TSUKUBA J. MATH.
Vol. 20 No. 2 (1996), 413–424

THE INTERSECTION OF QUADRICS AND DEFINING EQUATIONS OF A PROJECTIVE CURVE

By

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Abstract. Let C be a complete nonsingular curve over an algebrically closed field K and L a very ample invertible sheaf on C. We denote by $\phi_L : C \to P(H^0(L))$, the projective embedding of C by means of the vector space $H^0(C,L)$. There are two purposes in this paper. One is to the question: What is the intersection of quadrics through $\phi_L(C)$? The other is to answer the question: What degrees are the minimal generators of the associated homogeneous ideal?

0. Introduction

Let C be a complete nonsingular curve over an algebrically closed field K and L a very ample invertible sheaf on C. We denote by $\phi_L : C \to P(H^0(L))$, the projective embedding of C by means of the vector space $H^0(C,L)$.

Several authors have answered the questions of when $\phi_L(C)$ for a given invertible sheaf L is projectively normal and when the associated homogeneous ideal $I(L)$ of the embedded curve $\phi_L(C)$ is generated by quadrics. (*see* [3], [4], [8], [9]) Since it is well-known that if deg $L \geqq 2g+2$, then $I(L)$ is generated by quadrics (*see* [2], [9], [10]), they have treated low degree invertible sheaves (*i.e.* deg $L \leqq 2g+1$). For example, Green and Lazarsfeld proved that if deg $L = 2g$ and C is a hyperelliptic curve, then $\phi_L(C)$ is not projectively normal ([3]). Of course $I(L)$ is not generated by quadrics in this case. That is to say that $\phi_L(C)$ is not cut out by only quadrics. So two related questions arise:

(I) What is the intersection of quadrics $Q(\phi_L(C))$?

(II) What degrees are the minimal generators of $I(L)$?

For the questions above the theorem of Noether-Enriques-Petri (*cf.* [11]) is the answer for canonical sheaf ω of nonhyperelliptic curve. Serrano have reported some results about the first question ([12]), and Homma have answered for L on a curve of genus 3 ([6], [7]). In this paper, our purpose is to answer for

Received September 5, 1994.
Revised June 26, 1995.

case of $g \geqq 4$ (mainly $g = 4$).

First let C be a hyperelliptic curve. Our result about $Q(\phi_L(C))$ is as follows.

THEOREM 0.1. *Let C be a nonsingular hyperelliptic curve of genus g ($\geqq 3$) and L a nonspecial very ample invertible sheaf of degree d. If* (1) $d \leqq 2g$ *or* (2) $d = 2g+1$ *and* $\mathrm{h} = h^0(C, L \otimes \omega_c^{-1}) \leqq 1$, *then $Q(\phi_L(C))$ is coincides with rational ruled surface F_e embedded by $|D = C_0 + 1/2(d-g-1+e)F|$ for some invariant $e(< d-g-1)$. (where C_0 is a minimal section, and F is a fiber.)*

Furthermore in case of (2) *if* $g = (3), 4, 5$, *then* $e = g-4+2\mathrm{h}$.

By (0.1), $I(L)$ is not generated by quadrics under the condition above. It is known that if L is normally generated and $H^1(C, L) = (0)$, then $I(L)$ is generated by I_2 and I_3 (*cf.* [6]) (where I_m is $\mathrm{Ker}[S^m\Gamma(L) \to \Gamma(L^m)]$) If $\deg L \geqq 2g+1$, then L is normally generated ([9]). Therefore if $\deg L = 2g+1$, then $I(L)$ is generated strictly by I_2 and I_3. (we say that the homogeneous ideal $I(L)$ is generated strictly by its elements of degrees $v_1, \cdots, v_n$ if $I(L)$ is generated by its elements of degrees $v_1, \cdots, v_n$ and $I(L)$ is not generated by its elements of degrees $v_1, \cdots, \hat{v}_j, \cdots, v_n$ for any $v_j (1 \leqq j \leqq \mathrm{n})$, where $\hat{v}_j$ means that v_j is omitted.) But if $\deg L \leqq 2g$ and C is a hyperelliptic curve, then L is not normally generated. Therefore the question (II) arises. Our main results about $I(L)$ are the answers for the case of $\deg L = 2g, 2g-1$.

THEOREM 0.2. *Let C be a nonsingular hyperelliptic curve of genus g and L a very ample invertible sheaf of degree $2g$. Then $I(L)$ is generated by I_2, I_3 and I_4.*

THEOREM 0.3. *Let C be a nonsingular hyperelliptic curve of genus g and L a very ample invertible sheaf of degree $2g-1$. Then $I(L)$ is generated by I_2, I_3, I_4 and I_5 (Furthermore if $g = 4$, then $I(L)$ is generated strictly by I_2 and I_5. (see* (2.6))

Next let C be a nonhyperelliptic curve. Our results in this case are as follows.

THEOREM 0.4. *Let C be a nonsingular nonhyperelliptic curve of genus* 4 *and L a very ample invertible sheaf. If $\deg L$ is* 8, *then $Q(\phi_L(C))$ is a surface of degree* 4 *in $\boldsymbol{P}^4$. If $\deg L$ is* 7, *then $Q(\phi_L(C))$ coincides with $\boldsymbol{P}^3$ (see* (3.1) and (3.2))

The organization of the paper is as follows. In the first section (preliminaries), we summarize some facts about very ample invertible sheaves on C and rational scrolls. In the second section we prove theorems 0.1, 0.2 and 0.3. In the third section we prove theorem 0.4.

Notation. We fix an algebrically closed field K.

(1) For a finite dimensional vector space V over $K, S^m(V)$ means the m-th symmetric power of V. Let L be an invertible sheaf. The m-th tensor product of L (resp. $\Gamma(L)$) is denoted by L^m(resp. $\Gamma(L)^m$). For the vector space of global sections $\Gamma(L)$ we define $I_m(L)$(or I_m) and $I(L)$, by

$$I_m(L) = \mathrm{Ker}\,[S^m\Gamma(L) \to \Gamma(L^m)] \text{ and } I(L) = \oplus I_m(L).$$

We denote by ω_c the canonical invertible sheaf on C.

(2) If L is an invertible sheaf on a variety X which is generated by global sections, we may define a morphism $\phi_L : X \to \boldsymbol{P}(H^0(L))$ by means of the vector space $H^0(L)$.

(3) We denote by $\pi : F_e \to \boldsymbol{P}^1$, the geometrically rational ruled surface with invariant $e \geqq 0$. A minimal section of π is denoted by $\boldsymbol{C}_0$ and a fiber of π by $\boldsymbol{F}$.

(4) Let X be a closed subvariety of a projective space $\boldsymbol{P}^n$. We denote by $Q(X)$ the intersection of quadrics through X.

1. Preliminaries

First, we shall recall facts about very ample invertible sheaves on a curve, especially, of genus 4.

Let L be an invertible sheaf on a curve C of genus g. If $\deg L \geqq 2g+1$, then L is very ample. If $\deg L = 2g$, then L is not very ample if and only if L is isomorphic to ω_c (P+Q) for some points P,Q $\in$ C (may be P=Q). (see, for example, [1], I Exercises D-2) If $g \geqq 2$, then C has a very ample invertible sheaf L of degree d with $h^1(L)=0$ if and only if $d \geqq g+3$ (Halphen's Theorem) see, for example, [5], IV Proposition 6.1)

LEMMA 1.1 *Let C be a curve of genus 4 and L an invertible sheaf of degree $d \leqq 6$ on C. Then L is very ample if and only if C is nonhyperelliptic and $L \cong \omega_c$.*

PROOF. Let L be a very ample invertible sheaf of degree $d \leqq 6$. By virtue of Halphen's Theorem, we have $h^1(L) > 0$. Hence we have that $h^0(L) \leqq g = 4$ and equality occures if and only if $L \cong \omega_c$. It is clear that $h^0(L) \geqq 3$. In the case of $h^0(L) = 3$, C is a plane curve. It is a contradiction by the genus formula $g = 1/2(d-1)(d-2)$. Therefore L must be the canonical sheaf ω_c. On the other hand, ω_c is very ample if and only if C is nonhyperelliptic. This completes the proof.

Secondly, we shall state several facts about rational scrolls associating to a hyperelliptic curve C of genus $g \geqq 2$.

Let C be a hyperelliptic curve of genus $g \geqq 2$ with a unique linear system g_2^1 of degree 2 and of projective dimension 1. We denote by M_0 the invertible sheaf corresponding to g_2^1.

Let L be a nonspecial and very ample invertible sheaf on C. For every $y \in \boldsymbol{P}^1$ the linear span of the divisor $\phi^*(y)$ of C is a line $\ell_y \subseteq \boldsymbol{P}^{d-g} = \boldsymbol{P}(H^0(L))$. (where $\phi: C \to \boldsymbol{P}^1$ is a hyperelliptic double covering.) The union of these lines, $S = \bigcup \ell_y$, is a scroll in $\boldsymbol{P}^{d-g}$. S contains the curve $C \subseteq \boldsymbol{P}^{d-g}$ and, consequently, is nondegenerate. We call *the scroll associated to the double covering* $\phi: C \to \boldsymbol{P}^1$ *with respect to L.*

LEMMA 1.2. ([8]), Lemma 3.1) *Let* $\phi: C \to \boldsymbol{P}^1$ *be a hyperelliptic double covering of genus g* ($g \geqq 2$) *and L a nonspecial very ample line bundle of degree d on C. Then the scroll S associated to* ϕ *with respect to L is either a cone over a rational normal curve in* $\boldsymbol{P}^{d-g-1}$ *or smooth of degree* $d-g-1$ *in* $\boldsymbol{P}^{d-g}$.

REMARK 1.3. *If* $d \leqq 2g$ *or* $d \geqq 2g+3$ *in* Lemma 1.2, *then S is smooth.*

PROOF. Suppose that S is a cone F. Let $\tilde{F} \to F$ be the blowing up with a center vertex. Then $\tilde{F}$ coincides with the rational ruled surface F_{d-g-1} with invariant $d-g-1$. Let H be a hyperplane section on F and $\tilde{H}$ the strict transform of H on $\tilde{F}$. Since $\tilde{H} \cdot \boldsymbol{F} = 1$ and $\tilde{H} \cdot \boldsymbol{C}_0 = 0$, we have $\tilde{H} - \boldsymbol{C}_0 + (d-g-1)\boldsymbol{F}$. Suppose that the strict transform $\tilde{C}$ of $\phi_L(C)$ is linearly equivalent to $\alpha \boldsymbol{C}_0 + \beta \boldsymbol{F}$. Since $d = \deg \phi_L(C)$, we have

$$d = \tilde{C} \cdot \tilde{H} = \beta \tag{1}$$

On the other hand, using the adjunction formula, we have

$$\begin{aligned} 2g-2 &= (\tilde{C} + K_F) \cdot \tilde{C} \quad (\text{where } K_F \text{ is the canonical divisor on } F_{d-g-1}) \\ &= \alpha(\alpha-2)(-d+g+1) + \beta(\alpha-2) + \alpha(\beta-d+g-1). \end{aligned} \tag{2}$$

Solving (1) and (2), we have that $\tilde{C}$ is linearly equivalent to $2\boldsymbol{C}_0 + \mathrm{d}\boldsymbol{F}$. Therefore we have $\tilde{C} \cdot \boldsymbol{C}_0 = 2g+2-d$. Since $d \leqq 2g$ or $d \geqq 2g+3$, we have

$$\tilde{C} \cdot \boldsymbol{C}_0 \geqq 2, \tilde{C} \cdot \boldsymbol{C}_0 \leqq -1. \tag{3}$$

If the vertex of F does not lie on $\phi_L(C)$, then $\tilde{C} \cdot \boldsymbol{C}_0 = 0$. If not, then $\tilde{C} \cdot \boldsymbol{C}_0 = 1$.

This contradicts with (3). Therefore S is smooth in this condition.

REMARK 1.4. *If* $d=2g+1$ $(g\geqq 3)$ *and* $h^0(L\otimes\omega_c^{-1})\leqq 1$, *then* S *is smooth.*

PROOF. This result is owing to ([7], Theorem 3.1).

The following lemma will be used to calculate the dimension of $H^0(F_e, nC_0+mF)$ in the second section.

LEMMA 1.5. (*see*, for example, [7], Lemma 2.1) *Let* L *be the invertible sheaf* $\vartheta_F(nC_0+mF)$ *on* F_e *and* $n\geqq 0$ and $m\geqq ne-1$, *then* $h^1(L)=h^2(L)=0$ *and* $h^0(L)=(n+1)(m+1)-1/2n(n+1)e$.

2. Hyperelliptic case

LEMMA 2.1. *Let* M *and* N *be invertible sheaves on a curve* C. *If* $h^1(N)\leqq h^0(M)-1$, *then* $h^1(N\otimes M)=0$.

PROOF. Suppose that $h^1(N\otimes M)\geqq 1$. Then $h^0(M)\leqq h^0(M)+h^1(N\otimes M)-1=h^0(M)+h^0(\omega_c\otimes N^{-1}\otimes M^{-1})-1\leqq h^0(M\otimes\omega_c\otimes N^{-1}\otimes M^{-1})=h^1(N)$. It is a contradiction with the assumption.

THEOREM 2.2. *Let* C *be a nonsingular hyperelliptic curve of genus* 4 *and* L *a very ample invertible sheaf of degree* 8. *Then* $\phi_L(C)$ lies on F_1 *embedded by the complete linear system* $|C_0+2F|$ in P^4.

In this case, $Q(\phi_L(C))$ *coincides with* F_1.

PROOF. (Step 1) We shall claim that $h^1(L\otimes M_0^{-1})=0$ and $h^1(L\otimes M_0^{-2})=0$. In fact, since $h^1(L\otimes M_0^{-3})=1\leqq h^0(M_0)-1$, we get $h^1(L\otimes M_0^{-2})=0$ by using Lemma 2.1. In the same way we have $h^1(L\otimes M_0^{-1})=0$.

(Step 2) We will consider the natural map $\eta: H^0(L\otimes M_0^{-1})\otimes H^0(M_0)\to H^0(L)$. By the "base point free pencil trick" [11], dim $\mathrm{Ker}\,\eta=h^0(L\otimes M_0^{-2})=1$. Hence we have an exact sequence

$$0\to\mathrm{Ker}\,\eta=H^0(L\otimes M_0^{-1})\otimes H^0(M_0)\to H^0(L)\to 0.$$

Therefore we get the following commutative diagram.

$$\begin{array}{ccccc} C & \hookrightarrow & & & P(H^0(L)) \cong P^4 \\ \cap\downarrow & & & & \cap\downarrow \\ P(H^0(L\otimes M_0^{-1}))\times P(H^0(M_0)) & \hookrightarrow & & & P(H^0(L\otimes M_0^{-1})\otimes H^0(M_0)), \\ \wr\| & & f & & \wr\| \\ P^2\times P^1 & & & & P^5 \end{array}$$

where f is the Segre embedding.

Let F be an irreducible component of $P^2\times P^1\cap P^4$ containing $\phi_L(C)$. Since the Segre embedding of $P^2\times P^1$ does not lie on any hyperplane in P^5, we have dim F=2. From the degree of th Segre embedding of $P^2\times P^1$ and deg $F\geqq$ codim $F+1=3$ we get deg F=3. Varieties of degree 3 in P^n can be classified. (see [14])

By this fact, F is either F_1 or the cone over the 3-uple embedding of P^1 in P^3. The latter case does not occur by (1.3). So F must coincide with F_1.

(Step 3) Finally we will show that $I_2(L)=I_2(\mathscr{L})$ (where $\mathscr{L}=\vartheta_F(C_0+2F)$). If $I_2(L)=I_2(\mathscr{L})$, then $Q(\phi_L(C))$ coincides with F_1.

Now we shall chase the following commutative diagram (for n=2).

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & I_n(\mathscr{L}) & \to & S^n\Gamma(\mathscr{L}) & \to & \Gamma(\mathscr{L}^n) & \to & 0 \\ (2.2.1) & & \downarrow\gamma_n & & \wr\| & & \downarrow\phi_n & & \\ 0 & \to & I_n(L) & \to & S^n\Gamma(L) & \to & \Gamma(L^n) & & (n=2,3,4). \end{array}$$

(Since $\mathscr{L}$ is normally generated, $S^2\Gamma(\mathscr{L})\to\Gamma(\mathscr{L}^2)$ is surjective in this diagram.)

Let $\phi_L(C)$ be linearly equivalent to $\alpha C_0+\beta F$ on F_1. By using adjunction formula and $\phi_L(C)=8$, we have $\alpha=2$ and $\beta=6$. Then $\mathrm{Ker}\,\phi_2=H^0(F_1,\mathscr{L}^2\otimes\vartheta(-\phi_L(C)))=H^0(F_1,-2F)=(0)$. Therefore Coker $\gamma_2=(0)$ by snake's lemma. Hence we get the required assertion that $I_2(L)=I_2(\mathscr{L})$. This completes the proof.

PROOF OF THEOREM 0.1. First $\phi_L(C)$ lies on F_e embedded by $|C_0+1/2(d-g-1+e)F|$ by (1.3) and (1.4), where e satisfies $d-g-1>e$. By the same argument in (Step 3) of (2.2) we have $\phi_L(C)\sim 2C_0+(g+1+e)F$ and $\mathrm{Ker}(\Gamma(\mathscr{L}^2)\to\Gamma(\mathscr{L}^2))=H^0(F_e,(d-2g-2)F)=(0)$. Therefore we get similar results.

Next we shall apply the next lemmas to determining e uniquely in some cases.

LEMMA 2.3. ([13]), Theorem 2.5) *We define the number* $d_i (i \geqq 0)$:

$$d_i = h^0(L(-iD)) - h^0(L(-(i+1)D)) \quad (\text{where } D \in g_2^1).$$

Then $e = \#\{j \mid d_j = 1\}$.

In (2.3) we claim that $d_i \geqq d_j$ for $i < j$. Therefore we have $e = h^0(L(-\alpha D))$, where $\alpha = \max\{i \mid h^1(L(-iD)) = 0\}$.

LEMMA 2.4. $h^1(L(-iD)) = 0$ for $i \leqq d - 2g + 2 - h$ (where $h = h^0(L \otimes \omega_c^{-1})$).

PROOF. First we claim that $h^0(kD) = k + 1 (0 \leqq k \leqq g)$. Hence $h^0((g-1-i)D) - 1 = g - 1 - i \geqq h - d + 3g - 3 = h^1(L \otimes \omega_c^{-1})$. By using (2.1) we get the above result.

If $d = 2g+1$ and h=0, then $h^1(L(-iD)) = 0$ $(i \leqq 3)$ by (2.4). By using (2.3) we have that e = 0 (resp. 1) in the case of g = 4 (resp. 5). By the same way, if $d = 2g+1$ and h = 1, then e = 2 (resp. 3) in the case of g = 4 (resp. 5). This completes the proof.

Next we shall study $I(L)$ by using above results of $Q(\phi_L(C))$.

LEMMA 2.5. ([6]), COROLLARY 3.6) *Let L be a very ample invertible sheaf on an n-dimensional projective variety X. Assume that* $H^i(X, L^j) = (0)$ *for any integers* $i, j > 0$. *If* m = $\mathrm{Max}(n+3, n(L)+1)$, *then* $I(L)$ *is generated by* $I_2, \cdots, I_m$, *where* $n(L) = \mathrm{Min}\{n \in N \mid \Gamma(L)^i \to \Gamma(L^i)$ *is surjective for all* $i \geqq n\}$.

PROOF OF THEOREM 0.2. First we shall show that $\beta_m : \Gamma(L)^m \to \Gamma(L^m)$ is surjective for all $m \geqq 3$ by induction on m. For a given $m \geqq 3$, we consider the following commutative diagram.

$$\begin{tikzcd}
\Gamma(L)^{m+1} \arrow[r, "\beta_m \otimes 1"] \arrow[d, "\beta_{m+1}"'] & \Gamma(L^m) \otimes \Gamma(L) \arrow[dl, "\gamma_m"] \\
\Gamma(L^{m+1}) &
\end{tikzcd}$$

By the induction hypothesis β_m is surjective, and also $\beta_m \otimes 1$ is surjective. By "generalized lemma of Castelnuovo" (*see* [9], Theorem 2) γ_m is surjective, and also β_{m+1} is surjective. Therefore we have only to prove the surjectivity of β_3. We shall chase the commutative diagram (2.2.1) (for n=3 and $\mathscr{L} = \vartheta_F(C_0 + 1/2(g-1+e)F))$.

Since we recall $\phi_L(C) \sim 2C_0 + (g+1+e)F$ from the proof of (0.1), we have $\operatorname{Ker}\phi_3 = H^0(F_e, C_0 + 1/2(g-5+e)F)$. By (2.4) we have $h^1(L(-2D)) = 0$. Hence we get that $e \leqq h^0(L(-2D)) = g-3$ (*i.e.* $1/2(g-5+e) \geqq e-1$) by (2.3). Now using (1.5), we have dimKer $\phi_3 = g-3$. On the other hand, by the theorem of Riemann-Roch and (1.5), we have $\dim \Gamma(L^3) = 5g+1$ and $\dim \Gamma(\mathscr{L}^3) = 6g-2$. So we conclude that ϕ_3 and β_3 are surjective. Therefore we have $n(L) = 3$. By using (2.5) $I(L)$ is generated by I_2, I_3 and I_4.

PROOF OF THEOREM 0.3. First we shall show that $\beta_m : \Gamma(L)^m \to \Gamma(L^m)$ is surjective for all $m \geqq 4$ by induction on m. By an argument similar to the proof of (0.2), we have only to prove the surjectivity of β_4.

Secondly we claim that $h^1(L(-2D)) = 0$. Suppose that $h^1(L(-2D)) = h^0(\omega_c \otimes L^{-1}(2D)) > 0$. Then $\omega_c(2D) \cong L(P+Q+R)$ for some points P, Q, R on C. Hence we have that $\omega_c(P'+Q') \cong L(R)$ for some points P', Q' on C. That is to say $h^1(L(-P',-Q')) > 0$. Therefore $h^0(L) - h^0(L(-P'-Q')) \neq 2$. This contradicts with very ampleness of L.

Lastly we shall consider the commutative diagram (2.2.1) (for n = 4 and $\mathscr{L} = \vartheta_F(C_0 + 1/2(g-2+e)F))$.

In the way similar to the proof of (0.2) we have $\operatorname{Ker}\phi_4 = H^0(F_e, 2C_0 + (g-5+e)F)$. By (2.3) and $h^1(L(-2D)) = 0$ we get $e \leqq h^0(L(-2D)) = g-4$ (*i.e.* $(g-5+e) \geqq 2e-1$). Hence, by using (1.5), we have dim $\operatorname{Ker}\phi_4 = 3g-12$. On the other hand we have $\dim \Gamma(L^4) = 7g-3$ and $\dim \Gamma(\mathscr{L}^4) = 10g-15$. So we conclude that ϕ_4 and β_4 are surjective. Hence we get $n(L) = 4$. By (2.5) $I(L)$ is generated by I_2, I_3, I_4 and I_5.

COROLLARY 2.6. *Let C be a nonsingular hyperelliptic curve of genus 4 and L a very ample invertible sheaf of degree 7. Then $I(L)$ is generated strictly by I_2 and I_5.*

PROOF. From (0.3) $I(L)$ is generated by I_2, I_3, I_4 and I_5. We recall $I_2(L) = I_2(\mathscr{L})$ in (0.1). Since $\phi_L(C)$ is of degree 7 and lies on a quadric surface, it does not lie on any irreducible cubic surface. Hence we have $I_3(L) = I_3(\mathscr{L})$. Furthermore, we have $I_4(L) = I_4(\mathscr{L})$ because ϕ_4 is an isomorphism in (0.3). By the way, I_2 don't generate $I(L)$. This completes the proof.

3. Nonhyperelliptic case

THEOREM 3.1. *Let C be a nonsingular nonhyperelliptic curve of genus* 4 *and* L *a very ample invertible sheaf of degree* 8. *Then* $Q(\phi_L(C))$ *is a surface of degree* 4 *in* $\boldsymbol{P}^4$.

PROOF. By the projective normality of $\phi_L(C)$ (see [3], Corollary 1.4) we have $\dim I_2(L)=2$, and hence we have distinct quadric hypersurfaces Q_1 and Q_2 in $\boldsymbol{P}^4$. Since Q_i is irreducible, so dim $Q_1 \cap Q_2 = 2$. Let F be an irreducible component of $Q_1 \cap Q_2$ containing $\phi_L(C)$. Then we have deg F = 3 or 4, since deg $F \leqq 4$ and since F is nondegenerate. So we have only to show that deg F = 4. If deg F = 3, then F is the rational ruled surface F_1 embedded by $|C_0+2F|$ or the cone over the rational normal curve in $\boldsymbol{P}^3$. But F is not the cone over the rational normal curve in $\boldsymbol{P}^3$ by the argument of (1.3). Next if F coincides with F_1, we have $\phi_L(C) \sim 2C_0+6F$ by the argument in (Step 3) of (2.2). Then C is hyperelliptic curve. It contradicts the assumption. Hence we have deg F = 4 in $\boldsymbol{P}^4$.

THEOREM 3.2. *Let C be a nonsingular nonhyperelliptic curve of genus* 4 *and* L *a very ample invertible sheaf of degree* 7. *Then* $Q(\phi_L(C))$ *coincides with* $\boldsymbol{P}^3$.

PROOF. we have to show that $\phi_L(C)$ does not lie on a quadric hypersurface: including double plane. Indeed, obviously $\phi_L(C)$ does not lie on a union of planes; if $\phi_L(C)$ lies on a quadric cone, then g = 6, contradiction; if $\phi_L(C)$ lies on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\phi_L(C)$ is of type $(a,b)=(2,5)$ in the Picard group of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ by considering degree and genus, *i.e.*, $\deg L = a+b$ and $g=(a-1)(b-1)$. This means that C is hyperelliptic, which is a contradiction.

The following is a summary of the case of genus 4.

degree d of L	h	$Q(\phi_L(C))$	
		C is hyperelliptic curve	C is nonhyperelliptic curve
$d \geqq 10\,(=2g+2)$	(a)	$\phi_L(C)$	
$d=9\,(=2g+1)$	h=2	(b) *the projective come in* $\boldsymbol{P}^5$ *over the rational normal curve*	(e) F_0 *embedded by the linear system* $\lvert \boldsymbol{C}_0+2\boldsymbol{F}\rvert$ *in* $\boldsymbol{P}^5$
	h=1	(c) F_2 *embedded by the complete linear system* $\lvert \boldsymbol{C}_0+3\boldsymbol{F}\rvert$ *in* $\boldsymbol{P}^5$	
	h=0	(d) F_0 *embedded by the complete linear system* $\lvert \boldsymbol{C}_0+2\boldsymbol{F}\rvert$ *in* $\boldsymbol{P}^5$	(f) $\phi_L(C)$
$d=8\,(=2g)$		(g) F_1 *embedded by the linear system* $\lvert \boldsymbol{C}_0+2\boldsymbol{F}\rvert$ *in* $\boldsymbol{P}^4$	(h) *the surface of degree* 4 *in* $\boldsymbol{P}^4$
$d=7\,(=2g-1)$		(i) F_0 *embedded by the linear system* $\lvert \boldsymbol{C}_0+\boldsymbol{F}\rvert$ *in* $\boldsymbol{P}^3$	(j) $\boldsymbol{P}^3$
$d=6\,(=2g-2)$			(k) *an irreducible quadric surface in* $\boldsymbol{P}^3$

(where h is *the dimension of the vector space* $H^0(C, L\otimes\omega_C^{-1})$ *over* K)

Statements (b), (e) are Homma's results ([7]). Statement (k) is well-known. Statement (f) is Green-Lazersfeld's result ([4]).

<table>
<tr><td rowspan="2">degree of L</td><td rowspan="2">h</td><td colspan="2">$I(L)$ is generated by $I_n, I_{n+1}, \cdots I_m$.</td></tr>
<tr><td>C is hyperelliptic curve</td><td>C is nonhyperelliptic curve</td></tr>
<tr><td>$d \geqq 10\,(=2g+2)$</td><td>(1)</td><td colspan="2">$I(L)$ is generated strictly by I_2.</td></tr>
<tr><td>$d=9\,(=2g+1)$</td><td>h=2</td><td rowspan="3">(2) $I(L)$ is generated strictly by I_2 and I_3.</td><td>(3) $I(L)$ is generated strictly by I_2 and I_3.</td></tr>
<tr><td></td><td>h=1</td><td>(4) $I(L)$ is generated strictly by I_2 and I_3.</td></tr>
<tr><td></td><td>h=0</td><td>(5) $I(L)$ is generated strictly by I_2.</td></tr>
<tr><td>$d=8\,(=2g)$</td><td colspan="2">(6) $I(L)$ is generated by I_2, I_3 and I_4.</td><td>(7) $I(L)$ is generated strictly by I_2 and I_3.</td></tr>
<tr><td>$d=7\,(=2g-1)$</td><td colspan="2">(8) $I(L)$ is generated strictly by I_2 and I_5.</td><td>(9)</td></tr>
<tr><td>$d=6\,(=2g-2)$</td><td colspan="2"></td><td>(10) $I(\omega_C)$ is generated strictly by I_2 and I_3.</td></tr>
</table>

"*strictly*" in statements (2), (3), and (7) follow from (b), (c), (d), (e), and (h).

Statements (4), (5) are Green-Lazersfeld's results ([4]). Statement (10) is well-known.

Acknowledgements

I wish to express to all the professors who support me, my deep gratitude for their guidance, help and encouragement. Professor M. Homma advised me kindly. Professors H. Kaji and A. Noma gave me useful suggestions. Especially, professors N. Yamawaki and T. Morimoto have taught me basic knowledge patiently. My heart is filled with thanks to them.

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