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A REMARK ON TILED ORDERS OVER A LOCAL DEDEKIND DOMAIN

Dedicated to Professor Hisao Tominaga
on his 60th birthday

By

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Let R be a noetherian domain with the quotient ring K. An R-order in the full $n\times n$ matrix ring $(K)_n$ is called *tiled* if it contains n orthogonal idempotents (cf. [3]). There are many papers on noetherian ring theory which contain tiled R-orders as examples. Concerning global dimension, tiled R-orders are studied by K.L. Fields [1], R.B. Tarsy [10], [11], V.A. Jategaonkar [2], [3], [4] and K.W. Roggenkamp [8], [9].

In [5], B.J. Müller introduced the concept of *links* between prime ideals of Fully Bounded Noetherian (FBN) rings to study localizability of semiprime ideals. Recently in [6], he initiated a detailed study of the link graph and announced some results on FBN prime rings of Krull dimension one, especially, with self-injective dimension one.

In this note, we shall attempt a study on the link graph of tiled orders over a local Dedekind domain, which are FBN prime rings of Krull dimension one and have arbitrarily large global dimension (cf. [1], [7] and Example 3.5).

After recalling some definitions and notations, in Section 1, we shall point out that the link graph coincides with the quiver of orders introduced by A. Wiedemann and K.W. Roggenkamp [12].

Confining ourselves to tiled R-orders between $(R)_n$ and its radical, in Section 2, we shall prove the following.

THEOREM. *Let R be a local Dedekind domain with the maximal ideal πR and the quotient ring K. Let Λ be a basic tiled R-order between $(R)_n$ and $(\pi R)_n$, $Q(A)$ the quiver of the $R/\pi R$-algebra $A=\Lambda/(\pi R)_n$ and $M_1,\cdots,M_n$ the maximal ideals of Λ. Then, there is a link from M_i to M_j if and only if there is an arrow from i to j in $Q(A)$, or else i is a non-domain and j is a non-range in $Q(A)$.*

We shall give some remarks after proving the theorem. We shall add an

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Appendix in which we shall announce global dimension of some special Λ, i.e., $Q(A)$ is a tree, of A_n-type, a cycle and so on.

1. Preliminaries

Let R be a local Dedekind domain with the maximal ideal πR and the quotient ring K. Let $(K)_n$ be the full $n\times n$ matrix ring over K and Λ be a *tiled* R-order in $(K)_n$ (i.e., Λ contains n orthogonal primitive idempotents). By virtue of [4, Lemma 1], we may assume $\Lambda=(\pi^{\lambda_{ij}}R)\subset(R)_n$ where λ_{ij}'s are non-negative integers and $\lambda_{ii}=0$ for $1\leqq i\leqq n$. Since Λ is a subring of $(R)_n$, it holds that

(*-1) $\qquad \lambda_{ik}+\lambda_{kj}\geqq\lambda_{ij}$ for all $1\leqq i,\ j,\ k\leqq n$.

Since R is a local ring, Λ is semiperfect. Using (*-1), it is easily checked that Λ is basic if and only if Λ satisfies

(*-2) $\qquad$ If $i\neq j$, $\lambda_{ij}=0$ implies $\lambda_{ji}\neq 0$.

In what follows, $\Lambda=(\pi^{\lambda_{ij}}R)$ is a basic tiled R-order contained in $(R)_n$.

Since Λ is finitely generated over a local Dedekind domain, Λ is an FBN prime ring of Krull dimension one.

For $1\leqq k\leqq n$, let $M_k=(\pi^{m_{kij}}R)\subset\Lambda$ where $m_{kij}=1$ (if $i=j=k$) λ_{ij} (otherwise). Then $M_1,\cdots,M_n$ are the maximal ideals of Λ.

There exists a *link* from M_i to M_j (denoted by $M_i \rightsquigarrow M_j$) if $M_i\cap M_j\supsetneqq M_jM_i$ holds (cf. [5, Remark (2), p236]).

Let I_1, I_2 be ideals of R. Since R is a local Dedekind domain, $I_1=\pi^aR$, $I_2=\pi^bR$ for some integers $a,b\geqq 0$. Then $I_1\supset I_2$ if and only if $a\leqq b$. We define an order between ideals of R by $I_1\leqq I_2$ if and only if $a\leqq b$. (The symbol "$\leqq$" may not be confused in the context.) We shall use Max and Min among ideals of R under the above order.

Put $M_{kij}=\pi^{m_{kij}}R$ for $1\leqq i,\ j,\ k\leqq n$. For $1\leqq k,\ h\leqq n$, put

$$X_{kh}=\mathrm{Max}\{M_{ikh},\ M_{jkh}\},$$

$$Y_{kh}=\mathrm{Min}\{M_{jk\ell}\,M_{i\ell h}\,|\,1\leqq\ell\leqq n\}.$$

Then $M_i\cap M_j=(X_{kh})$ and $M_jM_i=(Y_{kh})$.

LEMMA 1.1. *If* $(k,\ h)\neq(j,\ i)$, *then* $X_{kh}=Y_{kh}$.

PROOF. It holds that $M_{ikh}\leqq X_{kh}\leqq Y_{kh}\leqq M_{jkk}\,M_{ikh}$. If $X_{kh}\neq Y_{kh}$, we have $M_{jkk}\neq R$. Hence $k=j$. Similarly $h=i$.

For $1\leqq i\leqq n$, let e_i be the matrix in Λ with (i,i)-entry equal to 1 and all others

0, and put $P_i=e_i\Lambda$, $J_i=\mathrm{rad}(P_i)=e_iJ$, where $J=\mathrm{rad}(\Lambda)$. $\boldsymbol{P}(X)$ denote a projective cover of a module X.

We now repeat the (right-handed) definition of the valued quiver of the tiled R-order given by A. Wiedemann and K.W. Roggenkamp [12]. (The links that we have been using are right-handed, but [12] is left-handed.)

A *valued quiver* $Q=(Q_0, Q_1, d, r, v)$ consists of a finite set Q_0 of vertices and a finite set Q_1 of arrows. d and r are maps from Q_1 to Q_0 such that $d(\alpha)$ is the domain and $r(\alpha)$ is the range of an arrow $\alpha\in Q_1$. v is a map from Q_1 to non-negative integers. Forgetting the valuation map v from a valued quiver, we call it a *quiver*.

For the tiled R-order $\Lambda=(\pi^{\lambda_{ij}}R)\subset(R)_n$, the *valued quiver* $Q(\Lambda)$ *of* Λ is defined by the vertices $Q(\Lambda)_0=\{1,\cdots,n\}$, there exists an arrow $\alpha\in Q(\Lambda)_1$ with $d(\alpha)=i$, $r(\alpha)=j$ if P_i is isomorphic to a direct summand of $\boldsymbol{P}(J_j)$, and $v(\alpha)=\lambda_{ji}$. In [12], a procedure is given to construct a tiled R-order $\Lambda(Q)$ from a certain valued quiver Q and it is shown that $\Lambda=\Lambda(Q(\Lambda))$ [12, §2, Theorem 1].

PROPOSITION 1.2. *The link graph between maximal ideals of* Λ *coincides with the quiver of* Λ.

PROOF. There is an arrow from i to j in $Q(\Lambda)\Longleftrightarrow P_i$ is isomorphic to a direct summand of $\boldsymbol{P}(J_j)\Longleftrightarrow e_jJe_i\not\subset e_jJ^2e_i\Longleftrightarrow X_{ji}\lneqq Y_{ji}\Longleftrightarrow M_i\rightsquigarrow M_j$ by Lemma 1.1.

COROLLARY 1.3. *If* Λ *has finite global dimension, then all maximal ideals of* Λ *are idempotent.*

PROOF. It follows from [12, §1, Lemma 3] that $Q(\Lambda)$ has no loops (i.e., there is no arrows from a vertex to itself). So, by the proposition, $M_i\not\rightsquigarrow M_i$, and hence $M_i=M_i^2$ for all $1\leqq i\leqq n$.

REMARK. There is a tiled R-order of infinite global dimension with all of whose maximal ideals are idempotent. See Example 3.4.

An ideal I is *eventually idempotent* if $I^m=I^{m+1}=\cdots$ for some integer $m>0$. Of course, there are many tiled R-orders with non-idempotent maximal ideals, however, we note the following.

PROPOSITION 1.4. *Let* Λ *be the tiled* R*-order in* $(R)_n$ $(n\geqq 2)$. *Then all maximal ideals of* Λ *are eventually idempotent.*

PROOF. Fix $1\leqq k\leqq n$. By induction on ℓ, define

$$\begin{cases} m^1_{kij} = m_{kij}, \\ m^{\ell+1}_{kij} = \min\{m^{\ell}_{kih} + m_{khj} | 1 \leqq h \leqq n\}. \end{cases}$$

Then it is easily shown that $m^{\ell}_{kij} \leqq \ell$ (if $i=j=k$) and $=\lambda_{ij}$ (otherwise). It follows that $M^d_k = M^{d+1}_k = \cdots$, where $d = \min\{\lambda_{kh} + \lambda_{hk} | 1 \leqq h \leqq n,\ h \neq k\}$.

2. The link graph of Λ and the quiver of the factor algebra

Let $\Lambda = (\pi^{\lambda_{ij}}R)$ be a basic tiled R-order between $(R)_n$ and $(\pi R)_n$ (i.e., $(R)_n \supset \Lambda \supset (\pi R)_n$). Put $\Lambda_{ij} = \pi^{\lambda_{ij}}R$, $A = \Lambda/(\pi R)_n$ and $N = \mathrm{rad}(A)$. Then A is a basic $R/\pi R$-algebra. The quiver $Q(A)$ of A is defined by the set of vertices $Q(A)_0 = \{1, \cdots, n\}$ and there is an arrow from i to j if

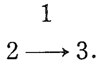
(*-3) $$e_j N e_i / e_j N^2 e_i \neq 0.$$

Let $J = (J_{ij})$ be the Jacobson radical of Λ and $B_{ij} = \mathrm{Min}\{J_{ik} J_{kj} | 1 \leqq k \leqq n\}$. Then (*-3) is equivalent to $J_{ji}/(B_{ji} + \pi R) \neq 0$. So $Q(A)$ has no loops. It follows from (*-2) that $Q(A)$ has no oriented cycles. Let $\mathscr{D}$ (resp. $\mathscr{R}$) denote a subset of $Q(A)_0$ consisting of non-domains (resp. non-ranges) in $Q(A)$.

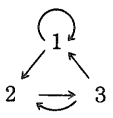
Here, we give an example which helps the reader's understanding of the theorem.

EXAMPLE 2.1. Let $\Lambda = \begin{pmatrix} R & \pi R & \pi R \\ \pi R & R & \pi R \\ \pi R & R & R \end{pmatrix}$. Then $A \cong \begin{pmatrix} \boldsymbol{k} & 0 & 0 \\ 0 & \boldsymbol{k} & 0 \\ 0 & \boldsymbol{k} & \boldsymbol{k} \end{pmatrix}$, where $\boldsymbol{k} = R/\pi R$.

The quiver $Q(A)$ is given by



So $\mathscr{D} = \{1, 3\}$ and $\mathscr{R} = \{1, 2\}$. Thus the link graph is given by



LEMMA 2.2. (1) *If* $d \in \mathscr{D}$, *then* $\Lambda_{jd} = \pi R$ *for all* $j \neq d$.
(2) *If* $r \in \mathscr{R}$, *then* $\Lambda_{ri} = \pi R$ *for all* $i \neq r$.

PROOF. (1) For each $j \neq d$, since $e_j N e_d / e_j N^2 e_d = 0$,

(i) $$\Lambda_{jd}/(B_{jd} + \pi R) = 0.$$

Assume that $\Lambda_{jd} = R$ for some $j \neq d$. Then by (i), $B_{jd} = R$. Since $B_{jd} = \mathrm{Min}\{J_{j\ell} J_{\ell d}|$

$1\leqq\ell\leqq n\}$, $J_{j\ell_1}J_{\ell_1 d}=R$ for some $1\leqq\ell_1\leqq n$. Clearly, $J_{j\ell_1}=J_{\ell_1 d}=R$ and $j\neq\ell_1\neq d$. Hence $\Lambda_{\ell_1 d}=J_{\ell_1 d}=R$, and by (i), $B_{\ell_1 d}=R$. Repeating the above argument, we obtain $\ell_0=j, \ell_1, \cdots, \ell_n$ $(\neq d)$ and $J_{\ell_{i-1}\ell_i}=R$ $(1\leqq i\leqq n)$. So $\ell_h=\ell_k$ $(=\ell$, say) for some $0\leqq h<k\leqq n$. We get $\pi R\supset J_{\ell\ell_{h+1}}\cdots J_{\ell_{k-1}\ell}=R$, a contradiction. This completes the proof of (1). Similarly, (2) is proved.

PROOF OF THE THEOREM. $(\Longrightarrow)$ It follows from Lemma 1.1 that $X_{ji}\lneqq Y_{ji}$.

Case 1. $X_{ji}=\pi R$.

Then $Y_{ji}\geqq\pi^2R$. It holds that

$$\text{(ii)}\qquad \begin{cases} M_{iji}=M_{jji}=\pi R & \text{if } i=j, \\ M_{iji}=\Lambda_{ji}=M_{jji}=\pi R & \text{if } j\neq i. \end{cases}$$

Hence $Y_{ji}=\mathrm{Min}\,\{M_{jjk}\,M_{iki}|1\leqq k\leqq n\}=\mathrm{Min}\,[\{\Lambda_{jk}\Lambda_{ki}|1\leqq k\leqq n,\ \ k\neq i,\ \ j\}\cup\{\pi^2R\}]\leqq\pi^2R$. Therefore $\pi^2R=Y_{ji}\leqq\Lambda_{jk}\Lambda_{ki}$ for $1\leqq k\leqq n$, $k\neq i$, j. Hence by (ii), $\Lambda_{jk}=\pi R$ if $k\neq j$ and $\Lambda_{ki}=\pi R$ if $k\neq i$. Thus

$$e_kNe_i/e_kN^2e_i\cong\Lambda_{ki}/(B_{ki}+\pi R)=\pi R/(B_{ki}+\pi R)=0 \qquad \text{if } \ k\neq i,$$

$$e_jNe_k/e_jN^2e_k\cong\Lambda_{jk}/(B_{jk}+\pi R)=\pi R/(B_{jk}+\pi R)=0 \qquad \text{if } \ k\neq j.$$

Consequently, $i\in\mathcal{D}$ and $j\in\mathcal{R}$.

Case 2. $X_{ji}=R$.

Then $Y_{ji}\geqq\pi R$. It holds that $i\neq j$, $M_{iji}=\Lambda_{ji}=M_{jji}=R$. Observe that

$$\text{(iii)}\qquad \begin{aligned} B_{ji}&=\mathrm{Min}\,\{J_{jk}\,J_{ki}|1\leqq k\leqq n\} \\ &=\mathrm{Min}\,\{M_{jjk}\,M_{iki}|1\leqq k\leqq n\} \\ &=Y_{ji}. \end{aligned}$$

Thus $e_jNe_i/e_jN^2e_i\cong\Lambda_{ji}/(B_{ji}+\pi R)=R/\pi R\neq 0$. Hence $i\rightarrow j\in Q(A)_1$.
$(\Longleftarrow)$ Case 1. $i\in\mathcal{D}$ and $j\in\mathcal{R}$.

It follows from Lemma 2.2 that $M_{jjk}=M_{iki}=\pi R$ for all $k=1,\cdots,n$. Hence $X_{ji}=\pi R<\pi^2R=Y_{ji}$. Therefore $M_i\rightsquigarrow M_j$.

Case 2. There is an arrow from i to j in $Q(A)$.

It holds that $i\neq j$, $\Lambda_{ji}/(B_{ji}+\pi R)\neq 0$, so that $\Lambda_{ji}=R$ and $B_{ji}\geqq\pi R$. Since $i\neq j$, $M_{iji}=\Lambda_{ji}=M_{jji}$. Hence $X_{ji}=\Lambda_{ji}=R$. By (iii), $Y_{ji}=B_{ji}\geqq\pi R>R=X_{ji}$. Therefore $M_i\rightsquigarrow M_j$.

REMARKS. (1) We note the following fact that is shown in the proof; If $M_i\rightsquigarrow M_j$ and $M_i\cap M_j=(X_{kh})$, then

$$\begin{cases} X_{ji}=\pi R \Longleftrightarrow i\in\mathscr{D} \text{ and } j\in\mathscr{R}, \\ X_{ji}=R \Longleftrightarrow i\rightarrow j\in Q(A)_1 . \end{cases}$$

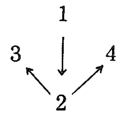
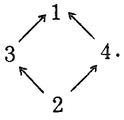
(2) In our vein, we note that the link graph of maximal ideals of Λ is connected.

(3) A maximal ideal M_i of Λ is not idempotent if and only if i is an isolated vertex in $Q(A)$.

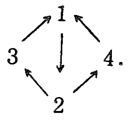
(4) $Q(A)$ has full information about Λ. As for the link graph, there are tiled R-orders Λ_1, Λ_2 with the same link graph, but $Q(A_1)$ is different from $Q(A_2)$ where $A_i=\Lambda_i/(\pi R)_n$ $(i=1,2)$.

EXAMPLE 2.3. Let

$$\Lambda_1=\begin{pmatrix} R & \pi R & \pi R & \pi R \\ R & R & \pi R & \pi R \\ R & R & R & \pi R \\ R & R & \pi R & R \end{pmatrix} \quad \text{and} \quad \Lambda_2=\begin{pmatrix} R & R & R & R \\ \pi R & R & \pi R & \pi R \\ \pi R & R & R & \pi R \\ \pi R & R & \pi R & R \end{pmatrix}.$$

Then $Q(A_1)=$  and $Q(A_2)=$ .

It follows from the theorem that the link graphs of Λ_1 and Λ_2 are given by



PROPOSITION 2.4. *Let Λ_1, Λ_2 be the tiled R-orders between $(R)_n$ and $(\pi R)_n$ with the same link graph and put $A_i=\Lambda_i/(\pi R)_n$ $(i=1,2)$. Then $Q(A_1)$ is connected if and only if $Q(A_2)$ is connected.*

PROOF. Suppose that $Q(A_1)$ is disconnected and put $Q(A_1)=\mathcal{U}\dot{\cup}\mathcal{V}$ (disjoint union) with $\mathcal{U}$ connected.

If $\mathcal{U}_0=\{m\}$, then the link graph has a loop on m. Assume that $Q(A_2)$ is connected. Then there exist a non-domain (or non-range) vertex v in $\mathcal{V}$ and an arrow $v\rightarrow m$ (or $m\rightarrow v$) in $Q(A_2)$. Hence the link graph of Λ_2 has no loops on m, a contradiction. Therefore $Q(A_2)$ is disconnected.

Let $\mathcal{U}_0$ be not a singleton and assume that $Q(A_2)$ is connected. Then (i) there exist a source s_0 in $\mathcal{U}$ and a non-domain d in $\mathcal{V}$ with $d\xrightarrow{\alpha} s_0\in Q(A_2)$, or (ii) there exist a sink s_i in $\mathcal{U}$ and a non-range r in $\mathcal{V}$ with $s_i\xrightarrow{\alpha} r\in Q(A_2)$.

In the case (i), since $Q(A_2)$ is connected, there exist a sink x and a path $s_0\to\cdots\to x$ in $Q(A_2)$. Assume $x\in\mathcal{V}_0$. Then there is an arrow β in the path from s_0 to x such that $d(\beta)\in\mathcal{U}_0$ and $r(\beta)\in\mathcal{V}_0$. Then $d(\beta)$ is a sink in $\mathcal{U}$. Since s_0 is a range of α in $Q(A_2)$, there must exist an arrow from s to s_0 in $Q(A_2)$ for each sink s in $\mathcal{U}$. Hence $Q(A_2)$ has an oriented cycle $\overleftarrow{s_0\to\cdots\to d(\beta)}$, a contradiction. Therefore $x\in\mathcal{U}_0$. Similarly, there exist a source y and a path $y\to\cdots\to d$ in $Q(A_2)$ with $y\in\mathcal{V}_0$. Hence there is an arrow $x\to y$ in the link graph of Λ_2. But, since x is not a sink in $\mathcal{U}$, the link graph of Λ_1 has no arrows from x to y, a contradiction. Similarly, we can deduce a contradiction in the case (ii). Therefore $Q(A_2)$ is disconnected.

3. Appendix: Global dimension of some special Λ

Let Λ be a basic tiled R-order between $(R)_n$ and $(\pi R)_n$ and put $A=\Lambda/(\pi R)_n$. Before attacking some special cases, we shall note the following proposition whose proof mainly depends on the infinite global dimensional criterion given by V.A. Jategaonkar [4].

Proposition 3.1. *If Λ has finite global dimension, then the quiver $Q(A)$ is connected.*

Proof. Suppose that $Q(A)$ is disconnected. Since there is a permutation matrix $u\in(R)_n$ such that

$$\Lambda\cong u\Lambda u^{-1}=\begin{pmatrix} R & & * & \pi R & \cdots & \pi R \\ & \ddots & & \vdots & & \vdots \\ * & & R & \pi R & \cdots & \pi R \\ \pi R & \cdots & \pi R & R & & * \\ \vdots & & \vdots & & \ddots & \\ \pi R & \cdots & \pi R & * & & R \end{pmatrix}(s,$$

we may assume that $\Lambda=u\Lambda u^{-1}$. Put $M=(R,\cdots,R)=R^n$ and

$$M\supset B_1=(\ R,\ \cdots,\ \overset{\overset{s}{\smile}}{R},\ \pi R,\ \cdots,\ \pi R)\supset\pi M,$$

$$M\supset B_2=(\pi R,\ \cdots,\ \pi R,\ \ R,\ \cdots,\ \ R)\supset\pi M.$$

Then B_1, B_2, M are right Λ-modules and $M/\pi M\cong B_1/\pi M\oplus B_2/\pi M$ as right $\Lambda/\pi\Lambda$-modules. Thus it follows from [4, Lemma 1.7] that proj. $\dim(M_\Lambda)=\infty$, so that gl. $\dim\Lambda=\infty$.

Proposition 3.2. *If $Q(A)$ is a tree, then* gl. $\dim\Lambda\leqq 3$.

PROOF. Put $i^{-}=\{j\in Q(A)_0 | d(\alpha)=j,\ r(\alpha)=i$ for some $\alpha\in Q(A)_1\}$, $\mathcal{S}=\{j\in Q(A)_0 | j$ is a sink in $Q(A)\}$ and $L=(R,\cdots,R)=R^n$.

Case 1 (i) $i^{-}=\phi$, $\mathcal{S}$ is a singleton.

Let $\mathcal{S}=\{j\}$. Then $P_j=L$. Since $J_i=(\pi R,\cdots,\pi R)=\pi L\cong L$, proj. $\dim(J_i)=0$.

Case 1 (ii) $i^{-}=\phi$, $\mathcal{S}=\{s_1,\cdots,s_t\}$ $(t\geqq 2)$.

For a subset $\mathcal{X}$ of $Q(A)_0$, put $\mathscr{P}re(\mathcal{X})=\{j\in Q(A)_0|$ there is a path $j\to\cdots\to x$ in $Q(A)$ or $j=x$ for some $x\in\mathcal{X}\}$. Pick up a sink s_1. Since $Q(A)$ is connected, there exist a sink s_2 outside $\mathscr{P}re(s_1)$ and $j_1\in\mathscr{P}re(s_1)$ such that there is a path $j_1\to\cdots\to s_2$ in $Q(A)$ which branches at the vertex j_1. Since $Q(A)$ has no cycles, such j_1 is unique. If $\mathscr{P}re(s_1, s_2)\neq Q(A)_0$, repeat the above procedure. After some repetitions, we reach $Q(A)_0=\mathscr{P}re(s_1,\cdots,s_t)$ with vertices $j_1,\cdots,j_{t-1}$. Then using canonical maps, we obtain short exact sequences

$$0\longrightarrow P_{j_1}\longrightarrow P_{s_1}\oplus P_{s_2}\longrightarrow M_2\longrightarrow 0,$$

$$0\longrightarrow P_{j_2}\longrightarrow M_2\oplus P_{s_3}\longrightarrow M_3\longrightarrow 0,$$

$$\vdots$$

$$0\longrightarrow P_{j_{t-1}}\longrightarrow M_{t-1}\oplus P_{s_t}\longrightarrow M_t\longrightarrow 0.$$

Then proj. $\dim(M_t)=$proj. $\dim(M_{t-1})=\cdots=$proj. $\dim(M_2)=1$ and $M_t=L$. Since $J_i=\pi L\cong L$, proj. $\dim(J_i)=1$.

Case 2 i^{-} is a singleton.

Let $i^{-}=\{j\}$. Then $J_i\cong P_j$. Hence proj. $\dim(J_i)=0$.

Case 3 (i) $i^{-}=\{j_1,\cdots,j_u\}$ $(u\geqq 2)$, $\mathcal{S}$ is a singleton.

Using canonical maps, we obtain short exact sequences

$$0\longrightarrow L\longrightarrow P_{j_1}\oplus P_{j_2}\longrightarrow N_2\longrightarrow 0,$$

$$0\longrightarrow L\longrightarrow N_2\oplus P_{j_3}\longrightarrow N_3\longrightarrow 0,$$

$$\vdots$$

$$0\longrightarrow L\longrightarrow N_{u-1}\oplus P_{j_u}\longrightarrow N_u\longrightarrow 0.$$

Then $N_u=J_i$. Hence proj. $\dim(J_i)=$proj. $\dim(L)+1=1$.

Case 3 (ii) $i^{-}=\{j_1,\cdots,j_u\}$, $\mathcal{S}=\{s_1,\cdots,s_t\}$ $(u,\ t\geqq 2)$.

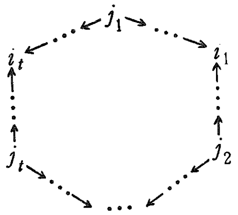
As in Case 3 (i), proj. $\dim(J_i)=$proj. $\dim(L)+1=2$ from Case 1 (ii).
Therefore gl. $\dim\Lambda=\sup\{$proj. $\dim(J_i)|1\leqq i\leqq n\}+1\leqq 3$.

REMARK. It follows from the proof that gl. $\dim\Lambda\leqq 2$ iff $Q(A)$ has a unique source or a unique sink. This is a special case of [8, Theorem].

EXAMPLE 3.3. Let $Q(A)$ be of A_n-type and put m be the number of vertices at which directions of arrows are changed. Then if $m=0,\ 1, \geqq 2$, then gl. dim $\Lambda=$ 1, 2, 3, respectively.

PROOF. This follows from the proof of Prop. 3.2.

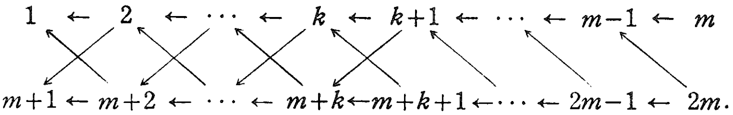
EXAMPLE 3.4. Let $Q(A)$ be a cycle



If $t=1$, $\geqq 2$, then gl. dim $\Lambda=2$, ∞, respectively. It follows from Remark (3) in § 2 that the maximal ideals of Λ are idempotent, while gl. dim $\Lambda=\infty$ if $t\geqq 2$.

In [1] and [7], there is an example which is a tiled R-order between $(R)_n$ and $(\pi R)_n$ with enough large global dimension. If $n=2^m$, then its global dimension is m. Next one is such an example with smaller n. Calculations of Examples 3.4 and 3.5 are left to the reader.

EXAMPLE 3.5. For $m\geqq 2$ and $0\leqq k\leqq m-1$, let $Q(A_k)$ be



Let Λ_k be the tiled R-order between $(R)_n$ and $(\pi R)_n$ such that $A_k=\Lambda_k/(\pi R)_n$ where $n=2m$. Then $\Lambda_0 \subset \Lambda_1 \subset \cdots \Lambda_{m-2} \subset \Lambda_{m-1}$ and gl. dim $\Lambda_k=k+3$ for $0\leqq k\leqq m-2$ and gl. dim $\Lambda_{m-1}=\infty$.

If $m=2(3)$ and $k=0(1)$, then $n=4(6)$ and gl. dim $\Lambda_k=3(4)$. It is verified by computation that 4(6) is the smallest n with global dimension 3(4).

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