TSUKUBA J. MATH.
Vol. 5 No. 1 (1981). 47–65

A GAMMA RING WITH MINIMUM CONDITIONS

By

Shoji KYUNO

Abstract. The aim of this note is to study the structure of a Γ-ring (not in the sense of Nobusawa) with minimum conditions. By ring theoretical techniques, we obtain various properties on the semi-prime Γ-ring and generalize Nobusawa's result which corresponds to the Wedderburn-Artin Theorem in ring theory. Using these results, we have that a Γ-ring with minimum right and left conditions is homomorphic onto the Γ_0-ring $\sum_{i=1}^{q} D_{n(i),m(i)}^{(i)}$, where $D_{n(i),m(i)}^{(i)}$ is the additive abelian group of the all rectangular matrices of type $n(i)\times m(i)$ over some division ring $D^{(i)}$, and Γ_0 is a subdirect sum of the Γ_i, $1\leqq i\leqq q$, which is a non-zero subgroup of $D_{m(i),n(i)}^{(i)}$ of type $m(i)\times n(i)$ over $D^{(i)}$.

1. Introduction.

Nobusawa [8] introduced the notion of a Γ-ring M as follows: Let M and Γ be additive abelian groups. If for all $a, b, c\in M$ and $\alpha, \beta, \gamma\in\Gamma$, the conditions

N_1. $a\alpha b\in M$, $\alpha a\beta\in\Gamma$

N_2. $(a+b)\alpha c=a\alpha c+b\alpha c$, $a(\alpha+\beta)b=a\alpha b+a\beta b$, $a\alpha(b+c)=a\alpha b+a\alpha c$

N_3. $(a\alpha b)\beta c=a(\alpha b\beta)c=a\alpha(b\beta c)$

N_4. $x\gamma y=0$ for all $x, y\in M$ implies $\gamma=0$,

are satisfied, then M is called a *Γ-ring*.

Barnes [1] weakened slightly defining conditions and gave the definition as follows:

If these conditions are weakened to

B_1. $a\alpha b\in M$

B_2. same as N_2

B_3. $(a\alpha b)\beta c=a\alpha(b\beta c)$,

then M is called a *Γ-ring*.

AMS (MOS) subject classifications (1970). Primary 16A78, 16A48, 16A46 and 16A12.
Received July 30, 1980.

In this paper, the former is called a *Γ-ring in the sense of Nobusawa* and the latter merely a *Γ-ring.*

Nobusawa [8] determined the structures of simple and semi-simple Γ-rings in the sense of Nobusawa with minimum right and left conditions as follows:

Using the notation introduced in [5], when M is simple, as a ring,

$$\begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} \cong \begin{pmatrix} D_m & D_{m,n} \\ D_{n,m} & D_n \end{pmatrix}$$

where D is a division ring ([8] Theorem 2); when M is semi-simple, as a ring,

$$\begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} \cong \sum_{i=1}^{q} \begin{pmatrix} D_{m(i)}^{(i)} & D_{m(i),n(i)}^{(i)} \\ D_{n(i),m(i)}^{(i)} & D_{n(i)}^{(i)} \end{pmatrix}$$

where $D^{(i)}$, $1 \leqq i \leqq q$, are division rings ([8] Theorem 3).

Nobusawa's definitions are in the following: M is *simple* if $a\Gamma b=0$ implies $a=0$ or $b=0$; M is *semi-simple* if $a\Gamma a=0$ implies $a=0$.

In [2], we defined that a Γ-ring M is *prime* if for any ideal A and B of M, $A\Gamma B=0$ implies $A=0$ or $B=0$; a Γ-ring M is *semi-prime* if for any ideal A of M, $A\Gamma A=0$ implies $A=0$.

When M is a Γ-ring in the sense of Nobusawa, one can easily verify that M is prime if and only if $a\Gamma b=0$ implies $a=0$ or $b=0$; M is semi-prime if and only if $a\Gamma a=0$ implies $a=0$ ([1] Theorem 5). Thus, when M is a Γ-ring in the sense of Nobusawa, Nobusawa's terms 'simple' or 'semi-simple' are equivalent to our 'prime' or 'semi-prime' respectively.

However, when M is a Γ-ring (not in the sense of Nobusawa), they are quite different notations. Following Luh [7] we call a Γ-ring M is *completely prime* if $a\Gamma b=0$ implies $a=0$ or $b=0$; M is *completely semi-prime* if $a\Gamma a=0$ implies $a=0$. Then, the primeness cannot imply the completely primeness, even for a finite Γ-ring ([7] Example 3.1). The semi-prime Γ-ring is one without non-zero strongly-nilpotent ideal (Theorem 2.10 below), while the completely semi-prime Γ-ring is one without non-zero strongly-nilpotent element (Definition 2.2). The gap between the primeness and completely primeness and the gap between semi-primeness and completely semi-primeness are caused by lack of a multiplication: $\Gamma \times M \times \Gamma \rightarrow \Gamma$. In the following we do not treat completely prime Γ-rings nor completely semi-prime ones, but prime and semi-prime Γ-rings.

Also, it should be noticed that a semi-prime Γ-ring with minimum right condition cannot always have the minimum left condition, nor $\dim({}_L M)$ can be equal to $\dim(M_R)$ even if it has both minimum right and left conditions, while a semi-prime ring R (an ordinary ring) with minimum right condition has the minimum left condition, and $\dim({}_R R)=\dim(R_R)$ (The comments followed Theorem 3.23).

The main aims of this paper are to study the structure of the semi-prime Γ-ring with minimum right condition and to generalize Nobusawa's results to the prime and semi-prime Γ-rings with minimum conditions and to determine the structure of the Γ-ring with minimum conditions.

Using ring theoretical techniques, we obtain various fundamental results on Γ-rings with minimum right condition. Then, using these results, we have the analogues of the Wedderburn-Artin Theorem for simple (Definition 3.9 and Theorem 3.15 below) and semi-prime Γ-rings with minimum right and left conditions. Also, these converses are considered. Nobusawa's results are obtained as corollaries of our theorems. Consequently, the structure of a Γ-ring with minimum right and left conditions is determined.

For the following notions we refer to [2]: the right operator ring R, the left operator ring L, a right (left, two-sided) ideal of M, a principal ideal $\langle a\rangle$, $[N, \Phi]$, where $N\subseteqq M$ and $\Phi\subseteqq\Gamma$, but for the prime radical $\mathscr{P}(M)$, a residue class Γ-ring, and the natural homomorphism to [3].

2. Strongly-nilpotent ideals.

DEFINITION 2.1. Let M be a Γ-ring and L be the left operator ring. Let S be a non-empty subset of M and denote $S_l=\{a\in L\,|\,aS=0\}$. Then S_l is a left ideal of L, called an *annihilator left ideal*. Let T be a non-empty subset of L and denote $T_r=\{x\in M\,|\,Tx=0\}$. Then T_r is a right ideal of M, called an *annihilator right ideal*. For singleton subsets we abbreviate this notation, for example, $\{a\}_r=a_r$, where a is an element of L.

DEFINITION 2.2. An element x of a Γ-ring M is *nilpotent* if for any $\gamma\in\Gamma$ there exists a positive integer $n=n(\gamma)$ such that $(x\gamma)^n x=(x\gamma)(x\gamma)\cdots(x\gamma)x=0$. A subset S of M is *nil* if each element of S is nilpotent. An element x of a Γ-ring M is *strongly-nilpotent* if there exists a positive integer n such that $(x\Gamma)^n x=(x\Gamma x\Gamma\cdots x\Gamma)x=0$. A subset of M is *strongly-nil* if each its element is strongly-nilpotent. S is *strongly-nilpotent* if there exists a positive integer n such that $(S\Gamma)^n S=(S\Gamma S\Gamma\cdots S\Gamma)S=0$.

By definitions for a subset S of M we have the following diagram of implication:

S is strongly-nilpotent. $\Rightarrow$ S is strongly-nil. $\Rightarrow$ S is nil.

LEMMA 2.3. *The sum of a finite number of strongly-nilpotent right (left) ideals of a Γ-ring M is a strongly-nilpotent right (left) ideal.*

PROOF. The proof needs only be given for two strongly-nilpotent right ideals A, B. Suppose $(A\Gamma)^m A=(B\Gamma)^n B=0$. Now we have $((A+B)\Gamma)^{m+n+1}(A+B)=(A+B)\Gamma(A+B)\Gamma\cdots\Gamma(A+B)$, with $m+n+2$ brackets, so that $((A+B)\Gamma)^{m+n+1}(A+B)$ is a sum of terms, each consisting of $m+n+2$ factors which are either A or B. Such a term T contains either $m+1$ factors A or $n+1$ factors B. In the former case, $T\subseteqq(A\Gamma)^m A$ or $T\subseteqq M\Gamma(A\Gamma)^m A$, because A is a right ideal; in the latter case, $T\subseteqq M\Gamma(B\Gamma)^n B$ or $T\subseteqq(B\Gamma)^n B$. Thus, $((A+B)\Gamma)^{m+n+1}(A+B)=0$ and $A+B$ is strongly-nilpotent.

COROLLARY 2.4. *The sum of any set of strongly-nilpotent right (left) ideals of a Γ-ring M is a strongly-nil right (left) ideal.*

PROOF. Each element x of the sum is in a finite sum of strongly-nilpotent right ideals of M, which by Lemma 2.3 is strongly-nilpotent. Therefore x is strongly-nilpotent, and the sum is strongly-nil.

LEMMA 2.5. *The sum $\mathcal{S}(M)'$ of all strongly-nilpotent right ideals of a Γ-ring M coincides with the sum $'\mathcal{S}(M)$ of all strongly-nilpotent left ideals and with the sum $\mathcal{S}(M)$ of all strongly-nilpotent ideals.*

PROOF. Let I be a strongly-nilpotent right ideal. The ideal $I+M\Gamma I$ is strongly-nilpotent, because $((I+M\Gamma I)\Gamma)^n(I+M\Gamma I)\subseteqq(I\Gamma)^n I+M\Gamma(I\Gamma)^n I$ for $n=1, 2, \cdots$. It follows $I\subseteqq\mathcal{S}(M)$ and hence that $\mathcal{S}(M)'\subseteqq\mathcal{S}(M)$. But $\mathcal{S}(M)\subseteqq\mathcal{S}(M)'$ trivially, and hence $\mathcal{S}(M)=\mathcal{S}(M)'$. Similarly, $\mathcal{S}(M)='\mathcal{S}(M)$.

When a Γ-ring M has the descending (or ascending) chain condition for right ideals, it is abbreviated to M has *min-r condition* (or *max-r condition*). The terms *min-l condition* or *max-l condition* on a Γ-ring M are likewise defined.

It is natural to ask whether $\mathcal{S}(M)$ is strongly-nilpotent. This is so when M has either the min-r or max-r conditions (min-l or max-l also serve). The case of max-r is trivial, because $\mathcal{S}(M)$ is a finite sum of strongly-nilpotent right ideals. When M has min-r condition, a strongly-nil right ideal is always strongly-nilpotent, which will be shown in the following theorem. We note that a non-strongly-nilpotent right ideal means the right ideal which is not strongly-nilpotent.

THEOREM 2.6. *Any non-strongly-nilpotent right ideal of a Γ-ring M with min-r condition contains an idempotent element.*

PROOF. Let I be a non-strongly-nilpotent right ideal of M and I_1 be minimal in the set of non-strongly-nilpotent right ideals which are contained in I. Then, $I_1=I_1\Gamma I_1$, since $I_1\Gamma I_1$ is not strongly-nilpotent. Let $\mathcal{S}$ be the set of right ideals S with properties (1) $S\Gamma I_1\neq 0$ and (2) $S\subseteqq I_1$.

The set $\mathcal{S}$ is not empty ($I_1\in\mathcal{S}$) and we suppose that S_1 is a minimal member of $\mathcal{S}$. Let $s\in S_1$, $\delta\in\Gamma$ with $s\delta I_1\neq 0$. Then, $s\delta I_1=S_1$, because $s\delta I_1\in\mathcal{S}$. It follows that $a\in I_1$ exists with $s\delta a=s$. Then a is not nilpotent, because if a is nilpotent, $s=s\delta a=s\delta a\delta a=\cdots=(s\delta)(a\delta)\cdots(a\delta)a=0$, a contradiction. Hence, I cannot be a nil right ideal. This proves that if I is a strongly-nil right ideal then I is strongly-nilpotent, since if I is strongly-nil then I is nil.

Now $a\Gamma M\subseteqq I_1$ and $a\Gamma M$ is not strongly-nilpotent, for a is not nilpotent. Hence $a\Gamma M=I_1$, because of the minimal property of I_1. Likewise, $a\Gamma a\Gamma M=I_1$ and hence $a\in a\Gamma a\Gamma M$, so that $a=a\omega a_1$, where $a_1\in a\Gamma M$. Note that $a\omega(a_1-a_1\omega a_1)=0$ and hence $a_1-a_1\omega a_1\in[a, \omega]_r\cap a\Gamma M$. Set $a_2=a+a_1-a_1\omega a$. Then, $a\omega a_2=a\omega a+a\omega a_1-(a\omega a_1)\omega a=a\omega a+a-a\omega a=a$. Also, $a_2\omega(a_1-a_1\omega a_1)=(a+a_1-a_1\omega a)\omega(a_1-a_1\omega a_1)=a_1\omega a_1-a_1\omega a_1\omega a_1$. Moreover, a_2 is not nilpotent, because $a\omega a_2=a$ and a is not zero. It follows that $a\Gamma M=a_2\Gamma M$, and that $[a_2, \omega]_r\cap a\Gamma M\subseteqq[a, \omega]_r\cap a\Gamma M$. However, either $a_1\omega a_1=a_1\omega a_1\omega a_1$, in which case I contains the idempotent $a_1\omega a_1$, or else $a_1\omega a_1\neq a_1\omega a_1\omega a_1$, in which case $a_1-a_1\omega a_1\in[a, \omega]_r$ and $a_1-a_1\omega a_1\notin[a_2, \omega]_r$. In the latter case, $[a_2, \omega]_r\cap a\Gamma M\subsetneqq[a, \omega]_r\cap a\Gamma M$. This process is repeated, if necessary, beginning with a_2 instead of a, and obtaining a_4; etc. The process ceases because of the minimum condition and this proves that I has an idempotent element.

COROLLARY 2.7. *The sum $\mathcal{S}(M)$ of all strongly-nilpotent ideals of the Γ-ring M with min-r or max-r conditions, is a strongly nilpotent ideal.*

DEFINITION 2.8. When the sum $\mathcal{S}(M)$ of all strongly-nilpotent ideals of M is strongly-nilpotent, $\mathcal{S}(M)$ is called the *Wedderburn radical* of M (or the *strongly-nilpotent radical*) and denoted by W.

DEFINITION 2.9. A Γ-ring M is *semi-prime* if, for any ideal U of M, $U\Gamma U=0$ implies $U=0$.

For a semi-prime Γ-ring we have the following theorem.

THEOREM 2.10. ([3] Theorem 1, 2 and 3). *If M is a Γ-ring, the following conditions are equivalent:*

(1) *M is semi-prime,*

(2) *If $a\in M$ and $a\Gamma M\Gamma a=0$, then $a=0$,*

(3) *If* $\langle a\rangle$ *is a principal ideal of* M *such that* $\langle a\rangle\Gamma\langle a\rangle=0$, *then* $a=0$,

(4) *If* U *is a right ideal of* M *such that* $U\Gamma U=0$, *then* $U=0$,

(5) *If* V *is a left ideal of* M *such that* $V\Gamma V=0$, *then* $V=0$,

(6) *The prime radical of* M, $\mathcal{P}(M)$, *is zero,*

(7) M *contains no non-zero strongly-nilpotent ideals (right ideals, left ideals),*

(8) *The sum* $\mathcal{S}(M)$ *of all strongly-nilpotent ideals of* M *is zero.*

THEOREM 2.11. *Let* M *be a* Γ*-ring which has a Wedderburn radical* W. *Then the residue class* Γ*-ring* M/W *is semi-prime.*

PROOF. Set $\bar{M}=M/W$ and suppose $\bar{N}$ is a strongly-nilpotent ideal of $\bar{M}$, and suppose that $(\bar{N}\Gamma)^m\bar{N}=\bar{O}$. Let N be the inverse image of $\bar{N}$ under the natural homomorphism $M\to\bar{M}$. Thus, $N=\{x\in M\,|\,x+W\in\bar{N}\}$. Clearly, $(N\Gamma)^mN\subseteq W$ and hence $(N\Gamma)^{mn+m+n}N=0$, where $(W\Gamma)^nW=0$. Thus, $N\subseteq W$ and $\bar{N}=\bar{O}$. Hence, $\bar{M}$ is semi-prime.

If M has min-r condition, then M/W has min-r condition ([3] Lemma 1), Corollary 2.7 and Theorem 2.11 yield the following theorem.

THEOREM 2.12. *Let* M *be a* Γ*-ring with min-r condition. Then the residue class* Γ*-ring* $M/\mathcal{S}(M)$ *is a semi-prime* Γ*-ring with min-r condition, where* $\mathcal{S}(M)$ *is the sum of all strongly-nilpotent ideals of* M.

3. Semi-prime Γ-rings with min-r condition.

For a right ideal I of a Γ-ring M, if there exists an idempotent element l of the left operator ring L such that $I=lM$, we say that I has the *idempotent generator* l. The idempotent generator plays an important role in the following.

THEOREM 3.1. *Any non-zero right ideal in a semi-prime* Γ*-ring* M *with min-r condition has an idempotent generator.*

PROOF. The result is first proved when the ideal is a minimal right ideal A. Since M is semi-prime, $A\Gamma A\neq 0$. Then, there exist $\delta\in\Gamma$, $a\in A$ such that $a\delta A=A$. Thus, there exists $e\in A$ such that $a=a\delta e$. Then, $e=e\delta e$, since from $a=a\delta e=(a\delta e)\delta e$ we have $a\delta(e-e\delta e)=0$ which means $e-e\delta e=0$, for the set $B=\{c\in A\,|\,a\delta c=0\}$ is a right ideal contained properly in the minimal right ideal A and is (0). Since $e\in A$, $0\neq e\delta M\subseteq A$ and hence $e\delta M=A$, where $[e,\delta]$ is an idempotent of L.

Let I be any non-zero right ideal of M. Since I contains one or more minimal right ideals, idempotent generators of the minimal right ideal(s) exist in $[I,\Gamma]$. Choose an idempotent $l\in[I,\Gamma]$ such that $l_r\cap I$ is as small as possible.

If $l_r \cap I \neq 0$, then $l_r \cap I \supseteqq l'M$, where l' is an idempotent of L. Then, $l' \in l'L = l'[M, \Gamma] \subseteqq [I, \Gamma]$ and $ll'=0$, for since $l'M \subseteqq l_r$, $ll'M=0$. Set $m=l+l'-l'l$ and then $m \in [I, \Gamma]$, for $[I, \Gamma]$ is an right ideal of L. Clearly, $m^2=m$, because $ll'=0$. Moreover, $m_r \cap I \subsetneqq l_r \cap I$, since we have $lm=l$ which implies $m_r \subseteqq l_r$, and $ll'=0$ but $ml'=l' \neq 0$ which implies $l'M \subseteqq l_r$ but $l'M \nsubseteq m_r$. This contradicts the minimality of $l_r \cap I$ and the contradiction arises from taking $l_r \cap I \neq 0$. Hence one has $l_r \cap I=0$. Now let $x \in I$, then $l(x-lx)=0$, where $x-lx \in I$, for $lx \in I\Gamma I \subseteqq I$. It follows that $I=lM$, for since $l \in [I, \Gamma]$, $lM \subseteqq I\Gamma M \subseteqq I$.

COROLLARY 3.2. *A semi-prime Γ-ring M with min-r condition has max-r condition.*

PROOF. The proof is analogous to that in ring theory but to tackle the situation that the generator does not exist in M but in $L=[M, \Gamma]$. For the sake of completeness, we write out it.

Suppose that the non-empty set S of some right ideals in M has no maximal elements. Take an element J_1 of S, then by the assumption there exists $J_2 \in S$ such that $J_1 \subsetneqq J_2$. Repeating this process, we have an infinite sequence of right ideals:

$$J_1 \subsetneqq J_2 \subsetneqq \cdots \subsetneqq J_n \subsetneqq \cdots .$$

Set $N=\cup_i J_i$. Then, by Theorem 3.1 $N=lM$, where l is an idempotent of L. Thus, $l=l^2 \in lL=l[M, \Gamma]=[N, \Gamma]=[\cup_i J_i, \Gamma]$ and hence there exists an integer m such that $l \in [J_m, \Gamma]$. Then, $N=lM \subseteqq J_m \Gamma M \subseteqq J_m$, so that $J_m=N=J_{m+1}$, a contradiction. Hence, every non-empty set of right ideals of M has a maximal element. Evidently, the max-r condition holds in M.

LEMMA 3.3. *If a Γ-ring M is semi-prime, then the right operator R and the left operator L are semi-prime.*

PROOF. Suppose $rRr=0$. Then $Mr\Gamma Mr=0$. Theorem 2.10 (5) implies $Mr=0$ and then $r=0$. Thus, R is semi-prime. Similarly, it may be verified that L is semi-prime.

THEOREM 3.4. *Let T be any non-zero ideal of semi-prime Γ-ring M with min-r condition. Then T has a unique idempotent generator.*

PROOF. Let $T=sM$, where $s=\sum_i [e_i, \delta_i]$ is an idempotent, be the given ideal. Then $s_l=T_l$ is a left ideal of the left operator ring L and $T_l \cap [T, \Gamma]=0$, because $(T_l \cap [T, \Gamma])^2 \subseteqq T_l[T, \Gamma]=0$ and L is semi-prime (Lemma 3.3). Hence

$s_l \cap [T, \Gamma]=0$. But for any $\sum_i [x_i, \gamma_i] \in [T, \Gamma]$ $(\sum_i [x_i, \gamma_i] - \sum_i [x_i, \gamma_i] s)s=0$ and hence $\sum_i [x_i, \gamma_i] - \sum_i [x_i, \gamma_i] s \in s_l \cap [T, \Gamma]$, which means that $\sum_i [x_i, \gamma_i] = \sum_i [x_i, \gamma_i] s$. It follows that $[T, \Gamma]=[T, \Gamma]s=sM\Gamma s$ and s is a two-sided identity for the ring $[T, \Gamma]$. The latter fact shows that s is unique.

DEFINITION 3.5. Let M be a Γ-ring and L be the left operator ring. If there exists an element $\sum_i [e_i, \delta_i] \in L$ such that $\sum_i e_i \delta_i x = x$ for every element x of M, then it is called that M has the *left unity* $\sum_i [e_i, \delta_i]$.

It can be verified easily that $\sum_i [e_i, \delta_i]$ is the unity of L. Similarly we can define the *right unity* which is the unity of the right operator ring R.

COROLLARY 3.6. *A semi-prime Γ-ring M with min-r condition has a left unity.*

PROOF. In Theorem 3.4 set $T=M$. Then, $L=[M, \Gamma]=sM\Gamma s$. Thus, s is the unity of L. Then for any x of M $[sx-x, \Gamma]=0$ and so $(sx-x)\Gamma M\Gamma(sx-x)=0$. Since M is semi-prime $sx-x=0$ or $sx=x$.

By symmetry we have

COROLLARY 3.7. *A semi-prime Γ-ring M with min-l condition has a right unity.*

COROLLARY 3.8. *Let T be any non-zero ideal of a semi-prime Γ-ring M with min-r condition. Then, the generating idempotent of T is the idempotent which lies in the center of L.*

PROOF. Let $T=(\sum_i [e_i, \delta_i])M$ and suppose the $l \in L$. Since $(\sum_i [e_i, \delta_i])l \in [T, \Gamma]$, we have $(\sum_i [e_i, \delta_i])l=((\sum_i [e_i, \delta_i])l)\sum_i [e_i, \delta_i]=\sum_i [e_i, \delta_i](l\sum_i [e_i, \delta_i])= l\sum_i [e_i, \delta_i]$, for $l\sum_i [e_i, \delta_i] \in L[T, \Gamma]=[M\Gamma T, \Gamma] \subseteqq [T, \Gamma]$. Thus, $\sum_i [e_i, \delta_i]$ is central in L.

DEFINITION 3.9. A Γ-ring M is said to be *simple* if $M\Gamma M \neq 0$ and M has no ideals other than 0 and M.

COROLLARY 3.10. (1) *Any non-zero ideal T of a semi-prime Γ-ring M with min-r condition is a semi-prime Γ-ring with min-r condition.* (2) *Any minimal ideals S of a semi-prime Γ-ring M with min-r condition is a simple Γ-ring.*

PROOF of (1). Let J be a right ideal of T (considered as a Γ-ring) $(J\Gamma T \subseteqq J)$. Let $T=sM$, where $s=\sum_i [e_i, \delta_i]$ is an idempotent. Since $[J, \Gamma] \subseteqq [T, \Gamma]$ Theorem 3.4 implies $[J, \Gamma]s=[J, \Gamma]$. Thus, $J\Gamma M=([J, \Gamma]s)M=J\Gamma(sM)=J\Gamma T\subseteqq J$ and hence J is a right ideal of M. It is immediate that the Γ-ring T has no strongly-nilpotent right ideals and satisfies the min-r condition.

PROOF of (2). Let T be any non-zero ideal of M. Then, as shown in the proof of (1), a right ideal of T is a right ideal of M. Now, we show that a left ideal Q of T is a left ideal of M. Suppose that $T=sM$, where s is an idempotent. Then, $M\Gamma Q=[M, \Gamma]Q=[M, \Gamma](sQ)=([M, \Gamma]s)Q=(s[M, \Gamma])Q=[T, \Gamma]Q \subseteqq Q$. So Q is a left ideal of M. Therefore, an ideal of T is an ideal of M. Since S is a minimal ideal of M, we deduce that S is a simple Γ-ring.

THEOREM 3.11. *If T is an ideal in a semi-prime Γ-ring M with min-r condition, then $M=T\oplus[T, \Gamma]_r$. If $M=T\oplus K$, where K is an ideal of M, then $K=[T, \Gamma]_r$.*

PROOF. Suppose that $T=sM$, where $s=\sum_i[e_i, \delta_i]$ is an idempotent, then $M=sM\oplus(1_L-s)M$, where 1_L denotes the left unity of M. $[T, \Gamma](1_L-s)M=[T, \Gamma]s(1_L-s)M=[T, \Gamma](s-s)M=0$. Hence, $(1_L-s)M\subseteqq[T, \Gamma]_r$. Conversely, suppose that $[T, \Gamma]x=0$ and $x=x'+x''$, where $x'\in T$, $x''\in(1_L-s)M$. Then, $sx=sx'+sx''=sx'$ and $0=[T, \Gamma]x=([T, \Gamma]s)x=[T, \Gamma]sx'=[T, \Gamma]x'$. Since $T\Gamma M\subseteqq T$, $T\Gamma M\Gamma x'=0$ and hence $x'\Gamma M\Gamma x'=0$, which implies $x'=0$. Thus, $x=x''\in(1_L-s)M$ and then $[T, \Gamma]_r\subseteqq(1_L-s)M$. Hence $[T, \Gamma]_r=(1_L-s)M$ and $M=T\oplus[T, \Gamma]_r$.

In the case when $M=T\oplus K$, it follows that $T\Gamma K=0$ (since $T\Gamma K\subseteqq T\cap K$) and hence $K\subseteqq[T, \Gamma]_r$. However $T\oplus K=T\oplus[T, \Gamma]_r$ and hence $K=[T, \Gamma]_r$.

We now prove the fundamental theorem on semi-prime Γ-rings with min-r condition.

THEOREM 3.12. *A semi-prime Γ-ring M with min-r condition has only a finite number of minimal ideals and is their direct sum.*

PROOF. Form $M_1\oplus M_2\oplus\cdots\oplus M_t$ of minimal ideals M_i of M. Because M has the max-r condition (Corollary 3.2), there is a sum S having maximal length q. Suppose that $[S, \Gamma]_r\neq 0$. Then $[S, \Gamma]_r$ contains a minimal ideal, which can be added directly to S, because $S\cap[S, \Gamma]_r=0$. This contradicts our supposition that S has maximal length of minimal ideals. Hence $[S, \Gamma]_r=0$ and $M=S\oplus[S, \Gamma]_r=S$, which proves that M is a direct sum of minimal ideals, $M=M_1\oplus M_2\oplus\cdots\oplus M_q$, say.

By Corollary 3.10 and Theorem 3.12 we have

THEOREM 3.13. *A semi-prime Γ-ring with min-r condition is a direct sum of a finite number of simple Γ-rings with min-r condition.*

DEFINITION 3.14. A Γ-ring M is *prime* if for all pairs of ideals S and T of M, $S\Gamma T=0$ implies $S=0$ or $T=0$. A Γ-ring M is *left (right) primitive* if (i) the left (right) operator ring of M is a left (right) primitive ring, and (ii) $x\Gamma M=0$ ($M\Gamma x=0$) implies $x=0$. M is a *two-sided primitive* Γ-ring (or simply a *primitive* Γ-ring) if both left and right primitive.

Luh proved the following theorem.

THEOREM 3.15 ([7] Theorem 3.6). *For a Γ-ring M with min-l condition, the three conditions*

(1) *M is prime,*

(2) *M is primitive,*

(3) *M is simple*

are equivalent.

Of course, Theorem 3.15 also holds when M has min-r condition instead of min-l condition. Thus, we can replace the term 'simple' in Theorem 3.13 by 'prime' or 'primitive'.

We will prove further results on the one sided ideal structure of a semi-prime Γ-ring with min-r condition.

LEMMA 3.16. *Let I be a right ideal in a semi-prime Γ-ring M with min-r condition and J_1 be a right ideal contained in I. Then there exists a right ideal J_2 in I such that $I=J_1\oplus J_2$.*

PROOF. Taking $I\neq 0$, $J_1\neq 0$ and $I=lM$ and $J_1=sM$, where $l=\sum_i[e_i, \delta_i]$, $s=\sum_j[f_j, \varepsilon_j]$ are idempotents. Write $x\in I$ as $x=sx+(l-s)x$. The set $J_2=\{x-sx \mid x\in I\}$ is a right ideal and $J_2\subseteqq I$. Clearly, $I=J_1\oplus J_2$.

DEFINITION 3.17. Idempotents $l_1, \cdots, l_k\in L$ are *mutually orthogonal* if $l_il_j=0$ for $i\neq j$.

The notation $l=l_1\oplus\cdots\oplus l_k$ indicates that $l=l_1+\cdots+l_k$, where $l_1, \cdots, l_k$ are mutually orthogonal idempotents.

In Lemma 3.16 we can choose generating idempotents s_1 of J_1, s_2 of J_2, so that $l=s_1\oplus s_2$. The proof is given in the following.

Take $I=lM$ and $J_1=sM$ as before, and set $s_1=sl$ and $s_2=l-sl$. Then $ls=s$ since $s\in l[M, \Gamma]$, and $s=s^2=s(ls)=(sl)s=s_1s$ so that $J_1=sM=s_1(sM)\subseteqq s_1M=s(lM)$ $\subseteqq sM=J_1$. Thus, $J_1=s_1M$. However, $J_2=\{x-sx\mid x\in I\}=\{la-sla\mid a\in M\}=$ $\{(l-sl)a\mid a\in M\}=s_2M$. We can easily verify that s_1, s_2 are idempotents and that $l=s_1\oplus s_2$. Q. E. D.

DEFINITION 3.18. An idempotent of the left operator ring L is *primitive* if it cannot be written as a sum of two orthogonal idempotents.

Lemma 3.16 and subsequent comments imply that in a semi-prime Γ-ring with min-r condition an idempotent of L is primitive if and only if it generates a minimal right ideal.

LEMMA 3.19. *Let M be a semi-prime Γ-ring with min-r condition. Then any idempotent element l of the left operator ring L is a sum of mutually orthogonal primitive idempotents.*

PROOF. Let $I=lM$ and M_1 be a minimal right ideal in I. There exists a right ideal $M_1'\subseteqq I$ such that $I=M_1\oplus M_1'$ (by Lemma 3.16). Then, either $M_1'=0$, in which case l is primitive (l generates the minimal right ideal), or we choose generating idempotents s_1 of M_1; s_1' of M_1' such that $l=s_1\oplus s_1'$ (by the above comment). Observe that s_1 is a primitive idempotent. If s_1' is not primitive, this process may be applied to $M_1'=s_1'M$, giving $s_1'=s_2\oplus s_2'$, where s_2 is primitive. Evidently, $l=s_1\oplus s_2\oplus s_2'$, and $s_1'M\supsetneqq s_2'M$. This process is continued and the sequence $s_1'M\supsetneqq s_2'M\supsetneqq s_3'M\supsetneqq\cdots$ being strictly decreasing, must be stop after a finite number of terms. Then, $l=s_1\oplus\cdots\oplus s_k$, say, which each s_i is a primitive idempotent.

COROLLARY 3.20. *Any non-zero right ideal in a semi-prime Γ-ring M with min-r condition is a direct sum of minimal right ideals.*

PROOF. Lemma 3.19 implies that $I=lM=s_1M\oplus\cdots\oplus s_kM$.

By symmetry, we have

COROLLARY 3.21. *Any non-zero left ideal in a semi-prime Γ-ring with min-l condition is a direct sum of minimal left ideals.*

Luh proved the following theorem.

THEOREM 3.22 ([6] Theorem 3.6). *Let M be a semi-prime Γ-ring and L and R be respectively the left and right operator rings of M. If $e\delta e=e$, where $e\in M$, $\delta\in\Gamma$, then the following statements are equivalent:*

(1) *$M\delta e$ is a minimal left ideal of M,*

(2) *$e\delta M$ is a minimal right ideal of M,*

(3) *$[M, \Gamma][e, \delta]$ is a minimal left ideal of L,*

(4) *$[\delta, e][\Gamma, M]$ is a minimal right ideal of R,*

(5) *$[e, \delta][M, \Gamma]$ is a minimal right ideal of L,*

(6) *$[\Gamma, M][\delta, e]$ is a minimal left ideal of R,*

(7) *$[e, \delta][M, \Gamma][e, \delta]$ is a division ring,*

(8) *$[\delta, e][\Gamma, M][\delta, e]$ is a division ring.*

Moreover, the division rings $[e, \delta][M, \Gamma][e, \delta]$ and $[\delta, e][\Gamma, M][\delta, e]$ are isomorphic if any of the above statements occurs.

Corollary 3.20 showed that every non-zero right ideal of a semi-prime Γ-ring M is a direct sum of minimal right ideals. This decomposition applies to M itself and gives a right dimension number for M, considered as an R-module.

THEOREM 3.23. *Let M be a semi-prime Γ-ring with min-r condition and let $M=I_1\oplus\cdots\oplus I_m=J_1\oplus\cdots\oplus J_n$, where I_t, J_s are minimal right ideals. Then, $m=n$.*

The proof is established by the quite similar fashion to that for an ordinary ring and so we omit it.

The integer $m=n$ in Theorem 3.23 is called the *right demension* of the semi-prime Γ-ring with min-r condition and denoted by $\dim(M_R)$. One can define the *left dimension* of a Γ-ring in a similar manner. But it should be noticed that a semi-prime Γ-ring with min-r condition cannot always have the min-l condition. For example, let D be a division ring and M be the discrete direct sum of the division rings $D_i=D$, $i\in N$ (the set of all natural numbers), and Γ be the set of all transposed elements of M. Then, the Γ-ring M is semi-prime and $\dim({}_LM)=\infty$, while $\dim(M_R)=1$. Even for a semi-prime Γ-ring with both min-r and min-l conditions, generally the right dimension cannot be equal to the left one. When $M=D_{2,1}$, the set of all matrices of type 2×1 over a division ring D, and $\Gamma=D_{1,2}$, $\dim(M_R)=2$ and $\dim({}_LM)=1$.

When M is a semi-prime Γ-ring with min-r condition, we consider the left operator ring L. Corollary 3.6 shows M has the left unity. Thus, by Lemma

3.19, $1_L=[e_1, \delta_1]+\cdots+[e_k, \delta_k]$, where $[e_1, \delta_1], \cdots, [e_k, \delta_k]$ are mutually orthogonal primitive idempotents. This implies that $L=[e_1, \delta_1]L\oplus\cdots\oplus[e_k, \delta_k]L$, where $[e_1, \delta_1]L, \cdots, [e_k, \delta_k]L$ are minimal right ideals. Also, we have $L=L[e_1, \delta_1]\oplus\cdots\oplus L[e_k, \delta_k]$, where $L[e_1, \delta_1], \cdots, L[e_k, \delta_k]$ are minimal left ideals (Theorem 3.22). Thus, we have $\dim(L_L)=\dim({}_LL)$. By symmetry, when M is a semi-prime Γ-ring with min-l condition, for the right operator ring R we have $\dim({}_RR)=\dim(R_R)$.

4. Simple Γ-rings with min-r and min-l conditions.

We note that if a Γ-ring M is simple, then the right operator ring R and the left operator ring L are simple.

Let I be an ideal of R such that $0\subsetneqq I\subsetneqq R$. Then MI is an ideal of M. Since M is simple, MI must be 0 or M. If $MI=M$, then $R=[\Gamma, MI]=[\Gamma, M]I=RI\subseteqq I$, a contradiction. If $MI=0$, then $I=0$, also a contradiction. Thus, R has only ideals 0 and R, and $R^2\neq 0$, for $MR^2=M[\Gamma, M\Gamma M]=M[\Gamma, M]=M\Gamma M=M\neq 0$. This proves R is simple. Similarly, it may be shown that L is simple.

If M is simple, then M is semi-prime. Indeed, for any ideal U of M we assume $U\Gamma U=0$. Since only ideals of M are 0 and M, $U=0$ or $U=M$. If $U=M$, then $M\Gamma M=M\neq 0$, a contradiction. Thus, $U=0$ and M is semi-prime.

DEFINITION 4.1. If M_i is a Γ_i-ring for $i=1, 2$, then an ordered pair (θ, ϕ) of mappings is called a *homomorphism of M_1 onto M_2* if it satisfies the following properties:

(1) θ is a group homomorphism from M_1 onto M_2,

(2) ϕ is a group homomorphism from Γ_1 onto Γ_2,

(3) For every $x, y\in M_1$, $\gamma\in\Gamma_1$, $(x\gamma y)\theta=(x\theta)(\gamma\phi)(y\theta)$.

Furthermore, if both θ and ϕ are injections, then (θ, ϕ) is called an *isomorphism from the Γ_1-ring M_1 onto the Γ_2-ring M_2.*

THEOREM 4.2. *Let M be a simple Γ-ring with min-r and min-l conditions and $\Gamma_0=\Gamma/\kappa$, where $\kappa=\{\gamma\in\Gamma \mid M\gamma M=0\}$. Then, the Γ_0-ring M is isomorphic onto the Γ'-ring $D_{n,m}$, where $D_{n,m}$ is the additive abelian group of all rectangular matrices of type $n\times m$ over a division ring D, and Γ' is a non-zero subgroup of the additive abelian group $D_{m,n}$ of all rectangular matrices of type $m\times n$, and $m=\dim({}_LM)$ and $n=\dim(M_R)$.*

PROOF. Let $e\delta M$, where $e\delta e=e$, be a minimal right ideal of M (Theorem 3.1) and let $D=[e\delta M\Gamma e, \delta]$; certainly D is a division ring (Theorem 3.22). Also,

$[e\delta M, \Gamma]=e\delta L$ is a minimal right ideal of L (Theorem 3.22). Since $(e\delta M\Gamma e\delta)e\delta L=e\delta L$ (for $0\neq(e\delta M\Gamma e\delta)e\delta L$) we see that $e\delta L$ is a vector space over D (a left D-space).

First we prove:

$l_1, \cdots, l_n \in e\delta L$ are linearly independent over D if and only if

$Ll_1\oplus\cdots\oplus Ll_n$, where $L=[M, \Gamma]$. ....(A)

Suppose $Ll_1+\cdots+Ll_n$ is not direct sum. Then, there exist $a_1, \cdots, a_n\in L$, not all a_il_i zero, such that $a_1l_1+\cdots+a_nl_n=0$. Set $L_i=\{a\in L[e, \delta] \mid al_i\in Ll_1+\cdots+Ll_{i-1}+Ll_{i+1}+\cdots+Ll_n\}$, where we suppose that $a_il_i\neq 0$. Then, $0\neq a_i[e, \delta]\in L_i$ and $L_i=L[e, \delta]$, because $L[e, \delta]$ is a minimal left ideal (Theorem 3.22). Hence, $[e, \delta]\in L[e, \delta]=L_i$ and then $l_i=e\delta l_i=y_1l_1+\cdots+y_{i-1}l_{i-1}+y_{i+1}l_{i+1}+\cdots+y_nl_n$, where $y_j\in L$. Then, $l_i=(e\delta y_1e\delta)l_1+\cdots+(e\delta y_{i-1}e\delta)l_{i-1}+(e\delta y_{i+1}e\delta)l_{i+1}+\cdots+(e\delta y_ne\delta)l_n$, which means that $l_1, \cdots, l_n$ are linearly dependent over D.

Conversely, if $Ll_1+\cdots+Ll_n$ is a direct sum, then $(e\delta Le\delta)l_1+\cdots+(e\delta Le\delta)l_n$ is a direct sum, which means $l_1, \cdots, l_n$ are linearly independent over D. Q. E. D.

Next, we prove:

$a_1\delta_1L\oplus\cdots\oplus a_k\delta_kL$ if and only if $a_1\delta_1M\oplus\cdots\oplus a_k\delta_kM$. ....(B)

Suppose $a_1\delta_1M+\cdots+a_k\delta_kM$ is a direct sum. If $\sum_{i=1}^k l_i=0$ with $l_i\in a_i\delta_iL$, then $\sum_{i=1}^k l_ix=0$ for all $x\in M$, where $l_ix\in l_iM\subseteq[a_i\delta_iM, \Gamma]M\subseteq a_i\delta_iM$. Thus, $l_ix=0$ for all $x\in M$ and for all i. Hence, $l_i=0$ for every i.

Conversely, assume that $a_1\delta_1L+\cdots+a_k\delta_kL$ is a direct sum. If $\sum_{i=1}^k x_i=0$, with $x_i\in a_i\delta_iM$, then $\sum_{i=1}^k[x_i, \gamma]=0$ for all $\gamma\in\Gamma$, where $[x_i, \gamma]\in[x_i, \Gamma]\subseteq[a_i\delta_iM, \Gamma]=a_i\delta_iL$. It follows that $[x_i, \gamma]=0$ for every $\gamma\in\Gamma$ and every i, and $x_i\Gamma M\Gamma x_i=0$ for every i. Since M is semi-prime, $x_i=0$ for every i. Thus, $a_1\delta_1M+\cdots+a_k\delta_kM$ is a direct sum. Q. E. D.

Thus, by (A), the comment (followed Theorem 3.23) on the dimensions of L, (B) and Theorem 3.22, we have $\dim({}_D[e\delta M, \Gamma])=\dim({}_LL)=\dim(L_L)=\dim(M_R)$. Similarly, we can prove $\dim({}_De\delta M)=\dim({}_LM)=\dim({}_RR)=\dim(R_R)$.

For $a\in M$ define a mapping ρ_a of $[e\delta M, \Gamma]$ to $e\delta M$ by $[x, \gamma]\rho_a=x\gamma a$, where $[x, \gamma]\in[e\delta M, \Gamma]$. Set $N=\{\rho_a \mid a\in M\}$.

For $\gamma\in\Gamma$ define a mapping ψ_γ of $e\delta M$ to $[e\delta M, \Gamma]$ by $x\psi_\gamma=[x, \gamma]$, where $x\in e\delta M$. Set $\Lambda=\{\psi_\gamma \mid \gamma\in\Gamma\}$.

Then one can easily verify that for all $a, b\in M$ and $\gamma, \delta\in\Gamma$

$$\rho_a+\rho_b=\rho_{a+b}, \quad \psi_\gamma+\psi_\delta=\psi_{\gamma+\delta}, \quad \text{and} \quad \rho_a\psi_\gamma\rho_b=\rho_{a\gamma b},$$

thus N becomes a Γ_1-ring, where $\Gamma_1=\Lambda$.

Set $\kappa=\{\gamma\in\Gamma \mid M\gamma M=0\}$, then κ is a subgroup of Γ. For any element $\bar{\gamma}\in\Gamma/\kappa$ we define $a\bar{\gamma}b=a\gamma b$ (well defined), where $\bar{\gamma}=\gamma+\kappa$. Then we get a Γ_0-ring M, where $\Gamma_0=\Gamma/\kappa$.

Let ρ be a mapping of M to N by $\rho(a)=\rho_a$, $a\in M$, and let ψ be a mapping from Γ_0 to Λ by $\psi(\bar{\gamma})=\psi_\gamma$ (well defined), where $\gamma+\kappa=\bar{\gamma}\in\Gamma_0$. Then $\rho(a)=0\Rightarrow\rho_a=0\Rightarrow e\delta M\Gamma a=0\Rightarrow M\delta e\delta M\Gamma a=0\Rightarrow M\Gamma a=0\Rightarrow a\Gamma M\Gamma a=0\Rightarrow a=0$, since $M\delta e\delta M=M$, due to M being simple, and M is semi-prime. Also, $\psi(\bar{\gamma})=0\Rightarrow\psi_\gamma=0\Rightarrow[e\delta M, \gamma]=0\Rightarrow[M\delta e\delta M, \gamma]=0\Rightarrow[M, \gamma]=0\Rightarrow M\gamma M=0\Rightarrow\bar{\gamma}=0$, since M is simple. Next, $\rho(a\bar{\gamma}b)=\rho(a\gamma b)=\rho_{a\gamma b}=\rho_a\psi_\gamma\rho_b=\rho(a)\psi(\bar{\gamma})\rho(b)$. Both, ρ and ψ are clearly surjections. Hence, the mapping (ρ, ψ) is a isomorphism from the Γ_0-ring M onto the Γ_1-ring N, where $\Gamma_1=\Lambda$.

Let $\dim({}_LM)=m$ and $\dim(M_R)=n$, and let $D_{n,m}$ and $D_{m,n}$ denote respectively the set of all matrices of type $n\times m$ over D and that of all matrices of type $m\times n$ over D. Similarly, D_n and D_m are respectively the total matrix ring of type $n\times n$ over D and that of type $m\times m$ over D.

Choose a basis $l_1, \cdots, l_n$ of the vector space $[e\delta M, \Gamma]$ and a basis $u_1, \cdots, u_m$ of the vector space $e\delta M$.

For $a\in M$ we have

$$l_ia=l_i\rho_a=\alpha_{i1}u_1+\cdots+\alpha_{im}u_m\,;\ i=1, 2, \cdots, n.$$

Now the correspondence

$$\rho_a\mapsto(\alpha_{ij})\,;\ 1\leqq i\leqq n,\ 1\leqq j\leqq m$$

is a group isomorphism from the additive abelian group N into the additive abelian group $D_{n,m}$. Thus, $\theta: a\mapsto(\alpha_{ij})$ is a group isomorphism of M into $D_{n,m}$. We show that this is an isomorphism onto $D_{n,m}$:

Along the similar fashion described in the above, ring theory shows that elements of the left operator L are linear transformations of the vector space $[e\delta M, \Gamma]$ and as a ring L is isomorphic onto D_n, and elements of the right operator ring R are linear transformations of the vector space $e\delta M$ and R isomorphic onto D_m. Since M is a left L-right R-bimodule, for any $l\in L$, $x\in M$, $r\in R$, $lxr\in M$. Let $l\mapsto(\sigma_{ij})\in D_n$, $x\mapsto(\alpha_{ij})\in D_{n,m}$, $r\mapsto(\tau_{ij})\in D_m$. Then for any $a\in[e\delta M, \Gamma]$,

$$a(lxr)=((al)x)r=((a(\sigma_{ij}))(\alpha_{ij}))(\tau_{ij})=a(\sigma_{ij})(\alpha_{ij})(\tau_{ij}),$$

and hence, $(lxr)\theta=(\sigma_{ij})(x)\theta(\tau_{ij})$. Thus, $LMR\subseteqq M$ implies $(LMR)\theta\subseteqq(M)\theta$, and so $D_n(M)\theta D_m\subseteqq(M)\theta$. It follows $D_{n,m}\subseteqq(M)\theta$, for $(M)\theta\subseteqq D_{n,m}$. Hence, $(M)\theta=D_{n,m}$.

Q. E. D.

By the similar argument, we obtain that the additive abelian group Γ_0 is isomorphic onto a subgroup of $D_{m,n}$, and we denote the isomorphism by ϕ.

We now prove $(a\bar{\gamma}b)\theta=a\theta\bar{\gamma}\phi b\theta$:

Let $a\theta=(\alpha_{ij})$, $b\theta=(\beta_{ij})$, $\bar{\gamma}\phi=(\omega_{uv})$. Then, for any $l\in[e\delta M, \Gamma]$ we have

$$l(a\bar{\gamma}b)=((la)\bar{\gamma})b=((l(\alpha_{ij}))(\omega_{uv}))(\beta_{ij})=l(\alpha_{ij})(\omega_{uv})(\beta_{ij}),$$

thus, $(a\bar{\gamma}b)\theta=(\alpha_{ij})(\omega_{uv})(\beta_{ij})=a\theta\bar{\gamma}\phi b\theta$.

Clearly, $D_{n,m}$ is a Γ'-ring, where Γ' is $(\Gamma_0)\phi$, which is a non-zero subgroup of $D_{m,n}$.

Therefore, the Γ_0-ring M is isomorphic onto the Γ'-ring $D_{n,m}$ and the proof is completed.

When M is a Γ-ring in the sense of Nobusawa, $\kappa=0$ and then $\Gamma_0=\Gamma$, and furthermore since Γ is a right L- left R-bimodule $D_m(\Gamma)\phi D_n\subseteqq(\Gamma)\phi$. On the other hand, $(\Gamma)\phi\subseteqq D_{m,n}$, and so $(\Gamma)\phi=D_{m,n}$, thus we have

COROLLARY 4.3 ([8] Theorem 2). *A simple Γ-ring M in the sense of Nobusawa with min-r and min-l conditions is isomorphic onto the Γ'-ring $D_{n,m}$, where $\Gamma'=D_{m,n}$.*

We note that the term 'simple' in this corollary is the one given in Definition 3.9. However, as shown already, since M has minimum condition, M becomes prime (Theorem 3.15). Then, since M is the prime Γ-ring in the sense of Nobusawa, M is completely prime ([1] Theorem 5), which coincides with 'M is simple' in Theorem 2 in Nobusawa [8].

5. Γ-rings with minimum right and left conditions.

First we consider the semi-prime Γ-ring with min-r and min-l conditions. Let $\Gamma_0=\Gamma/\kappa$, where $\kappa=\{\gamma\in\Gamma| M\gamma M=0\}$, and $M=M_1\oplus\cdots\oplus M_q$, where $M_1, \cdots, M_q$ are simple Γ-rings with min-r and min-l conditions (Theorem 3.13). Let $\kappa_i=\{\gamma\in\Gamma| M_i\gamma M_i=0\}$, $1\leqq i\leqq q$, then $\kappa=\kappa_1\cap\cdots\cap\kappa_q$. Thus, $\Gamma_0=\Gamma/\kappa$ is isomorphic to a subgroup of $\Gamma/\kappa_1\oplus\cdots\oplus\Gamma/\kappa_q$. Set $\Gamma/\kappa_i=\Gamma_i$. This means that Γ_0 is isomorphic to a subdirect sum of the Γ_i, $1\leqq i\leqq q$. Theorem 4.2 implies that M_i is isomorphic onto $D^{(i)}_{n(i),m(i)}$ over a division ring $D^{(i)}$ and Γ_i is isomorphic to a non-zero subgroup of $D^{(i)}_{m(i),n(i)}$ over $D^{(i)}$. Thus, we have

$$M=\sum_{i=1}^{q}D^{(i)}_{n(i),m(i)} \text{ (direct sum) and}$$

$\Gamma_0=\Gamma/\kappa$ is a subdirect sum of the Γ_i, where $\Gamma_i\subseteqq D^{(i)}_{m(i),n(i)}$, $1\leqq i\leqq q$, where the product of elements of $D^{(i)}_{m(i),n(i)}$ and of $D^{(j)}_{n(j),m(j)}$ is performed as usual if $i=j$

and is 0 if $i \neq j$.

Thus we have

THEOREM 5.1. *Let M be a semi-prime Γ-ring with min-r and min-l conditions. Then, the Γ-ring M is homomorphic onto the Γ_0-ring $\sum_{i=1}^{q} D_{n(i),m(i)}^{(i)}$ where Γ_0 is a subdirect sum of the Γ_i, $1 \leqq i \leqq q$, which is a non-zero subgroup of $D_{m(i),n(i)}^{(i)}$.*

Theorem 2.12 and Theorem 5.1 yield the following corollary.

COROLLARY 5.2. *Let M be a Γ-ring with min-r and min-l conditions. Then, the Γ-ring M is homomorphic onto the Γ_0-ring $\sum_{i=1}^{q} D_{n(i),m(i)}^{(i)}$ where Γ_0 is a subdirect sum of the Γ_i, $1 \leqq i \leqq q$, which is a non-zero subgroup of $D_{m(i),n(i)}^{(i)}$.*

We consider the converse of the preceding comment to Theorem 5.1. First we prove the converse of Theorem 4.2.

THEOREM 5.3. *$D_{n,m}$, D is a division ring, is a simple Γ-ring with min-r and min-l conditions, where Γ is a non-zero subgroup of $D_{m,n}$ and $[\Gamma, D_{n,m}] = D_m$ and $[D_{n,m}, \Gamma] = D_n$.*

PROOF. Denote the elementary matrices by $E_{ij} \in D_{n,m}$, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$; $G_{st} \in D_m$, $1 \leqq s, t \leqq m$; $H_{pq} \in D_n$, $1 \leqq p, q \leqq n$. Let $A = (\alpha_{ij})$ belong to $D_{n,m}$, then $A = \sum_{i,j} \alpha_{ij} E_{ij}$.

The ideal generated by A contains $H_{pq} A G_{st} = \alpha_{qs} E_{pt}$. If $A \neq 0$, then $\alpha_{qs} \neq 0$ for some (q, s) and the E_{pt} is in the ideal generated by A. This is true for all $p = 1, \cdots, n$; $t = 1, \cdots, m$, and hence the ideal is equal to $D_{n,m}$, so that $D_{n,m}$ is simple. To verify the min-r condition, observe that $D_{n,m}$ is a right vector space of dimension nm over D. Every right ideal J of $D_{n,m}$ is a subspace, since $A \in J \Rightarrow Ad = A(dE_m) \in J$, where E_m the identity matrix and $d \in D$. The min-r condition holds. Similarly, the min-l condition holds.

THEOREM 5.4. *If $M = M_1 \oplus \cdots \oplus M_q$, where $M_1, \cdots, M_q$ are simple Γ_i-rings with min-r and min-l conditions, then M is a semi-prime Γ-ring with min-r and min-l conditions, where Γ is a subdirect sum of the Γ_i's, $M_i \Gamma M_j = 0$ $(i \neq j)$ and $M_i \Gamma_j M_i = 0$ $(i \neq j)$.*

PROOF. Let S be a strongly-nilpotent ideal of M and let $S_1, \cdots, S_q$ be its component ideals in $M_1, \cdots, M_q$, respectively. If $(S\Gamma)^n S = 0$ then $(S_i \Gamma_i)^n S_i = 0$ for each i. Since M_i is simple $S_i = M_i$ or $S_i = 0$. If $S_i = M_i$, then $(S_i \Gamma_i)^n S_i = M_i = 0$, a contradiction. Thus, $S_i = 0$ and hence $S = S_1 \oplus \cdots \oplus S_q = 0$ and M is semi-prime.

To verify the min-r condition, suppose $J^{(1)} \supseteqq J^{(2)} \supseteqq \cdots$ is a descending sequence of right ideals of M. The components $J_i^{(n)}$ in the Γ_i-ring M_i are a descending sequence in M_i ($J_i^{(1)} \supseteqq J_i^{(2)} \supseteqq \cdots \supseteqq J_i^{(n)} \supseteqq \cdots$) and hence $J_i^{(n)}$ is fixed for $n \geqq n(i)$, say. It followed that $J^{(n)}$ is fixed for $n \geqq \max[n(1), \cdots, n(q)]$, and hence the min-$r$ condition holds in M. Similarly, the min-l condition can be verified.

We consider the Γ-rings in the sense of Nobusawa.

Let M be a Γ-ring in the sense of Nobusawa and M be semi-prime with min-r and min-l conditions. Let $M = M_1 \oplus \cdots \oplus M_q$, where $M_1, \cdots, M_q$ are simple Γ-rings with min-r and min-l conditions (Theorem 3.13). Let $\Gamma_i = \Gamma/\kappa_i$, where $\kappa_i = \{\gamma \in \Gamma \mid M_i \gamma M_i = 0\}$. We show that each Γ-ring M_i is the Γ_i-ring in the sense of Nobusawa. Since $\Gamma M_i \Gamma \subseteqq \Gamma$, κ_i is an ideal of Γ. Indeed, $M_i(\Gamma M_i \kappa_i) M_i = (M_i \Gamma M_i)\kappa_i M_i = M_i \kappa_i M_i = 0$ and then $\Gamma M_i \kappa_i \subseteqq \kappa_i$. Similarly, $\kappa_i M_i \Gamma \subseteqq \kappa_i$. Hence, we can define a multiplication: $\Gamma_i \times M_i \times \Gamma_i \to \Gamma_i$ as follows:

For any $\bar{\gamma}, \bar{\delta} \in \Gamma_i$, $a \in M_i$, where $\bar{\gamma} = \gamma + \kappa_i$, $\bar{\delta} = \delta + \kappa_i$,

$$\bar{\gamma} a \bar{\delta} = \overline{\gamma a \delta} \quad \text{(well defined)}.$$

Clearly, $M_i \bar{\gamma} M_i = 0$ implies $\bar{\gamma} = 0$. Q. E. D.

Therefore, by Corollary 4.3, we have $\Gamma_i = D_{m(i), n(i)}^{(i)}$. Since $\kappa = 0$ and so $\Gamma_0 = \Gamma$, Γ is isomorphic to the subgroup of $\sum_{i=1}^{q} D_{m(i), n(i)}^{(i)}$. Let this isomorphism be ϕ, then

$$\gamma\phi = \gamma_1 + \cdots + \gamma_q, \text{ where } \gamma_i = \gamma + \kappa_i,\ 1 \leqq i \leqq q.$$

We show that the subgroup coincides with the group $\sum_{i=1}^{q} D_{m(i), n(i)}^{(i)}$. Fix an element i of the index set $\{1, 2, \cdots, q\}$. For any $\sigma_i \in \Gamma_i = D_{m(i), n(i)}^{(i)}$, choose an element $\sigma \in \Gamma$ such that $\sigma_i = \sigma + \kappa_i$. Let $\sigma\phi = \sigma_1 + \cdots + \sigma_i + \cdots + \sigma_q$, where $\sigma_k = \sigma + \kappa_k$, $1 \leqq k \leqq q$, and E_{ii} be the unit matrix of $D_{m(i)}^{(i)}$, and F_{ii} be the unit matrix of $D_{n(i)}^{(i)}$. Then, since Γ is the right L- left R-bimodule and $D_{n(i)}^{(i)} = [M_i, \Gamma_i] \subseteqq L$ and $D_{m(i)}^{(i)} = [\Gamma_i, M_i] \subseteqq R$, $\sigma_i = E_{ii}(\sigma\phi)F_{ii} \in (\Gamma)\phi$, $1 \leqq i \leqq q$. Now let i be free. Then, $\sum_{i=1}^{q} \sigma_i \in (\Gamma)\phi$, where each σ_i is an arbitrary element of Γ_i. This means $\sum_{i=1}^{q} D_{m(i), n(i)}^{(i)} \subseteqq (\Gamma)\phi$, and $(\Gamma)\phi = \sum_{i=1}^{q} D_{m(i), n(i)}^{(i)}$.

Thus, we have

$$M = \sum_{i=1}^{q} D_{n(i), m(i)}^{(i)} \quad \text{and} \quad \Gamma = \sum_{i=1}^{q} D_{m(i), n(i)}^{(i)},$$

which is Theorem 3 of Nobusawa [8].

ACKNOWLEDGEMENT. I would like to thank the refree for several valuable and helpful remarks.

References

[1] Barnes, W. E., On the Γ-rings of Nobusawa, Pacific J. Math., **18** (1966), 411-422.
[2] Kyuno, S., On the radicals of Γ-rings, Osaka J. Math., **12** (1975), 639-645.
[3] Kyuno, S., On prime gamma rings, Pacific J. Math., **75** (1978), 185-190.
[4] Kyuno, S., A gamma ring with right and left unities, Math. Japonica, **24** (1979), 191-193.
[5] Kyuno, S., Nobusawa's gamma rings with right and left unities, Math. Japonica, **25** (1980), 179-190.
[6] Luh, J., On primitive Γ-rings with minimal one-sided ideals, Osaka J. Math., **5** (1968), 165-173.
[7] Luh, J., On the theory of simple Γ-rings, Michigan Math. J., **16** (1969), 65-75.
[8] Nobusawa, N., On a generalization of the ring theory, Osaka J. Math., **1** (1964), 81-89.

Department of Technology
Tohoku Gakuin University
Tagajo, Miyagi, 985
Japan