$

REMARK. We can rewrite (2.14) as

$$\mathrm{Her}_{0}(T^{*})\text{-part of } D^{*}Z=-\sqrt{-1}\{\bar{\partial}(\partial^{*}\varphi)+\partial(\bar{\partial}^{*}\bar{\varphi})\}\,, \tag{2.15}$$

when we regard it as section of $\Omega_{0}^{1,1}$. Here ∂^{*}, $\bar{\partial}^{*}$ are the adjoint of $\partial:\Omega^{1,0}\to\Omega^{2,0}$, $\bar{\partial}:\Omega^{0,1}\to\Omega^{0,2}$, respectively; $(\partial^{*}\varphi)_{\alpha}=-\nabla^{\beta}\varphi_{\beta\alpha}$, $\bar{\partial}^{*}\bar{\varphi}=\overline{\partial^{*}\varphi}$.

The skew hermitian part of $D^{*}Z$, $Z=\theta(\varphi+\bar{\varphi})$ is

$$(\nabla^{B}\nabla^{C}+\nabla^{C}\nabla^{B})Z_{\alpha BC\delta}=(\nabla^{\beta}\nabla^{\bar{\gamma}}+\nabla^{\bar{\gamma}}\nabla^{\beta})Z_{\alpha\beta\bar{\gamma}\delta}+(\nabla^{\bar{\beta}}\nabla^{\gamma}+\nabla^{\gamma}\nabla^{\bar{\beta}})Z_{\alpha\bar{\beta}\gamma\delta}\,.$$

So substituting $Z_{\alpha\bar{\beta}\gamma\delta}=\sqrt{-1}/2\, g_{\alpha\bar{\beta}}\varphi_{\gamma\delta}$ we get

LEMMA 2.6. *For* $Z=\theta(\varphi+\bar{\varphi})$, $\varphi\in C^{\infty}(M, K_{M})$ $D^{*}Z$ *has the following skew hermitian part*

$$(D^{*}Z)_{\alpha\delta}=\sqrt{-1}/2\,(\nabla^{\gamma}\nabla_{\alpha}+\nabla_{\alpha}\nabla^{\gamma})\varphi_{\gamma\delta}+\sqrt{-1}/2\,(\nabla^{\gamma}\nabla_{\delta}+\nabla_{\delta}\nabla^{\gamma})\varphi_{\gamma\alpha}\,. \tag{2.6}$$

Now we will show

PROPOSITION 2.7. *For* $Z=\theta(\varphi+\bar{\varphi})\in C^{\infty}(M, V^{1})$, *Z is in* Ker D^* *if and only if* φ *is a holomorphic section of* K_M, *i.e.,* $\varphi\in H^0(M, \mathcal{O}(K_M))$.

PROOF. Assume $D^*Z=0$. We have from (2.15) $\bar{\partial}(\partial^*\varphi)+\partial(\bar{\partial}^*\bar{\varphi})=0$. To this we operate $\bar{\partial}^*: \Omega^{1,1}\to\Omega^{1,0}$, the adjoint of $\bar{\partial}: \Omega^{1,0}\to\Omega^{1,1}$. Then $\bar{\partial}^*\bar{\partial}\partial^*\varphi+\bar{\partial}^*\partial(\bar{\partial}^*\bar{\varphi})=0$. Here the second term vanishes since it reduces to $-g^{\beta\bar{\delta}}\nabla_{\beta}\nabla_{\alpha}\nabla^{\bar{\gamma}}\bar{\varphi}_{\bar{\gamma}\bar{\delta}}=-\nabla_{\alpha}\nabla^{\bar{\delta}}\nabla^{\bar{\gamma}}\bar{\varphi}_{\bar{\gamma}\bar{\delta}}=0$. Hence $\bar{\partial}^*\bar{\partial}\partial^*\varphi=0$.

Since $0=\langle\bar{\partial}^*\bar{\partial}\partial^*\varphi, \partial^*\varphi\rangle=\langle\bar{\partial}(\partial^*\varphi), \bar{\partial}(\partial^*\varphi)\rangle$ if follows that $\bar{\partial}\partial^*\varphi=0$, in other words, $\partial^*\varphi$ is holomorphic as a (1, 0)-form. So $\partial\partial^*\varphi$ is holomorphic and then is written as $\partial\partial^*\varphi=\varphi_0$ for some $\varphi_0\in H^0(M, \mathcal{O}(K_M))$.

Together with $\partial\varphi=0$ this implies $(\partial\partial^*+\partial^*\partial)\varphi=\varphi_0$ to which we use the fact that $H^0(M, \mathcal{O}(K_M))=\{$parallel sections of $K_M\}$ (see Lemma 2.10). Then $\|(\partial\partial^*+\partial^*\partial)\varphi\|^2=\langle\partial^*\varphi, \partial^*\varphi_0\rangle=0$, namely $(\partial\partial^*+\partial^*\partial)\varphi=0$. To this we apply the Weitzenböck-Bochner formula and then conclude that φ is parallel, that is, $\varphi\in H^0(M, \mathcal{O}(K_M))$.

It is obvious that conversely any φ in $H^0(M, \mathcal{O})K_M))$ satisfies (2.14) and (2.16), since φ is parallel.

In the rest of this section we will show (iii) of Theorem 1.

For Z of form $Z=\phi+\bar{\phi}$, $\phi\in C^{\infty}(M, K_M^2)$ components Z_{ABCD} are all zero except for $Z_{\alpha\beta\gamma\delta}$, $Z_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}$. So the Hermitian part of D^*Z vanishes since $\int_M(h, D^*Z)dv=\int_M(Dh, Z)dv=\int_M(U(h)^+, Z)dv$ and $U(h)^+$, $h\in C^{\infty}(M, \mathrm{Her}_0(T^*))$, has neither $\alpha\beta\gamma\delta$-components nor $\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}$-components.

Therefore, $Z\in$ Ker D^* if and only if

$$\int(D(h), Z)dv=\int((U(h)^+, Z)dv=0$$

for all $h\in C^{\infty}(M, \mathrm{Sk}_0(T^*))$.

Since every $h\in\mathrm{Sk}_0(T^*)$ satisfies $h(JX, JY)=-h(X, Y)$, there exist an endomorphism I of $T=T_M$ satisfying that $h(X, Y)=g(IX, Y)+g(X, IY)$ and $IJ+JI=0$ hold (J is the complex structure of (M, g)).

From the last relation I is represented in complex coordinates as

$$I=I_{\bar{\beta}}^{\alpha}\frac{\partial}{\partial z^{\alpha}}\otimes dz^{\bar{\beta}}+\overline{I_{\bar{\beta}}^{\alpha}}\frac{\partial}{\partial z^{\bar{\alpha}}}\otimes dz^{\beta}$$

so that $\bar{\alpha}\bar{\beta}$-component of h is $h_{\bar{\alpha}\bar{\beta}}=I_{\bar{\alpha}\bar{\beta}}+I_{\bar{\beta}\bar{\alpha}}$, where $I_{\bar{\alpha}\bar{\beta}}=g_{\gamma\bar{\alpha}}I_{\bar{\beta}}^{\gamma}$.

From Lemma 2.3 the $\Omega^{2,0}\otimes\Omega^{2,0}$-component of $U(h)^+$ is $U(h)_{1212}=1/2(\nabla_1\nabla_2h_{12}+\nabla_2\nabla_1h_{12}-\nabla_2\nabla_2h_{11}-\nabla_1\nabla_1h_{22})$. Since from the Kähler property we have $\nabla_1\nabla_2h_{12}=\nabla_2\nabla_1h_{12}$, this reduces to $1/2(2\nabla_1\nabla_2h_{12}-\nabla_2\nabla_2h_{11}-\nabla_1\nabla_1h_{22})$ and its conjugate is the

$\Omega^{0,2}\otimes\Omega^{0,2}$-component $U(h)_{\bar{1}\bar{2}\bar{1}\bar{2}}$.

Thus

$$U(h)_{\bar{1}\bar{2}\bar{1}\bar{2}}=\nabla_{\bar{1}}\nabla_{\bar{2}}I_{\bar{2}\bar{1}}+\nabla_{\bar{1}}\nabla_{\bar{2}}I_{\bar{1}\bar{2}}-\nabla_{\bar{1}}\nabla_{\bar{1}}I_{\bar{2}\bar{2}}-\nabla_{\bar{2}}\nabla_{\bar{2}}I_{\bar{1}\bar{1}}\,. \tag{2.17}$$

On the other hand, because of $IJ+JI=0$, I is a deformation of complex structures and is regarded as a section of the bundle $T^{1,0}\otimes\Omega^{0,1}$, $I=I^{\alpha}_{\bar{\beta}}(\partial/\partial z^{\alpha})\otimes dz^{\bar{\beta}}$. The bundle $T^{1,0}\otimes\Omega^{0,1}$ is equipped with the $\bar{\partial}$-operator $\bar{\partial}: C^{\infty}(M, T^{1,0}\otimes\Omega^{0,1})\to C^{\infty}(M, T^{1,0}\otimes\Omega^{0,2})$. So $\bar{\partial}I\in C^{\infty}(M, T^{1,0}\otimes\Omega^{0,2})$.

Define an operator

$$\vartheta: C^{\infty}(M, T^{1,0}\otimes\Omega^{0,2})\longrightarrow C^{\infty}(M, \Omega^{0,2}\otimes\Omega^{0,2}) \tag{2.18}$$

by

$$\vartheta(L)=b(\bar{\partial}(\# L)),\quad L\in C^{\infty}(M, T^{1,0}\otimes\Omega^{0,2}) \tag{2.19}$$

where $\#: T^{1,0}\otimes\Omega^{0,2}\to\Omega^{0,1}\otimes\Lambda^2 T^{1,0}$, $L^{\alpha}_{\bar{\beta}\bar{\gamma}}\mapsto L^{\beta\gamma}_{\bar{\alpha}}$ and $b: \Omega^{0,2}\otimes\Lambda^2 T^{1,0}\to\Omega^{0,2}\otimes\Omega^{0,2}$, $N^{\beta\gamma}_{\bar{\alpha}\bar{\delta}}\mapsto N_{\bar{\alpha}\bar{\delta}\bar{\beta}\bar{\gamma}}$.

LEMMA 2.8. *The $\Omega^{2,0}\otimes\Omega^{2,0}\oplus\Omega^{0,2}\otimes\Omega^{0,2}$-component of $U(h)^{+}$, $h\in C^{\infty}(M, \mathrm{Sk}_0(T^*))$ coincides with $\overline{(\vartheta\bar{\partial}I)}+(\vartheta\bar{\partial}I)$. Here I is $I=I^{\alpha}_{\bar{\beta}}(\partial/\partial z^{\alpha})\otimes dz^{\bar{\beta}}\in C^{\infty}(M, T^{1,0}\otimes\Omega^{0,1})$ satisfying $h_{\bar{\alpha}\bar{\beta}}=I_{\bar{\alpha}\bar{\beta}}+I_{\bar{\beta}\bar{\alpha}}$ and $I_{\bar{\alpha}\bar{\beta}}=g_{\gamma\bar{\alpha}}I^{\gamma}_{\bar{\beta}}$.*

From this lemma

$$\int(Dh, Z)dv=\int(U(h)^{+}, Z)dv=\int(\overline{\vartheta\bar{\partial}I}+\vartheta\bar{\partial}I, Z)dv=0$$

for all $I\in C^{\infty}(M, T^{1,0}\otimes\Omega^{0,1})$ and this is equivalent to that the $\Omega^{0,2}\otimes\Omega^{0,2}$-component Σ of Z satisfies $\int(\Sigma, \vartheta\bar{\partial}I)dv=\int(\vartheta^*\Sigma, \bar{\partial}I)dv=0$ for all I, in other words, $\vartheta^*\Sigma\in\mathrm{Ker}\,\bar{\partial}^*=H^2(M, \mathcal{O}(T^{1,0}))$.

We assume now $H^2(M, \mathcal{O}(T^{1,0}))=0$. Then $\vartheta^*\Sigma=0$. Since $T^{1,0}\cong\Omega^{0,1}$, the operator ϑ is considered as

$$\vartheta: C^{\infty}(M, \Omega^{0,1}\otimes\Omega^{0,2})\longrightarrow C^{\infty}(M, \Omega^{0,2}\otimes\Omega^{0,2})\,.$$

So for $a=1/2\, a_{\bar{\alpha}\bar{\beta}\bar{\gamma}}dz^{\bar{\alpha}}\otimes(dz^{\bar{\alpha}}\wedge dz^{\bar{\gamma}})\in C^{\infty}(M, \Omega^{0,1}\otimes\Omega^{0,2})$

$$(\vartheta a)_{\bar{\alpha}\bar{\delta}\bar{\beta}\bar{\gamma}}=\nabla_{\bar{\alpha}}a_{\bar{\delta}\bar{\beta}\bar{\gamma}}-\nabla_{\bar{\delta}}a_{\alpha\bar{\beta}\bar{\gamma}}$$

and for $\Sigma=1/4\sum_{\bar{\alpha}\bar{\delta}\bar{\beta}\bar{\gamma}}(dz^{\bar{\alpha}}\wedge dz^{\bar{\delta}})\otimes(dz^{\bar{\beta}}\wedge dz^{\bar{\gamma}})$ the adjoint ϑ^* is

$$(\vartheta^*\Sigma)_{\bar{\alpha}\bar{\beta}\bar{\gamma}}=-g^{\sigma\bar{\delta}}\nabla_{\sigma}\Sigma_{\bar{\delta}\bar{\alpha}\bar{\beta}\bar{\gamma}} \tag{2.20}$$

LEMMA 2.9. *For any $\Psi\in C^{\infty}(M, \Omega^{0,2}\otimes\Omega^{0,2})$*

(2.21) $$(\vartheta\vartheta^*+\vartheta^*\vartheta)\Psi=\nabla''^*\nabla''\Psi\,,$$

where $\vartheta: C^\infty(M, \Omega^{0,2}\otimes\Omega^{0,2})\to C^\infty(M, \Omega^{0,3}\otimes\Omega^{0,2})$ *is the extended operator of* ϑ *on* $\Omega^{0,1}\otimes\Omega^{0,2}$ *and* $\nabla''^*\nabla''$ *is the rough Laplacian* $-g^{\sigma\bar\tau}\nabla_\sigma\nabla_{\bar\tau}$.

In fact, from the Weitzenböck-Bochner formula

$$(\vartheta\vartheta^*+\vartheta^*\vartheta)\Psi_{\bar\alpha\bar\delta\bar\beta\bar\gamma}=-g^{\sigma\bar\tau}\nabla_\sigma\nabla_{\bar\tau}\Psi_{\bar\alpha\bar\delta\bar\beta\bar\gamma}+g^{\sigma\bar\tau}[\nabla_\sigma, \nabla_{\bar\alpha}]\Psi_{\bar\tau\bar\delta\bar\beta\bar\gamma}$$
$$-g^{\sigma\bar\tau}[\nabla_\sigma, \nabla_{\bar\delta}]\Psi_{\bar\alpha\bar\tau\bar\beta\bar\gamma}\,.$$

We apply the Ricci identity and $2\rho=g^{\alpha\bar\beta}R_{\alpha\bar\beta}$ to have

(2.22) $$(\vartheta\vartheta^*+\vartheta^*\vartheta)\Psi=\nabla''^*\nabla''\Psi+\rho\Psi$$

from which (2.21) follows.

From Lemma 2.9 the $\Omega^{0,2}\otimes\Omega^{0,2}$-component Σ of Z satisfies $\nabla_{\bar\tau}\Sigma_{\bar\alpha\bar\delta\bar\beta\bar\gamma}=0$ and then by applying the Ricci identity again $\nabla_\sigma\Sigma_{\bar\alpha\bar\delta\bar\beta\bar\gamma}=0$ so that Σ and its complex conjugate $\bar\Sigma$ are parallel. So $\bar\Sigma$ gives a holomorphic section of K_M^2.

Conversely, given $\bar\Sigma\in H^0(M, \mathcal{O}(K_M^2))$. Then $\bar\Sigma$ is parallel from Lemma 2.10, (iii) so that $Z=\bar\Sigma+\Sigma$ turns out to be in the kernel of D^*, which proves (iii), Theorem 1.

The proof of (iv) of Theorem 1 is easily done, since for Z of form $\theta(\varphi+\bar\varphi)$ $(\phi+\bar\phi)$, $\varphi\in C^\infty(M, K_M)$, $\phi\in C^\infty(M, K_M^2)$ we have

$$\int(D(h), Z)dv=\int(D(h), \theta(\varphi+\bar\varphi))dv$$

for $h\in C^\infty(M, \mathrm{Her}_0(T^*))$ and

$$\int(D(h), Z)dv=\int(D(h), \phi+\bar\phi)dv$$

for $h\in C^\infty(M, \mathrm{Sk}_0(T^*))$.

Thus by these arguments we can prove Theorem 1 completely.

(iii) We would like to show the following parallel lemma which was applied to the proof of (iii), Theorem 1.

LEMMA 2.10. *Let* (M, g) *be a compact Kähler surface of zero scalar curvature. Then for any positive integer* $m>0$

$$H^0(M, \mathcal{O}(K_M^m))=\{\textit{parallel sections of } K_M^m\}\,.$$

PROOF. The space of holomorphic sections of K_M^m, $H^0(M, \mathcal{O}(K_M^m))$, is $\{\Psi\in C^\infty(M, K_M^m)\,;\,\bar\partial^*\bar\partial\Psi=0\}$, where $\bar\partial: C^\infty(M, K_M^m)\to C^\infty(M, K_M^m\otimes\Omega^{0,1})$ and, $\bar\partial^*$ is its adjoint with respect to the naturally induced fibre metric on the m-th power of K_M. The Weitzenböck-Bochner formula is then

$$\bar{\partial}^*\bar{\partial}\Psi=-g^{\alpha\bar{\beta}}\nabla_\alpha\nabla_{\bar{\beta}}\Psi+m/2\rho\Psi \tag{2.23}$$

(see Theorem 6.2 in [12], we applied a similar argument to our case). From $\rho=0$ we have $\nabla_{\bar{\beta}}\Psi=0$ for any $\bar{\beta}$ and further from the Ricci identity $\nabla_\alpha\Psi=0$ and thus $\nabla\Psi=0$.

3. Cohomologies of K_M^i, $i=1, 2$ and $T_M^{1,0}$.

For each type of compact Kähler surface of zero scalar curvature we can evaluate cohomologies $H^0(M, \mathcal{O}(K_M^i))$, $i=1, 2$ and $H^2(M, \mathcal{O}(T_M^{1,0}))$.

		$H^0(M, \mathcal{O}(K_M))$	$H^0(M, \mathcal{O}(K_M^2))$	$H^2(M, \mathcal{O}(T^{1,0}))$
(i)	complex 2-torus	$\boldsymbol{C}$	$\boldsymbol{C}$	$\boldsymbol{C}^2$
(ii)	K3 surface	$\boldsymbol{C}$	$\boldsymbol{C}$	0
	Enriques surface	0	$\boldsymbol{C}$	0
(iii)	ruled surface M_k	0	0	0
(iv)	blown up of M_k	0	0	0

For the evaluation of $H^0(M, \mathcal{O}(K_M^i))$, $i=1, 2$ we apply Lemma 2.8 to the first three cases. $H^0(M, \mathcal{O}(K_M^i))=0$, $i=1, 2$ for the last case since $\dim H^0(M, \mathcal{O}(K_M^i))$, $i>0$, is a birational invariant.

The cohomology group $H^2(M, \mathcal{O}(T^{1,0}))$ which is the obstruction space for deformation of complex structure is isomorphic by Serre's duality to $H^0(M, \mathcal{O}(K_M\otimes\Omega^{1,0}))$. For the first two surfaces this is isomorphic to $H^0(M, \mathcal{O}(\Omega^{1,0}))$ because $\mathcal{O}(K_M)=\mathcal{O}$ and then $H^2(M, \mathcal{O}(T^{1,0}))$ is isomorphic to $\boldsymbol{C}^2$ for a complex 2-torus and is zero for a K 3 surface.

For an Enriques snrface with a Ricci flat Kähler metric we have similarly $H^2(M, \mathcal{O}(T^{1,0}))=H^0(M, \mathcal{O}(K_M\otimes\Omega^{1,0}))=0$ since it is a $\boldsymbol{Z}_2$-quotient of a K 3 surface.

That $H^2(M, \mathcal{O}(T^{1,0}))$ vanishes for the case of ruled surface is obtained by restricting $\phi\in H^0(M, \mathcal{O}(K_M\otimes\Omega^{1,0}))$ to each fibre, a complex projective line CP^1, since any holomorphic covariant tensor on CP^1 must vanish.

Vanishing of $H^2(M, \mathcal{O}(T^{1,0}))$ for the last case is derived from the fact that there is a one-to-one correpondence between holomorphic covariant tensors on M and those on $\tilde{M}$, a one point blown up of M ([p. 225, 13]).

REMARK. A hyperelliptic surface M is a finite group quotient of a product of elliptic curves. As a smooth 4-manifold the surface M is a quotient of complex 2-torus and it is seen that M admits a flat Kähler metric. It is shown that the canonical line bundle K_M is a torsion bundle of order 2, 3, 4 and 6 according to the type of M ([p. 148, 2]). So we have $H^0(M, \mathcal{O}(K_M))=0$ and $H^2(M, \mathcal{O}(T^{1,0}))=0$. Moreover $H^0(M, \mathcal{O}(K_M^2))=\boldsymbol{C}$ when $\mathcal{O}(K_M^2)=\mathcal{O}$ and $H^0(M, \mathcal{O}(K_M^2))=0$ when $\mathcal{O}(K_M^2)\neq\mathcal{O}$. That $H^2(M, \mathcal{O}(T^{1,0}))$ vanishes follows from the fact that $H^0(M, \mathcal{O}(\Omega^{1,0}))=\boldsymbol{C}$ together with Lemma 2.10.

4. The case of trivial K_M.

Let M be a complex 2-torus or a K 3 surface with a Ricci flat Kähler metric. Since $K_M=\mathcal{O}$, it admits a holomorphic section ϕ which is parallel with respect to the Ricci flat Kähler metric from Lemma 2.10.

We now show Theorem 2, namely that the second cohomology group $\boldsymbol{H}^2$ is isomorphic to $\boldsymbol{R}^5$ for such a 4-manifold M. For this it suffices to verify from the proof of iv), Theorem 1 that $\mathrm{Ker}\,(D^*|_{C^\infty(M, V^0\oplus V^1)})\cong\boldsymbol{R}^3$.

Suppose that $f\Phi+Z$ is in $\mathrm{Ker}\,D^*$, for $Z=\theta(\varphi+\bar{\varphi})\in C^\infty(M, V^1)$. Then we have (2.11) and (2.14).

Since $\varphi\in C^\infty(M, K_M)$ is written as $\varphi=F\phi$ for a complex valued function F on M. So combining the formulae (2.11) and (2.14) we get the equation

$$-\nabla_\alpha\nabla_{\bar\beta}f-14\,\Delta f\cdot g_{\alpha\bar\beta}+\sqrt{-1}\{\nabla_{\bar\beta}\nabla^{\delta}F\cdot\phi_{\alpha\delta}+\nabla_{\bar\alpha}\nabla^{\bar\gamma}\bar F\cdot\bar\phi_{\gamma\beta}\}=0\,. \tag{4.1}$$

Operating the partial covariant derivation $\nabla^{\bar\beta}$ to both sides and taking the contraction we have

$$-\nabla^{\bar\beta}\nabla_\alpha\nabla_{\bar\beta}f-1/4\,\nabla_\alpha\Delta f+\sqrt{-1}\{\nabla^{\bar\beta}\nabla_{\bar\beta}\nabla^{\delta}F\cdot\phi_{\alpha\delta}\}=0\,, \tag{4.2}$$

since in the forth term $\nabla^{\bar\beta}\nabla_\alpha\nabla^{\bar\gamma}\bar F\cdot\bar\phi_{\gamma\beta}=\nabla_\alpha\nabla^{\bar\beta}\nabla^{\bar\gamma}\bar F\cdot\bar\phi_{\gamma\beta}$ and $[\nabla^{\bar\beta}, \nabla^{\bar\gamma}]=0$ and $\phi_{\gamma\beta}$ is skew symmetric.

Since M is Ricci flat, (4.2) reduces to

$$1/4\,\nabla_\alpha\Delta f-\sqrt{-1}/2\,(\nabla^{\delta}\Delta F)\phi_{\alpha\delta}=0\,. \tag{4.3}$$

So, by integrating, we derive the following over M

$$1/4\int_M\|\partial\Delta f\|^2dv-\sqrt{-1}/2\int_M\nabla^{\delta}\Delta F\nabla^{\alpha}\Delta f\phi_{\alpha\delta}=0\,. \tag{4,4}$$

The second term is $\int_M\Delta f(\nabla^{\alpha}\nabla^{\delta}\Delta F)\phi_{\alpha\delta}$ and then vanishes. So $\partial\Delta f=0$, that is, Δf is constant and then by integrating it over M space f must be constant.

Therefore, it follows from $D^*(f\Phi+Z)_{\alpha\bar\beta}=0$ that f is constant and that $D^*(Z)_{\alpha\bar\beta}=0$. We can then apply the statement of (ii), Theorem 1 and have Theorem 2.

5. The Hessian equation.

Let (M, g) be a compact Kähler surface of zero scalar curvature.

Consider on M the equation (2.13),

$$\mathrm{Hes}(f)=-1/4\,\Delta f\cdot g-1/2\,f\cdot B\,,$$

$B=Ric-\rho/4\,g=Ric.$

Except for the case of a complex 2-torus or a K 3 surface as is seen in Theorem 2, so far we do not have exact knowledge of the solution space of the above equation.

Now we assume that (M, g) is a ruled surface of genus $k\geqq 2$. M is then a compact quotient of the product Kähler surface $D^1\times \boldsymbol{C}P^1$ by a subgroup Γ of $SL(2, \boldsymbol{R})\times SU(2)/\boldsymbol{Z}_2$ acting freely and properly discontinuously.

Let (z^1, z^2), $z^1=z$, $z^2=w$, be the complex coordinate of $D^1\times \boldsymbol{C}P^1$ in such a way that z and w represent complex coordinates of D^1 and $\boldsymbol{C}P^1$, respectively. Then the metric g is $g=g_1+g_2$, $g_1=g_1(z)dzd\bar{z}$, $g_2=g_2(w)dwd\bar{w}$ and the Ricci tensor is $Ric=-g_1+g_2$.

Suppose now that $f\in C^\infty(M)$ is a solution of (2.13).

We consider f as a function on $D^1\times \boldsymbol{C}P^1$ invariant under the action of Γ. So by taking (2, 0) and (0, 2) parts of (2.13) we have

$$\partial_z\partial_w f=0\,,\qquad \partial_{\bar{z}}\partial_{\bar{w}} f=0 \tag{5.1}$$

and also

$$\partial_z\partial_{\bar{w}} f=\partial_w\partial_{\bar{z}} f=0 \tag{5.2}$$

by substituting $(\partial/\partial z, \partial/\partial\bar{w})$ into (2.13).

(5.1) and (5.2) are the second order partial differential equations and then f must be written in the form $f=F(z, \bar{z})+G(w, \bar{w})$ where F and G are real valued functions on D^1 and $\boldsymbol{C}P^1$, respectively.

We put $f=F+G$ into (2.13). We have then

$$\partial_z\partial_{\bar{z}}F=1/2\,(g_1^{-1}\partial_z\partial_{\bar{z}}F+g_2^{-1}\partial_w\partial_{\bar{w}}G)\cdot g_1+1/2\,g_1(F+G)\,, \tag{5.3}$$

$$\partial_w\partial_{\bar{w}}G=1/2\,(g_1^{-1}\partial_z\partial_{\bar{z}}F+g_2^{-1}\partial_w\partial_{\bar{w}}G)\cdot g_2-1/2\,g_2(F+G)\,. \tag{5.4}$$

So $\square_1 F=-g_1^{-1}\partial_z\partial_{\bar{z}}F$ and $\square_2 G=-g_2^{-1}\partial_w\partial_{\bar{w}}G$ satisfy the following equation:

$$\square_1 F-\square_2 G=-(F+G) \tag{5.5}$$

from which

$$\square_1 F+F=\square_2 G-G=\text{constant }\lambda \tag{5.6}$$

holds. Thus $F-\lambda$ is an eigenfunction of $\square_1$ corresponding to the eigenvalue -1. Since f is Γ-invariant, F is considered as a function on a compact Riemann surface D^1/Γ. Hence $F-\lambda$ is an eigenfunction of negative eigenvalue and must be zero, in other words, $F=\lambda$ and then $f=G(w, \bar{w})+\lambda$.

Since $\square_2(G+\lambda)-(G+\lambda)=0$ and $\square_2=1/2\,\Delta_2$, $f=G+\lambda$ is an eigenfunction of the real Laplacian Δ_2 on CP^1 corresponding to the eigenvalue 2. These eigenfunctions are obtained by restricting each coordinate function x, y and z of $\boldsymbol{R}^3$ to the unit sphere $S^2=\{(x, y, z)\in\boldsymbol{R}^3\,;\,x^2+y^2+z^2=1\}\cong\boldsymbol{C}P^1$ (see [p. 160, 3]). It is easily shown that these functions satisfy the rest of the equation (2.13) because of the symmetry of S^2.

We have thus

PROPOSITION 5.1. *Let (M, g) be a ruled surface of genus $k\geqq 2$ with a Kähler metric of zero scalar curvature. Then the solution space of the equation* (2.13) *is isomorphic to $\boldsymbol{R}^3$, namely the kernel of D^* restricted to $C^\infty(M, V^0)$ is* $\mathrm{Ker}\, D^*|_{C^\infty(M, W^0)}\cong\boldsymbol{R}^3$.

The proof of Theorem 2 given in §4 can be applied to the case of Ricci flat Kähler surfaces with nontrivial canonical line bundle (for example, Enriques surface and hyperelliptic surface). In fact we have

PROPOSITION 5.2. *For a Ricci flat Kähler surface* $\mathrm{Ker}\, D^*|_{C^\infty(M, V^0)}\cong\boldsymbol{R}$.

PROOF. Suppose that $D^*(f\Phi)=0$ for $f\in C^\infty(M)$. Then from (2.11) we have

$$-\nabla_\alpha\nabla_{\bar\beta}f-1/4\Delta f\cdot g_{\alpha\bar\beta}=0\,,$$

since $Ric=0$. So

$$-\nabla^{\bar\beta}\nabla_\alpha\nabla_{\bar\beta}f-1/4\nabla_\alpha\Delta f=0$$

which reduces to $1/4\nabla_\alpha\Delta f=0$. Thus f must be constant.

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Institute of Mathematics
University of Tsukuba
305, Tsukuba, Japan