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PROPAGATION OF SINGULARITIES FOR HYPERBOLIC OPERATORS WITH CONSTANT COEFFICIENT PRINCIPAL PART

By

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1. Introduction

The wave front sets of solutions of hyperbolic Cauchy problems have been studied by many authors. For hyperbolic operators with constant coefficients Atiyah, Bott and Gårding [1] studied the singular supports (wave front sets) and the lacunas of fundamental solutions. In variable coefficient cases the wave front sets of solutions are studied by using Fourier integral operators (see [2]). In this paper we shall investigate the wave front sets of solutions for hyperbolic operators with constant coefficient principal part by the same arguments as in [1].

Let $P(x, D)$ be a partial differential operator of order m in n independent variables where $x=(x_1, \cdots, x_n)$ and $D=i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$.
We assume that

$$P(x, D)=P_m(D)+Q(x, D),$$
$$Q(x, D)=\sum_{|\alpha|<m}a_\alpha(x)D^\alpha,\ a_\alpha\in C^\infty(\boldsymbol{R}^n),$$

where α is a multi-index $(\alpha_1, \cdots, \alpha_n)$. Furthermore we assume that

(A) for each fixed x in $\boldsymbol{R}^n$ $P(x, \xi)$ is a hyperbolic polynominal with respect to $\vartheta=(1,0,\cdots,0)$, where $\xi=(\xi_1, \cdots, \xi_n)$ (see [1]).

From Svensson [6] it follows that this condition (A) is equivalent to the condition

(A)$'$ $P_m(\xi)$ is a hyperbolic polynomial with respect to ϑ and $Q(x, \xi)\prec P_m(\xi)$ for every fixed x in $\boldsymbol{R}^n$, where $q(\xi)\prec p(\xi)$ means that there is a positive number C such that

$$\tilde{q}(\xi)\equiv(\sum_\alpha|q^{(\alpha)}(\xi)|^2)^{1/2}\leq C\tilde{p}(\xi) \text{ for every } \xi \text{ in } \boldsymbol{R}^n.$$

Here we have used the notation $q^{(\alpha)}(\xi)=(\partial^\alpha/\partial\xi^\alpha)q(\xi)$.

Well-posedness of Cauchy problems for operators $P(x, D)$ of this type was proved by Dunn [3], obtaining energy inequalities. We note that $P(x, D)$ satisfying the

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condition (A) has constant strength in $\mathbf{R}^n$ (see [4]). So one can construct locally a fundamental solution for $P(x, D)$ in the same way as in [4].

The remainder of this paper is organized as follows. In § 2 we shall construct solutions of the Cauchy problems for $P(x, D)$ by successive iteration. Dunn [3] proved the convergence of the iteration by means of energy inequalities. However, we shall prove the same result by obtaining the uniform convergence of the Fourier-Laplace transforms in the iteration. Moreover we shall obtain energy inequalities for $P(x, D)$ if its coefficients are in $\mathscr{B}(\mathbf{R}^n)$. In § 3 the wave front sets of solutions will be considered, using the same arguments as in [1]. Localization theorem will be give in § 3.1 and outer estimates for wave front sets of solutions will be given in § 3.2. When the coefficients of $P(x, D)$ are in $C^L(\mathbf{R}^n)$† we can study the wave front sets of solutions with respect to C^L. This will be done in § 4.

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2. Construction of solution

First we shall state some lemmas.

LEMMA 2.1. *Let $a(\xi)$ and $b(\xi)$ be measurable functions of ξ in $\mathbf{R}^n$ which satisfy the inequalities*

$$|a(\xi)| \le A_M \langle\xi\rangle^{-M}, \quad |b(\xi)| \le B_N \langle\xi\rangle^{-N},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. If $M+N-n>0$, $s<M+N-n$ and $s \le \min\,(M, N)$, then there is a positive number $C \equiv C(M, N, s, n)$ such that

$$|a*b(\xi)| \le C A_M B_N \langle\xi\rangle^{-s}.$$

We omit the proof of Lemma 2.1. We remark that a less precise result is sufficient for our discussion below.

LEMMA 2.2 (Svensson [6]). *Let $P_m(\xi)$ be a hyperbolic polynomial with respect to ϑ and homogeneous of degree m and $p(\xi)$ a polynomial of degree less than m. Then the following three conditions are equivalent:*

(i) *$P_m(\xi)+p(\xi)$ is a hyperbolic polynomial with respect to ϑ.*

(ii) *$p \prec P_m$.*

(iii) *There is a positive number $C \equiv C(P_m, p)$ such that*

$$|p(\xi+s\vartheta)/P_m(\xi+s\vartheta)| \le C|\mathrm{Im}\, s|^{-1} \text{ when } \xi \text{ in } \mathbf{R}^n \text{ and } |\mathrm{Im}\, s| \ge 1.$$

† The definition of $C^L(\mathbf{R}^n)$ is given in [5].

Set $V=\{p(\xi)\in \boldsymbol{C}[\xi]^{\dagger};\ \deg p<m \text{ and } p\prec P_m\}$ and let $\{p_j\}$ be a basis of the finite dimensional vector space V.

The following lemma is obvious.

LEMMA 2.3. *Under the condition* (A) *we can write*

$$Q(x, \xi)=\sum_j q_j(x)p_j(\xi),\quad q_j\in C^{\infty}(\boldsymbol{R}^n).$$

Moreover the q_j *are in* $C_0^{\infty}(\boldsymbol{R}^n)$ *(resp.* $C^L(\boldsymbol{R}^n)$*) if the coefficients of* $P(x, \xi)$ *are in* $C_0^{\infty}(\boldsymbol{R}^n)$ *(resp.* $C^L(\boldsymbol{R}^n)$*).*

Let us consider the equation

$$\text{(2.1)}\qquad \begin{cases} P(x, D)u(x)=f(x), \\ \operatorname{supp} u(x)\subset\{x\in\boldsymbol{R}^n;\ x\cdot\vartheta\geq 0\}, \end{cases}$$

where f is in $\mathscr{D}'(\boldsymbol{R}^n)$ and $\operatorname{supp} f\subset\{x\in\boldsymbol{R}^n;\ x\cdot\vartheta\geq 0\}$. First we construct a solution of (2.1) by successive iteration when the $q_j(x)$ are in $C_0^{\infty}(\boldsymbol{R}^n)$. Assume that f is in $\mathscr{E}'$ and that the q_j are in C_0^{∞}. Then there are a positive number $C(f)$ and a real number s such that

$$|\hat{f}(\xi)|^{\dagger\dagger}\leq C(f)<\xi>^s.$$

Define u_l, $l=0, 1, 2, \cdots$, by

$$P_m(D)u_0(x)=f(x),$$

$$P_m(D)u_{l+1}(x)=-Q(x, D)u_l(x),$$

$$\operatorname{supp} u_l\subset\operatorname{supp} f+\Gamma(P_m, \vartheta)^*,$$

Where $\Gamma(P_m, \vartheta)^*$ is the dual cone of $\Gamma(P_m, \vartheta)$ which is the component of the set $\{\xi\in\boldsymbol{R}^n;\ P_m(\xi)\neq 0\}$ containing ϑ.

LEMMA 2.4. *Assume that the condition* (A) *is satisfied, the coefficients of* $P(x, D)$ *are in* $C_0^{\infty}(\boldsymbol{R}^n)$ *and that* f *is in* $\mathscr{E}'$ *and has its support in* $\{x\in\boldsymbol{R}^n;\ x\cdot\vartheta\geq 0\}$. *Then*

$$\text{(2.2)}\qquad |\hat{u}_l(\xi-i\gamma\vartheta)|\leq C'(f)C(P_m)e^{\gamma}\gamma^{-1}(C(P_m, Q, s)/\gamma)^l<\xi>^s,$$

$$l=0, 1, 2, \cdots, \gamma\geq 1.$$

Thus we can define $u(x)$ *in* $\mathscr{D}'(\boldsymbol{R}^n)$ *by*

$$\text{(2.3)}\qquad \hat{u}(\xi-i\gamma\vartheta)=\sum_{l=0}^{\infty}\hat{u}_l(\xi-i\gamma\vartheta),\quad \gamma>C(P_m, Q, s).$$

Then $u(x)$ *is a solution of* (2.1), $\operatorname{supp} u(x)\subset\operatorname{supp} f(x)+\Gamma(P_m, \vartheta)^*$ *and*

† $\boldsymbol{C}[\xi]$ denotes the space of polynomials of ξ with complex coefficients.

†† $\hat{f}(\xi)$ denotes the Fourier-Laplace transform of f.

$$|\hat{u}(\xi-i\gamma\vartheta)|\leq C'(f)C(P_m)e^{\gamma}\gamma^{-1}(1-C(P_m, Q, s)/\gamma)^{-1}<\xi>^s, \ \gamma>C(P_m, Q, s).$$

REMARK. We can also show that the right-hand side on (2.3) is convergent for $\gamma>C$, where C is independent of s. This can be proved by the same methods as in the proof of Lemma 2.8.

PROOF. We have

$$\hat{u}_{l+1}(\xi-i\gamma\vartheta)=-P_m(\xi-i\gamma\vartheta)^{-1}\mathscr{F}[Q(x, D)u_l](\xi-i\gamma\vartheta),$$

$$\mathscr{F}[Q(x, D)u_{l+1}](\xi-i\gamma\vartheta)$$
$$=-(2\pi)^{-n}\sum_j\hat{q}_j(\xi)*\{p_j(\xi-i\gamma\vartheta)P_m(\xi-i\gamma\vartheta)^{-1}\mathscr{F}[Q(x, D)u_l](\xi-i\gamma\vartheta)\},$$
$$l=-1, 0, 1, \cdots,$$

where $-Q(x, D)u_{-1}(x)=f(x)$. So the estimates (2.2) follow from Lemmas 2.1 and 2.2. Q.E.D.

THEOREM 2.5 (finite propagation property). *Assume that the condition* (A) *is satisfied. If a distribution u with support in $\{x\in\boldsymbol{R}^n;\ x\cdot\vartheta\geq 0\}$ satisfies the equation $P(x, D)u(x)=0$ in a neighborhood of $x^0-\Gamma(P_m, \vartheta)^*$, then $u=0$ in a neighborhood of x^0.*

PROOF. Let U be a neighborhood of x^0 such that $P(x, D)u(x)=0$ in a neighborhood of $U-\Gamma(P_m, \vartheta)^*$. We note that the tranposed operator ${}^tP(x, D)$ of $P(x, D)$ satisfies the condition (A). Choose ψ in C_0^∞ such that $\psi(x)=1$ in a neighborhood of $\{U-\Gamma(P_m, \vartheta)^*\}\cap\{x\cdot\vartheta\geq 0\}$. Then we have

$$(P_m(D)+\psi(x)Q(x, D))u(x)=0 \text{ in a neighborhood of } x^0-\Gamma(P_m, \vartheta)^*.$$

Since Lemma 2.4 is applicable to the operator ${}^t(P_m(D)+\psi(x)Q(x, D))$, to every ϕ in $C_0^\infty(U)$ there is a smooth function v such that

$${}^t(P_m(D)+\psi(x)Q(x, D))v(x)=\phi(x),$$

$$\operatorname{supp} v\subset\operatorname{supp}\phi-\Gamma(P_m, \vartheta)^*\subset U-\Gamma(P_m, \vartheta)^*.$$

Since ${}^t(P_m+Q)v=\phi$ in a neighborhood of $\{x\cdot\vartheta\geq 0\}$, we have

$$<u, \phi>=<u, {}^t(P_m+Q)v>=<(P_m+Q)u, v>=0.$$

This implies that $u=0$ in U. Q.E.D.

From Lemma 2.4 and Theorem 2.5 we have the following

THEOREM 2.6. *Under the condition* (A) *the equation* (2.1) *has a unique solution u in $\mathscr{D}'(\boldsymbol{R}^n)$.*

REMARK. (i) This theorem was proved in [3]. (ii) The Cauchy problem

$$(2.4)\qquad \begin{cases} P(x, D)u(x)=0, \ x_1>0, \\ D_1^{j-1}u|_{x_1=0}=g_j(x')\in\mathscr{D}'(\boldsymbol{R}^{n-1}), \ x'=(x_2, \cdots, x_n)\in\boldsymbol{R}^{n-1}, \ 1\leq j\leq m, \end{cases}$$

also has a unique solution in $C^\infty([0, \infty]; \mathscr{D}'(\boldsymbol{R}^{n-1}))$†. This easily follows from Theorem 2.6 and the following lemma.

LEMMA 2.7. *Assume that the hyperplane $x\cdot\vartheta=0$ is non-characteristic with respect to $P(x, D)$. Let $v\in\mathscr{D}'(\boldsymbol{R}^n)$ be a solution of the equation*

$$(2.5)\qquad P(x, D)v(x)=\mathscr{P}(g_1, \cdots, g_m),$$

$$\operatorname{supp} v\subset\{x\cdot\vartheta\geq 0\},$$

where the g_j are in $\mathscr{D}'(\boldsymbol{R}^{n-1})$ and

$$\mathscr{P}(g_1, \cdots, g_m)=\sum_{j=1}^m\sum_{k=1}^j i^{k-j-1}\delta^{(j-k)}(x_1)\otimes b_j(0, x', D')g_k(x'),$$

$$P(x, D)=\sum_{j=0}^m b_j(x,D')D_1^j, \ D'=i^{-1}(\partial/\partial x_2, \cdots, \partial/\partial x_n).$$

Then $u(x)\equiv v(x)|_{x_1>0}\in C^\infty([0, \infty); \mathscr{D}'(\boldsymbol{R}^{n-1}))$ is a solution af (2.4). Conversely $\tilde{u}(x)$ is a solution of (2.5) if $u(x)$ in $C^\infty([0, \infty); \mathscr{D}'(\boldsymbol{R}^{n-1}))$ is a solution of (2.4). Here $\tilde{u}(x)$ is the distribution defined by

$$(2.6)\qquad <\tilde{u}, \phi>=\int_0^\infty<u(x_1, x'), \phi(x_1, x')>_{x'}dx_1 \text{ for every } \phi \text{ in } C_0^\infty(\boldsymbol{R}^n).$$

REMARK. Since the hyperplane $x\cdot\vartheta=0$ is non-characteristic with respect to $P(x, D)$, it follows from partial hypoellipticity that a solution u in $\mathscr{D}'(\overline{\boldsymbol{R}_+^n})$ belongs to $C^\infty([0, \infty); \mathscr{D}'(\boldsymbol{R}^{n-1}))$††.

Next let us prove the energy inequalities. We assume that the coefficients of $P(x, D)$ are in $C_0^\infty(\boldsymbol{R}^n)$. Let $E(x, y)$ be a fundamental solution for $P(x, D)$ with support in $\{(x, y)\in\boldsymbol{R}^{2n};\ x-y\in\Gamma(P_m, \vartheta)^*\}$ and set

$$F(x, y;\ \gamma)=\exp[-\gamma(x_1-y_1)]E(x, y).$$

Define $F_l(x, y;\ \gamma)$, $l=0,1,2,\cdots$, by

$$(2.7)\qquad P_m(D_x-i\gamma\vartheta)F_0(x, y;\ \gamma)=\delta(x-y),$$

$$P_m(D_x-i\gamma\vartheta)F_{l+1}(x, y;\ \gamma)=-Q(x, D_x-i\gamma\vartheta)F_l(x, y;\ \gamma),$$

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† $f\in C^\infty([0,\infty); \mathscr{D}'(\boldsymbol{R}^{n-1}))$ implies that $<f(x_1, x'), \phi(x')>_{x'}$ is in $C^\infty([0,\infty))$ for each ϕ in $C_0^\infty(\boldsymbol{R}^{n-1})$.

†† $f\in\mathscr{D}'(\overline{\boldsymbol{R}_+^n})$ implies that there exists a ditribution F in $\mathscr{D}'(\boldsymbol{R}^n)$ such that $F|_{x_1>0}=f$. Moreover one can regard $C^\infty([0,\infty); \mathscr{D}'(\boldsymbol{R}^{n-1}))$ as a subspace of $\mathscr{D}'(\overline{\boldsymbol{R}_+^n})$.

$$\operatorname{supp} F_l(x, y; \gamma) \subset \{(x, y) \in \mathbf{R}^{2n};\ x-y \in \Gamma(P_m, \vartheta)^*\}.$$

Then by the same method as in Lemma 2.4 we have $F(x, y; \gamma)=\sum_{l=0}^{\infty} F_l(x, y; \gamma)$ in $\mathcal{S}'(\mathbf{R}^{2n})$ for sufficiently large γ. Denote by $\hat{F}(\xi, \eta; \gamma)$ the Fourier transform of $F(x, y,; \gamma)$ with respect to (x, y).

LEMMA 2.8. *Assume that the condition* (A) *is satisfied and that the q_j are in $C_0^\infty(\mathbf{R}^n)$. Then we have*

$$\hat{F}_0(\xi, \eta; \gamma)=(2\pi)^n P_m(\xi-i\gamma\vartheta)^{-1}\delta(\xi+\eta),$$

$$|\hat{F}_l(\xi, \eta; \gamma)| \le C(Q, N)(C(P_m, Q)/\gamma)^l (l-1)^N |P_m(\xi-i\gamma\vartheta)|^{-1} \langle \xi+\eta \rangle^{-N},$$

$$N=0,1,2,\cdots,\quad l=1,2,\cdots,\quad \gamma \ge 1.$$

Here $C(Q, N)$ depends on $\operatorname{supp} q_j$ *and the supremum norms of the derivatives of q_j of order $\le N+n+1$. Furthermore we have*

$$|\textstyle\sum_{l=1}^{\infty} \hat{F}_l(\xi, \eta; \gamma)| \le C(P_m, Q, \gamma, N) |P_m(\xi-i\gamma\vartheta)|^{-1} \langle \xi+\eta \rangle^{-N},$$
$$N=0,1,2,\cdots,\quad \textit{if } \gamma > C(P_m, Q).$$

PROOF. For $l=1,2,\cdots$ we have

$$\begin{aligned}
(2.8)\qquad \hat{F}_l(\xi, \eta; \gamma) = & \textstyle\sum_{j_1,\cdots,j_l,\pm} (-1)^l (2\pi)^{-(l-1)n} P_m(\xi-i\gamma\vartheta)^{-1} \\
& \times \int d\zeta^1 \hat{q}^{\pm}_{j_1 l}(\zeta^1; |\xi+\eta|) p_{j_1}(\xi-\zeta^1-i\gamma\vartheta) P_m(\xi-\zeta^1-i\gamma\vartheta)^{-1} \\
& \times \{ \int d\zeta^2 \cdots \\
& \times \{ \int d\zeta^{l-1} \hat{q}^{\pm}_{j_{l-1} l}(\zeta^{l-1}; |\xi+\eta|) p_{j_{l-1}}(\xi-\zeta^1-\cdots-\zeta^{l-1}-i\gamma\vartheta) \\
& \times P_m(\xi-\zeta^1-\cdots-\zeta^{l-1}-i\gamma\vartheta)^{-1} \hat{q}_{j_l}(\xi+\eta-\zeta^1-\cdots-\zeta^{l-1}) \\
& \times p_{j_l}(-\eta-i\gamma\vartheta) P_m(-\eta-i\gamma\vartheta)^{-1}\} \cdots \},
\end{aligned}$$

where

$$\hat{q}^{+}_{j l}(\zeta; s)=\chi_l(\zeta; s)\hat{q}_j(\zeta),$$

$$\hat{q}^{-}_{j l}(\zeta; s)=(1-\chi_l(\zeta; s))\hat{q}_j(\zeta),$$

$$\chi_l(\zeta; s)=\begin{cases} 1, & |\zeta| < s/2(l-1)^{-1}, \\ 0, & |\zeta| > s/2(l-1)^{-1}, \end{cases}$$

Set

$$I^+=\textstyle\sum_{j_1,\cdots,j_l,+} (\textit{each term on the right-hand side of } (2.8)),$$

$$I^-=\hat{F}_l(\xi, \eta; \gamma)-I^+.$$

Then it follows from Lemma 2.2 that

$$|I^+| \le C(Q, N)(C(P_m, Q)/\gamma)^l |P_m(\xi-i\gamma\vartheta)|^{-1} \langle \xi+\eta \rangle^{-N},$$

$$|I^-|\leq C(Q, N)(C(P_m, Q)/\gamma)^l(l-1)^N|P_m(\xi-i\gamma\vartheta)|^{-1}\langle\xi+\eta\rangle^{-N}.$$

In fact, the $\hat{q}_j(\xi)$ are rapidly decreasing and

$$|\xi+\eta-\zeta^1-\cdots-\zeta^{l-1}|\geq 1/2|\xi+\eta| \text{ if } \hat{q}^+_{j_kl}(\zeta^k;\ |\xi+\eta|)\neq 0,\ 1\leq k\leq l-1.$$

Q.E.D.

LEMMA 2.9. *Let $P(x, D)$ be as described in Lemma 2.8 and $R(\xi)$ a polynomial such that $R\prec P_m$ and $\deg R<m$. Then to every real s there are positive numbers $C\equiv C(P_m, Q, R, s)$ and $C'(P_m, Q)$ such that for every f in $C_0^\infty(\boldsymbol{R}^n)$*

$$||\exp[-\gamma x_1]R(D_x)\int E(x, y)f(y)dy||_s$$
$$\leq C(P_m, Q, R, s)\gamma^{-1}||\exp[-\gamma x_1]f(x)||_s \text{ if } \gamma>C'(P_m, Q),$$

where

$$||f||_s=(\int\langle\xi\rangle^{2s}|\hat{f}(\xi)|^2d\xi)^{1/2}.$$

Here $C(P_m, Q, R, s)$ depends on P_m, R, s, $\operatorname{supp} q_j$ and the supremum norms of the derivatives of q_j.

REMARK. With a simple modification of Lemma 2.9 we have

$$||\exp[-\gamma x_1]\int E(x, y)f(y)dy||_{p,\tilde{P}k}\leq C(\gamma)||\exp[-\gamma x_1]f(x)||_{p,k},$$

where $k\in\mathscr{K}$ and $1\leq p\leq\infty$ (see § 2.2 in [4])†.

PROOF. It is obvious that

$$\exp[-\gamma x_1]R(D_x)\int E(x, y)f(y)dy \tag{2.9}$$
$$=\mathscr{F}_\xi^{-1}[(2\pi)^{-n}R(\xi-i\gamma\vartheta)\langle\hat{F}(\xi, -\eta;\gamma),\ \hat{h}(\eta;\gamma)\rangle_\eta],$$

where $\hat{h}(\eta;\gamma)=\mathscr{F}_y[\exp[-\gamma y_1]f(y)](\eta)$. From Lemma 2.8 it follows that

$$||(2\pi)^{-n}\langle\xi\rangle^sR(\xi-i\gamma\vartheta)\langle\hat{F}(\xi, -\eta;\gamma),\ \hat{h}(\eta;\gamma)\rangle_\eta||_{L^2_\xi}$$
$$\leq C'(P_m, Q, R, s)\gamma^{-1}(||\langle\xi\rangle^s\hat{h}(\xi;\gamma)||_{L^2}$$
$$+||\langle\langle\xi\rangle^s\langle\xi-\eta\rangle^{-n-1-|s|}\langle\eta\rangle^{-s},\ \langle\eta\rangle^s\hat{h}(\eta;\gamma)\rangle_\eta||_{L^2_\xi})$$
$$\leq C(P_m, Q, R, s)\gamma^{-1}||\exp[-\gamma x_1]f(x)||_s.$$

Here we have used Hausdorff-Young's inequality and the inequlity

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† $k\in\mathscr{K}$ means that k is a positive function defined in $\boldsymbol{R}^n$ and that there exist positive numbers C and N such that $k(\xi+\eta)\leq(1+C|\xi|)^Nk(\eta)$ for ξ, η in $\boldsymbol{R}^n$. Moreover we denote for $k\in\mathscr{K}$ and $1\leq p\leq\infty$

$$||f||_{p,k}=||(2\pi)^{-n/p}k(\xi)\hat{f}(\xi)||_{L^p},\ f\in\mathscr{S}'.$$

$$\langle\xi\rangle^{s}\langle\eta\rangle^{-s}\leq 2^{|s|/2}\langle\xi-\eta\rangle^{|s|}. \qquad \text{Q.E.D.}$$

THEOREM 2.10. *Assume that the condition* (A) *is satisfied and that the coefficients of $P(x, D)$ are in $\mathscr{B}(\boldsymbol{R}^{n})$. Let R be a polynomial such that $R\prec P_{m}$. Then for any positive T and any non-negative integer s there is a positive number $C\equiv C(R, T, s)$ such that for every $u\in C_{0}^{\infty}(\boldsymbol{R}^{n})$ with support in $\{x_{1}\geq 0\}$*

$$||R(D)u||_{s,T}\leq C(R, T, s)||P(x, D)u||_{s,T},$$

where

$$||u||^{2}_{s,T}=\sum_{|\alpha|\leq s}\int_{x_{1}\leq T}|D^{\alpha}u(x)|^{2}dx.$$

PROOF. Using partition of unity it follows from Theorem 2.5 and Lemma 2.9 that

$$||R(D)u||_{s,T}\leq C(R, T, s)||P(x, D)u||_{s}.$$

Let f be a function such that $P(x, D)u(x)=f(x)$ when $x_{1}<T$ and $\langle\xi\rangle^{s}\hat{f}(\xi)\in L^{2}(\boldsymbol{R}^{n})$. Then $u(x)=w(x)$ when $x_{1}<T$ if $w(x)$ is a solution of $P(x, D)w=f$ and $\operatorname{supp} w\subset\{x_{1}\geq 0\}$. Thus

$$||R(D)u||_{s,T}\leq C(R, T, s)\inf\{||f||_{s};\ P(x, D)u(x)=f(x) \text{ when } x_{1}<T\}.$$

This completes the proof. Q.E.D.

THEOREM 2.11. *Let $P(x, D)$ be as described in Theorem 2.10 and let $u(x)$ be a solution of* (2.4). *Then for any positive number T and any non-negative integer s there is a positive number $C\equiv C(T, s)$ such that for every $g_{k}(x')\in C_{0}^{\infty}(\boldsymbol{R}^{n-1})$, $1\leq k\leq m$, we have*

$$||u||_{(0,s),T}\leq C(T, s)\sum_{k=1}^{m}||g_{k}||_{s+m-k},$$

where

$$||u||^{2}_{(0,s),T}=\int_{0}^{T}dx_{1}\int d\xi'\langle\xi'\rangle^{2s}|\mathscr{F}_{x'}[u(x_{1}, x')](\xi')|^{2}.$$

REMARK. From partial hypoellipticity we have some results on the regularity of solutions (see Theorem 4.3.1 in [4]).

PROOF. Assume that the coefficients of $P(x, D)$ are in $C_{0}^{\infty}(\boldsymbol{R}^{n})$. $\tilde{u}(x)$ defined by (2.6) satisfies the equation

$$P(x, D)\tilde{u}(x)=f(x),\ \operatorname{supp}\tilde{u}\subset\{x_{1}\geq 0\},$$

where

$$f(x)=\sum_{j=1}^{m}\sum_{k=1}^{j} i^{k-j-1}\delta^{(j-k)}(x_1)\otimes b_j(0, x', D')g_k(x').$$

Since

$$\mathscr{F}[\exp[-\gamma x_1]f(x)]=\hat{f}(\xi)$$
$$=\sum_{j=1}^{m}\sum_{k=1}^{j} i^{-1}\xi_1^{j-k}\mathscr{F}_{x'}[b_j(0, x'\ D')g_k(x')](\xi'),$$

it follows from (2.9) with $R(\xi)=1$ and Lemma 2.8 that

$$\begin{aligned}&||\exp[-\gamma x_1]\langle E(x, y), f(y)\rangle_y||_{(0,s)}\\ &\le C'(P_m, Q, s)\sum_{j=1}^{m}\sum_{k=1}^{j}\{||\gamma^{-m+1+j-k}(\xi_1-i\gamma-\lambda_m(\xi'))^{-1}\langle\xi'\rangle^s\\ &\quad\times\Pi_{\nu=1}^{j-k}(1+|\lambda_\nu(\xi')|/\gamma)\mathscr{F}_{x'}[b_j(0, x', D')g_k(x')](\xi')||_{L^2_\xi}\\ &\quad+||\gamma^{-m+1+j-k}(\xi_1-i\gamma-\lambda_m(\xi'))^{-1}\langle\xi_1\rangle^{-j+k}\langle\xi'\rangle^s\ \Pi_{\nu=1}^{j-k}(1+|\lambda_\nu(\xi')|/\gamma)\\ &\quad\times\langle\langle\xi-\eta\rangle^{-2j+2k-n-2-|s|}, \langle\eta_1\rangle^{j-k}\mathscr{F}_{x'}[b_j(0, x', D')g_k(x')](\eta')\rangle_\eta||_{L^2_\xi}\}\\ &\quad \textit{if } \gamma>C'(P_m, Q),\end{aligned}$$

where

$$||f||^2_{(0,s)}=\int\langle\xi'\rangle^{2s}|\hat{f}(\xi)|^2d\xi.$$

Here we have used the fact that

$$|\xi_1(\xi_1-i\gamma-\lambda_j(\xi'))^{-1}|\le 1+|\lambda_j(\xi')|/\gamma,$$

where

$$P_m(\xi-i\gamma\vartheta)=\Pi_{j=1}^{m}(\xi_1-i\gamma-\lambda_j(\xi')).$$

Since

$$1+|\lambda_j(\xi')|/\gamma\le C(1+\langle\xi'\rangle/\gamma),$$
$$\langle\xi_1\rangle^{-j+k}\langle\xi'\rangle^s\langle\xi-\eta\rangle^{-j+k-|s|}\langle\eta_1\rangle^{j-k}\langle\eta'\rangle^{-s}\le C$$

and

$$\int_{-\infty}^{\infty}|\xi_1-i\gamma-\lambda_m(\xi')|^{-2}d\xi_1\le C/\gamma,$$

we have

$$\begin{aligned}(2.10)\qquad &||\exp[-\gamma x_1]\langle E(x, y), f(y)\rangle_y||_{(0,s)}\\ &\le C''(P_m, Q, s)\sum_{j=1}^{m}\sum_{k=1}^{j}\sum_{\nu=0}^{j-k}\gamma^{-m+1+j-k-\nu-1/2}||g_k||_{s+\nu+m-j}\\ &\le C(P_m, Q, s)\sum_{k=1}^{m}\gamma^{1/2-k}||g_k||_{s+m-k} \text{ if } \gamma>C'(P_m, Q).\end{aligned}$$

In fact,

$$||b_j(0, x', D')g_k(x')||_s\le C(b_j, s)||g_k||_{s+m-j},$$

$$\|\langle\xi'\rangle^{-n} \underset{\xi'}{*} \langle\xi'\rangle^{\nu+s}\mathscr{F}_{x'}[b_j(0, x', D')g_k(x')](\xi')\|_{L^2_{\xi'}} \leq C'(b_j, s)\|g_k\|_{s+\nu+m-j}.$$

Note that $C(P_m, Q, s)$ has the same properties as one in Lemma 2.9. Therefore (2.10) proves Theorem 2.11, using Theorem 2.5 and partition of unity.

Q.E.D.

3. Wawe front sets of solutions

3.1. Localization theorem. An inner estimate of the wave front set of the fundamental solution $E(x, y)$ for $P(x, D)$ can be given by Localization theorem. For each ξ^0 in $\boldsymbol{R}^n$ we can write $P(x, t\xi^0+\xi)$ in the from

$$P(x, t\xi^0+\xi)=t^{r_{\xi^0}}P_{\xi^0}(x, \xi)+\sum_{j=1}^{\infty}t^{r_{\xi^0}-j}R^j_{\xi^0}(x, \xi),$$

$$R^j_{\xi^0}(x, \xi)\equiv 0 \text{ if } j>r_{\xi^0},$$

where r_{ξ^0} is a non-negative integer, $P_{\xi^0}(x, \xi)$ is the localization of $P(x, \xi)$ at ξ^0 and the $R^j_{\xi^0}(x, \xi)$ are polynomials of ξ. Then $P_{\xi^0}(x, \xi)$ also satisfies the condition (A) if $P(x, \xi)$ satisfies the condition (A) (see [1]). Assume without loss of generality that the coefficients of $P(x, D)$ are in $C_0^\infty(\boldsymbol{R}^n)$. Define $E_j(x, y; \xi^0)$, $j=0, 1, 2, \cdots$, by the equations

$$P_{\xi^0}(x, D)E_0(x, y; \xi^0)=\delta(x-y),$$

$$P_{\xi^0}(x, D)E_j(x, y; \xi^0)=-\sum_{k=1}^{\infty}R^k_{\xi^0}(x, D)E_{j-k}(x, y; \xi^0),$$

$$\operatorname{supp} E_j\subset\{(x, y)\in\boldsymbol{R}^n;\ x-y\in\Gamma(P_{m\xi^0}, \vartheta)^*\},$$

where $E_j(x, y; \xi^0)\equiv 0$ if $j<0$. Then it follows that

$$P(x, D_x+t\xi^0)G_N(x, y; t, \xi^0)=t^{r_{\xi^0}-N-1}F^N(x, y; t, \xi^0),$$

where

$$G^N(x, y; t, \xi^0)=t^{r_{\xi^0}}\exp[-it(x-y)\cdot\xi^0]\,E(x,y)-\sum_{j=0}^{N}t^{-j}E_j(x, y; \xi^0),$$

$$F_N(x, y; t, \xi^0)=-\sum_{k=1}^{\infty}\sum_{i=1}^{k-1}t^{-j}R^k_{\xi^0}(x, D)E_{N+1+j-k}(x, y; \xi^0)$$

and $E(x, y)$ is the fundamental solution for $P(x, D)$. By the same method as in the proof of Lemma 2.4 (or Lemma 2.8) we have

$$|\mathscr{F}_x[\exp[-\gamma(x, y)]E_j(x, y; \xi^0)]|\leq C(\gamma, j, \xi^0)\langle\xi\rangle^{j(m-r_{\xi^0}+1)},$$

$$j=0, 1, 2, \cdots, \gamma>C(P_{\xi^0}).$$

In fact, $\deg R^k_{\xi^0}(x, \xi)=m-r_{\xi^0}+k$. Thus we have

$$|\mathscr{F}_x[\exp[-\gamma(x-y)]F_N(x, y; t, \xi^0)]|\leq C(\gamma, N, \xi^0)\langle\xi\rangle^{(N+r_{\xi^0})(m-r_{\xi^0}+1)},$$

$$N=0, 1, 2, \cdots, \gamma>C(P_{\xi^0}),\ t\geq 1.$$

This implies that

$$|\mathcal{F}_x[\exp[-\gamma(x-y)]G_N(x, y; t, \xi^0)]|\leq C(\gamma, N, \xi^0)t^{r_{\xi^0}-N-1}$$
$$\times\langle\xi\rangle^{(N+r_{\xi^0})(m-r_{\xi^0}+1)},\ N=0,1,2,\cdots,\gamma>C(P),\ t\geq 1.$$

In fact, this follows from the proof of Lemma 2.4 (or Lemma 2.8) and the fact that

$$|p_j(\xi+t\xi^0-i\gamma\vartheta)/P_m(\xi+t\xi^0-i\gamma\vartheta)|\leq C(P_m, p_j)/\gamma \text{ if } \gamma\geq 1.$$

Therefore we have the following

Theorem 3.1. *Assume that the condition* (A) *is satisfied and* $\xi^0\in\boldsymbol{R}^n$. *Then we have*

$$t^{N-r_{\xi^0}}G_N(x, y; t, \xi^0)\to 0, \text{ as } t\to\infty, \text{ in } \mathcal{D}'(\boldsymbol{R}^{2n}),\ N=0,1,2,\cdots.$$

Moreover we have

$$\cup_{j=0}^{\infty}\{((x, y), (\xi^0, -\xi^0))\in T^*\boldsymbol{R}^{2n}\backslash 0;\ (x, y)\in \operatorname{supp} E_j(\cdot, \cdot; \xi^0)\}$$
$$\subset WF(E(x, y)) \text{ for } \xi^0\neq 0$$

and

$$\mathrm{ch}[\operatorname{supp} E_j(\cdot, \cdot; \xi^0)]^\dagger\subset\{(x, y)\in\boldsymbol{R}^{2n};\ x-y\in\Gamma(P_{m\xi^0}, \vartheta)^*\}.$$

Remark. We can prove by using Seidenberg's lemma that there is a real number δ such that

$$|P_m(\xi+t\xi^0-i\gamma\vartheta)|^{-1}\leq C(P_m)t^{-r_{\xi^0}}\langle\xi-i\gamma\vartheta\rangle^{\delta} \text{ if } t\geq 1,\ \gamma\geq 1.$$

Therefore we can obtain a more precise result

$$t^N G_N(x, y; t, \xi^0)\to 0, \text{ as } t\to\infty, \text{ in } \mathcal{D}'(\boldsymbol{R}^{2n}),\ N=0,1,2,\cdots.$$

3.2. Outer estimate. We shall prove the following theorem in this section.

Theorem 3.2. *Under the condition* (A) *we have*

$$WF(E(x, y))\subset\{((x, y), (\xi, \eta))\in T^*\boldsymbol{R}^{2n}\backslash 0;\ ((x, \xi), (y, -\eta))\in C\},$$

where $E(x, y)$ *is the fundamental solution for* $P(x, D)$ *and*

$$C=\{((x, \xi), (y, \eta))\in T^*\boldsymbol{R}^n\times T^*\boldsymbol{R}^n\backslash 0;\ \xi=\eta \text{ and } x-y\in\Gamma(P_{m\xi}, \vartheta)^*\}.$$

Corollary. *Assume that the condition* (A) *is satisfied and let* $u(x)$ *be a solution of* (2.1). *Then we have*

† $\mathrm{ch}[M]$ denotes the convex hull of M.

$$WF(u)\subset C\circ WF(f)\equiv\{(x,\xi)\in T^*\boldsymbol{R}^n\backslash 0;\ ((x,\xi),(y,\eta))\in C \text{ for some } (y,\eta)\in WF(f)\}$$

Let x^0, $y^0\in\boldsymbol{R}^n$ and $\xi^0\in\boldsymbol{R}^n\backslash\{0\}$. First we shall prove that $((x^0, y^0), (\xi^0, -\xi^0))\notin WF(E(x, y))$ when $x^0-y^0\notin\Gamma(P_{m\xi^0}, \vartheta)^*$. Therefore we assume that $x^0-y^0\notin\Gamma(P_{m\xi^0}, \vartheta)^*$. Then there exist a neighborhood U_1 of x^0, a neighborhood U_2 of y^0 and η^0 in $\Gamma(P_{m\xi^0}, \vartheta)$ such that

$$(x-y)\cdot\eta^0<0 \text{ for every } x\in U_1 \text{ and } y\in U_2. \tag{3.1}$$

LEMMA 3.3. *There are an open convex conic neighborhood Γ of ξ^0 and positive numbers δ and t_0 such that*

$$P_m(\xi-i(t|\xi|\zeta+\vartheta))\neq 0 \text{ when } 0<t\leq t_0,\ \xi\in\Gamma,\ \zeta\in\boldsymbol{R}^n \text{ and } |\zeta-\eta^0|\leq\delta.$$

PROOF. $M\equiv\mathrm{ch}[\{\vartheta\}\cup\{\zeta\in\boldsymbol{R}^n;\ |\zeta-\eta^0|\leq\delta\}]$ is contained in $\Gamma(P_{m\xi^0}, \vartheta)$ if δ is sufficiently small. Since M is compact, it follows from Lemma 5.1 in [1] that there exist a neighborhood V of ξ^0 and a positive number t_1 such that

$$P_m(\xi-it\zeta)\neq 0 \text{ when } \xi\in V,\ \zeta\in M \text{ and } 0<t\leq t_1. \tag{3.2}$$

We can assume without loss of generality that $|\xi^0|=1$. Set $\Gamma=\{\lambda\xi;\ |\xi|=1,\ \xi\in V$ and $\lambda>0\}$. Then we may assume that Γ is an open convex cone. (3.2) implies that for every positive number $t_0<t_1$

$$P_m(\xi-i(t|\xi|\zeta+\vartheta))\neq 0 \text{ when } \xi\in\Gamma,\ \zeta\in\boldsymbol{R}^n,\ |\xi|\geq 1/(t_1-t_0),\ |\zeta-\eta^0|\leq\delta \text{ and } 0\leq t\leq t_0.$$

In fact, we have $0<t+|\xi|^{-1}\leq t_1$, $t/(t+|\xi|^{-1})\zeta+|\xi|^{-1}/(t+|\xi|^{-1})\vartheta\in M$ and $|\xi|^{-1}\xi\in V$. Choose t_0 so small that $(t_1-t_0)\vartheta+t_0\zeta\in\Gamma(P_m, \vartheta)$ when $\zeta\in\boldsymbol{R}^n$ and $|\zeta-\eta^0|\leq\delta$. Then it follows from hyperbolicity of $P_m(\xi)$ that $P_m(\xi-i(t|\xi|\zeta+\vartheta))\neq 0$ when $\xi\in\Gamma$, $\zeta\in\boldsymbol{R}^n$, $|\xi|<1/(t_1-t_0)$, $|\zeta-\eta^0|\leq\delta$ and $0\leq t\leq t_0$. Q.E.D.

LEMMA 3.4. *Let $\xi\in\Gamma$ and $0\leq t\leq t_0$. Then we have*

$$\tilde{P}_m(\xi-i(t|\xi|\eta^0+\vartheta))\leq C|P_m(\xi-i(t|\xi|\eta^0+\vartheta))|, \tag{3.3}$$

$$\tilde{P}_m(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta))\leq C|P_m(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta))| \text{ if } t|\xi|>0 \tag{3.4}$$

PROOF. Let ρ be a positive number such that $\rho\eta^0+\vartheta\in\Gamma(P_m, \vartheta)$. Since $t|\xi|\eta^0+\vartheta$ is contained in some compact subset of $\Gamma(P_m, \vartheta)$ if $t|\xi|\leq\rho$, there is a positive number c such that $d(\xi)\equiv\mathrm{distance}(0, \{z\in\boldsymbol{C}^n;\ P_m(z+\xi-i(t|\xi|\eta^0+\vartheta))=0\})>c$ when $t|\xi|\leq\rho$. So (3.3) follows from Lemma 4.1.1 in [4] if $t|\xi|\leq\rho$. Modifying Γ and

t_0 if necessary, it follows from Lemma 3.3 that

$$P_m(\xi+t|\xi|\mathrm{Re}\,\zeta-i(t|\xi|(\eta^0-\mathrm{Im}\,\zeta)+\vartheta))\neq 0$$

$$\textit{when } \xi\in\Gamma,\ \zeta\in\boldsymbol{C}^n,\ |\zeta|\leq\delta \textit{ and } 0\leq t\leq t_0.$$

Thus we have $d(\xi)\geq t|\xi|\delta$ if $\xi\in\Gamma$ and $0\leq t\leq t_0$, From Lemma 4.1.1 in [4] it follows that

$$(3.5)\qquad |P_m^{(\alpha)}(\xi-i(t|\xi|\eta^0+\vartheta))|\leq(C_m/(t|\xi|\delta))^{|\alpha|}|P_m(\xi-i(t|\xi|\eta^0+\vartheta))|$$

$$\textit{if } \xi\in\Gamma,\ t|\xi|>0 \textit{ and } 0\leq t\leq t_0.$$

This proves (3.3) with $t|\xi|>\rho$. Multiplying (3.5) by $(t|\xi|)^{|\alpha|-m}$ we have (3.4).

Q.E.D.

Lemma 3.5. *Let p be a polynomial such that $p\prec P_m$ and* $\deg p<m$. Then we have

$$|p(\xi-i(t|\xi|\eta^0+\gamma\vartheta))P_m(\xi-i(t|\xi|\eta^0+\gamma\vartheta))^{-1}|\leq C(P_m,p)/\gamma$$

$$\textit{when } \xi\in\Gamma,\ 0\leq t\leq t_0 \textit{ and } \gamma\geq 1.$$

Proof. Let ρ be a fixed positive number. If $t|\xi|\leq\rho$, we have

$$\begin{aligned}\tilde{p}(\xi-i(t|\xi|\eta^0+\vartheta))&\leq C(1+C|t|\xi|\eta^0+\vartheta|)^{m+j}\tilde{P}_m(\xi-i(t|\xi|\eta^0+\vartheta))\\ &\leq C'\tilde{P}_m(\xi-i(t|\xi|\eta^0+\vartheta)),\end{aligned}$$

where $\deg p=j$ (see (2.1.10) in [4]). Thus (3.3) implies that

$$(3.6)\qquad |p(\xi-i(t|\xi|\eta^0+\vartheta))|\leq C(P_m,p)|P_m(\xi-i(t|\xi|\eta^0+\vartheta))|$$

$$\textit{if } \xi\in\Gamma,\ t|\xi|\leq\rho \textit{ and } 0\leq t\leq t_0.$$

If $t|\xi|>\rho$, we have

$$\begin{aligned}&\tilde{p}(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta))\\ &\leq C(1+C|\eta^0+t^{-1}|\xi|^{-1}\vartheta|)^{m+j}\tilde{P}_m(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta))\\ &\leq C'\tilde{P}_m(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta)).\end{aligned}$$

Thus (3.4) implies that

$$\begin{aligned}(3.7)\qquad &|p(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta))|\\ &\leq C'(P_m,p)|P_m(t^{-1}|\xi|^{-1}\xi-i(\eta^0+t^{-1}|\xi|^{-1}\vartheta))|\end{aligned}$$

$$\textit{if } \xi\in\Gamma,\ t|\xi|>\rho \textit{ and } 0\leq t\leq t_0.$$

Now we may assume without loss of generality that p is homogeneous (see Lemma 5.5.1 in [4]). Multiplying (3.7) by $(t|\xi|)^m$ we have

(3.8) $$|p(\xi-i(t|\xi|\eta^0+\vartheta))|\leq C(P_m, p)|P_m(\xi-i(t|\xi|\eta^0+\vartheta))|$$
$$\text{if } \xi\in\Gamma,\ t|\xi|>\rho \text{ and } 0\leq t\leq t_0.$$

(3.6) and (3.8) imply that if $\xi\in\Gamma$, $0\leq t\leq t_0$ and $\gamma\geq 1$,

$$|p(\xi-i(t|\xi|\eta^0+\gamma\vartheta))|\leq C(P_m, p)\gamma^{j-m}|P_m(\xi-i(t|\xi|\eta^0+\gamma\vartheta))|,$$

which proves Lemma 3.5. Q.E.D.

Proof of the theorem. We can asume without loss of generality that the coefficients of $P(x, D)$ are in C_0^∞. Let $F_l(x, y; \gamma)$, $l=0, 1, 2, \cdots$, be distributions defined by (2.7) and $\phi_j\in C_0^\infty(U_j)$, $j=1, 2$. Set $\phi(x, y)=\phi_1(x)\phi_2(y)$ and choose a positive number ε and a conic neighborhood Γ^1 of ξ^0 such that $1/2\,\xi+\zeta\in\Gamma$ if $\xi\in\Gamma^1$ and $|\zeta|<\varepsilon|\xi|$. Then we have

(3.9)
$$\mathscr{F}_{(x,y)}[\phi F_l](\xi, \eta)=\sum_{j_1,\cdots,j_l,\pm}(-1)^l(2\pi)^{-(1+l)n}$$
$$\times\int d\zeta^0\hat{\phi}_1(1/2\,\xi-\zeta^0)P_m(1/2\,\xi+\zeta^0-i\gamma\vartheta)^{-1}$$
$$\times\{\int d\zeta^1\,\hat{q}^{\varepsilon\pm}_{j_1 l}(\zeta^1; |\xi|)p_{j_1}(1/2\,\xi+\zeta^0-\zeta^1-i\gamma\vartheta)P_m(1/2\,\xi+\zeta^0-\zeta^1-i\gamma\vartheta)^{-1}$$
$$\times\{\cdots$$
$$\times\{\int d\zeta^l\,\hat{q}^{\varepsilon\pm}_{j_l l}(\zeta^l; |\xi|)p_{j_l}(1/2\,\xi+\zeta^0-\zeta^1-\cdots-\zeta^l-i\gamma\vartheta)$$
$$\times P_m(1/2\,\xi+\zeta^0-\zeta^1-\cdots-\zeta^l-i\gamma\vartheta)^{-1}\hat{\phi}_2(1/2\,\xi+\eta+\zeta^0-\zeta^1-\cdots-\zeta^l)\}\cdots\},$$

where $\hat{q}^{\varepsilon+}_{jl}(\zeta; s)=\chi^{\varepsilon}_l(\zeta; s)\hat{q}_j(\zeta)$, $\hat{q}^{\varepsilon-}_{jl}(\zeta; s)=(1-\chi^{\varepsilon}_l(\zeta; s))\hat{q}_j(\zeta)$ and $\chi^{\varepsilon}_l(\zeta; s)=1$ if $|\zeta|<\varepsilon s/l$, $=0$ if $|\zeta|>\varepsilon s/l$. Set

$$I^+=\sum_{j_1,\cdots,j_l,+}(\textit{each term on the right-hand side of } (3.9)),$$
$$I^-=\mathscr{F}_{(x,y)}[\phi F_l](\xi, \eta)-I^+.$$

Then it follows from Lemma 2.2 that

(3.10) $$|I^-|\leq C(Q)C(P_m)\gamma^{-1}l^N(C(P_m, Q)/\gamma)^l|\phi_1|_{n+1}|\phi_2|_0|Q|_{N+n+1}\langle\xi\rangle^{-N},$$
$$N=0, 1, 2, \cdots,\ \gamma\geq 1,$$

where $|Q|_k=\sup_j|q_j|_k$ and $|f|_k=\sup_{|\alpha|\leq k, x\in R^n}|D^\alpha f(x)|$. Here $C(Q)$ depends on supp q_j and $|Q|_{n+1}$. Next let us estimate I^+. Let V_t, $0\leq t\leq t_0$, be the chain $\{\zeta^0\in \boldsymbol{C}^n; \zeta^0=\zeta-iv_t(\zeta), \zeta\in\boldsymbol{R}^n\}$, where $v_t(\zeta)=t|\zeta|\Phi(\zeta)\eta^0$ and $\Phi(\zeta)\in C^\infty(\boldsymbol{R}^n)$ is positively homogeneous of degree 0 in $|\zeta|\geq 1$ and $\Phi(\zeta)=1$ on a neighborhood of $\Gamma^1\cap\{|\zeta|\geq 1\}$, supp $\Phi(\zeta)\subset\Gamma\cap\{|\zeta|>1/2\}$, $0\leq\Phi(\zeta)\leq 1$. It follows from (3.1) that

(3.11) $$|\hat{\phi}_1(1/2\,\xi-\zeta^0)\hat{\phi}_2(1/2\,\xi+\eta+\zeta^0-\zeta^1-\cdots-\zeta^l)|\le C|\phi_1|_N|\phi_2|_0\langle 1/2\,\xi-\zeta^0\rangle^{-N},$$
$$N=0,1,2,\cdots,\ \text{if}\ \zeta^0\in V_t,\ 0\le t\le t_0.$$

It is obvious that there is a positive number δ such that

(3.12) $$|1/2\,\xi-\zeta^0|^2\ge\delta^2(|\xi|^2+|\zeta|^2)\ \ if\ \xi\in\Gamma^1\ and\ \zeta^0=\zeta-iv_{t_0}(\zeta).$$

Moreover Lemmas 2.2 and 3.5 show that

(3.13) $$|p_j(1/2\,\xi+\zeta^0+z-i\gamma\vartheta)P_m(1/2\,\xi+\zeta^0+z-i\gamma\vartheta)^{-1}|\le C(P_m,\,p_j)/\gamma$$

if $\xi\in\Gamma^1$, $z\in\boldsymbol{R}^n$, $\zeta^0\in V_t$, $|z|<\varepsilon|\xi|$, $0\le t\le t_0$ and $\gamma\ge1$. In fact we have $1/2\,\xi+\zeta+z\in\Gamma$ and $|\zeta|\Phi(\zeta)\le|1/2\,\xi+\zeta+z|$ if $\xi\in\Gamma^1$, $\zeta\in\Gamma$, $z\in\boldsymbol{R}^n$ and $|z|<\varepsilon|\xi|$, modifying Γ. From (3.11), (3.13) and Stokes' formula it follows that

$$I^+=\sum_{j_1,\cdots,j_l}(-1)^l(2\pi)^{-(1+l)n}\int_{V_{t_0}}d\zeta^0\int_{\boldsymbol{R}^n}d\zeta^1\cdots\int_{\boldsymbol{R}^n}d\zeta^l\cdots.$$

Therefore (3.11)—(3.13) give the estimates

(3.14) $$|I^+|\le C(Q)C(P_m)\gamma^{-1}(C(P_m,\,Q)/\gamma)^l|\phi_1|_{N+n+1}|\phi_2|_0\langle\xi\rangle^{-N},$$
$$N=0,1,2,\cdots,\ \text{if}\ \xi\in\Gamma^1\ \text{and}\ \gamma\ge1.$$

From (3.10) and (3.14) it follows that

(3.15) $$|\mathscr{F}_{(x,y)}[\phi F](\xi,\,\eta)|\le C(P_m,\,Q,\,\gamma)(|\phi|_{n+1}|Q|_{N+n+1}+|\phi|_{N+n+1})\langle\xi\rangle^{-N},$$
$$N=0,1,2,\cdots,\ \text{if}\ \xi\in\Gamma^1\ \text{and}\ \gamma>C(P_m,\,Q),$$

which implies that $((x^0,\,y^0),\,(\xi^0,\,\eta))\notin WF(E(x,\,y))$ when $x^0-y^0\notin\Gamma(P_{m\xi^0},\,\vartheta)^*$. Finally let us prove that $((x^0,\,y^0),\,(\xi^0,\,\eta^1))\notin WF(E(x,\,y))$ if $\xi^0\neq-\eta^1$. Here we do not suppose that $x^0-y^0\notin\Gamma(P_{m\xi^0},\,\vartheta)^*$. Let $\tilde{\Gamma}$ be a conic neighborhood of $(\xi^0,\,\eta^1)$ in $\boldsymbol{R}^{2n}\backslash\{0\}$. Then there is a positive number ε such that $|\xi+\eta|\ge2\varepsilon|(\xi,\,\eta)|$ for $(\xi,\,\eta)\in\tilde{\Gamma}$. Since

(3.16) $$\mathscr{F}_{(x,y)}[\phi F_l](\xi,\,\eta)=\sum_{j_1,\cdots,j_l,\pm}(-1)^l(2\pi)^{-(1+l)n}$$
$$\times\int d\zeta^0\hat{\phi}_1(\xi+\eta-\zeta^0)P_m(\zeta^0-\eta-i\gamma\vartheta)^{-1}$$
$$\times\Big\{\int d\zeta^1\hat{q}^{\varepsilon\pm}_{j_1 l}(\zeta^1;\,|(\xi,\,\eta)|)p_{j_1}(\zeta^0-\eta-\zeta^1-i\gamma\vartheta)P_m(\zeta^0-\eta-\zeta^1-i\gamma\vartheta)^{-1}$$
$$\times\Big\{\int\cdots$$
$$\times\Big\{\int d\zeta^l\hat{q}^{\varepsilon\pm}_{j_l l}(\zeta^l;\,|(\xi,\,\eta)|)p_{j_l}(\zeta^0-\eta-\zeta^1-\cdots-\zeta^l-i\gamma\vartheta)$$
$$\times P_m(\zeta^0-\eta-\zeta^1-\cdots-\zeta^l-i\gamma\vartheta)^{-1}\hat{\phi}_2(\zeta^0-\zeta^1-\cdots-\zeta^l)\Big\}\cdots\Big\},$$

it follows from the application of the same arguments as in the proof of Lemma 2.8 that

(3.17) $$|\mathscr{F}_{(x,y)}[\phi F](\xi,\eta)|\leq C(P_m, Q, \gamma)(|\phi|_{n+1}|Q|_{N+n+1}+|\phi|_{N+n+1})\langle(\xi,\eta)\rangle^{-N},$$
$$N=0,1,2,\cdots,\ \text{if } (\xi,\eta)\in\tilde{\Gamma} \text{ and } \gamma>C(P_m, Q).$$

In fact we have

$$|\hat{\phi}_1(\xi+\eta-\zeta^0)\hat{\phi}_2(\zeta^0-z)|\leq C|\phi|_{N+n+1}\langle\xi+\eta\rangle^{-N}\langle\zeta^0\rangle^{-n-1},$$
$$N=0,1,2,\cdots,\ \text{if } \zeta^0\in\boldsymbol{R}^n,\ z\in\boldsymbol{R}^n \text{ and } |z|<1/2|\xi+\eta|.$$

(3.17) implies that $((x^0, y^0), (\xi^0, \eta^1))\notin WF(E(x, y))$ when $\xi^0\neq-\eta^1$. Therefore Theorem 3.2 has been proved.

4. Some remarks

First we shall consider the wave front sets of solutions with respect to $C^L(\boldsymbol{R}^n)$ when the coefficients of $P(x, D)$ are in $C^L(\boldsymbol{R}^n)$. We assume that there exists a positive number C such that $L_k\leq CkL_N/N$ and $N\geq k\geq 0$ and that $\sum_{k=1}^{\infty}1/L_k<\infty$. Let $\phi_j\in C_0^{\infty}(U_j)\cap C^L(\boldsymbol{R}^n)$, $j=1,2$, where U_1 (resp. U_2) is a neighborhood of x^0 (resp. y^0). Assume that the coefficients of $Q(x, D)$ have compact supports. Then we have

(4.1) $$|\mathscr{F}_{(x,y)}[\phi_1(x)\phi_2(y)F(x, y; \gamma)](\xi,\eta)|$$
$$\leq C(CL_N)^N\langle(\xi,\eta)\rangle^{-N},\ N=0,1,2,\cdots,$$

if (ξ,η) belongs to some conic neighborhood of (ξ^0,η^1), $((x^0,\xi^0), (y^0,-\eta^1))\notin C$ and $\gamma>C(P_m, Q)$. In fact, we divide the integral on the right-hand side of (3.9) into two parts:

$$\mathscr{F}_{(x,y)}[\phi_1(x)\phi_2(y)F_l(x, y, ; \gamma)](\xi,\eta)$$
$$=\sum_{j_1,\cdots,j_l}(-1)^l(2\pi)^{-(1+l)n}\{\int_{|\zeta^1|+\cdots+|\zeta^l|\leq\varepsilon|\xi|}$$
$$+\int_{|\zeta^1|+\cdots+|\zeta^l|\geq\varepsilon|\xi|}\}d\zeta^0\cdots d\zeta^l\equiv I_1+I_2.$$

We can estimate I_1 in the same way as for I^+. Since $|\hat{q}_{j_1}(\zeta^1)\cdots\hat{q}_{j_l}(\zeta^l)|\leq C^l(CL_N)^N\langle\xi\rangle^{-N}$ if $|\zeta^1|+\cdots+|\zeta^l|\geq\varepsilon|\xi|$ and $N=0,1,2,\cdots$, we can estimate I_2. Applying the same argument to (3.16) we have (4.1). Then by Theorem 2.5 and (4.1) we have the following

THEOREM 4.1. *Assume that the condition* (A) *is satisfied and the coefficients of $P(x, D)$ are in $C^L(\boldsymbol{R}^n)$. Then we have*

$$WF_L(E(x, y))\subset\{((x, y), (\xi,\eta))\in T^*\boldsymbol{R}^{2n}\backslash 0;\ ((x,\xi), (y,-\eta))\in C\}^{\dagger}$$

For hyperbolic systems we can apply the same arguments when every entries of the operator ${}^t\mathrm{cof}\, P_m(D)\cdot Q(x, D)$ are weaker than $\det P_m(D)$ for every fixed $x\in \boldsymbol{R}^n$. In fact,

$$ {}^t\mathrm{cof}\, P_m(D)\cdot P(x, D)=\det P_m(D)I_r+{}^t\mathrm{cof}\, P_m(D)\cdot Q(x, D)$$

and ${}^t\mathrm{cof}\, P_m(D)$ is a hyperbolic operator with constant coefficients and $\det({}^t\mathrm{cof}\, P_m(\xi)) =(\det P_m(\xi))^{r-1}$, where $P(x, \xi)$ is an $r\times r$ matrix (see [3]). However, this condition is not a necessary condition for hyperbolicity of $P(x, D)$ when x is fixed (see [6]).

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† The definition of $WF_L(u)$ is given in [5].