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A UNIFIED APPROACH OF CHARACTERIZATIONS AND RESOLUTIONS FOR COHOMOLOGICAL DIMENSION MODULO p

Dedicated to Professor Akihiro Okuyama on his 60th birthday

By

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Abstract. We prove characterization and resolution theorems for compact spaces and metrizable spaces with respect to cohomological dimension modulo p.

1. Introduction and preliminary

In the last ten years, cohomological dimension theory has striking development. A motivation of the development is surely the Edwards-Walsh theorem, [24], as follows:

1.1. THEOREM. *Every compact metric space X of cohomological dimension $c\text{-}\dim_Z X \leq n$ (integer coefficient) is the image of a cell-like map $f: Z \to X$ from a compact metric space Z of* $\dim Z \leq n$.

Not only the result but also techniques of the proof gave an important influence to the development. After them, L. R. Rubin and P. J. Schapiro [22] showed the noncompact version of the Edwards-Walsh theorem and S. Mardešić and L. R. Rubin [17] gave the nonmetrizable version. On the other hand, A. N. Dranishnikov, [5] and [6], characterized cohomological dimension with respect to $\boldsymbol{Z}_p$ by the Edwards-Walsh's way and showed the Edwards-Walsh-like theorem:

1.2. THEOREM. *Every compact metric space X of cohomological dimension with respect to $\boldsymbol{Z}_p$, $c\text{-}\dim_{Z_p} X \leq n$, is the image of a map $f: Z \to X$ from a compact metric space Z of* $\dim Z \leq n$ *whose fibers are acyclic modulo p.*

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Motivated above results and Mardešić's characterization of $c\text{-dim}_Z X \leqq n$, we will show a characterization of $c\text{-dim}_{Z_p} X \leqq n$ for both nonmetrizable and noncompact cases. Using the characterization, we will give the existence of an acyclic resolution modulo p. In fact, our characterization suggests a dimension-like function, called approximable dimension, and can obtain the following more general results.

1.3. THEOREM. *Let X be a compact Hausdorff space or a metrizable space having approximable dimension with respect to an arbitrary coefficients $G \leqq n$. Then there exists a proper map $f: Z \to X$ from a compact Hausdorff space or a metrizable space Z, respectively, of* $\dim Z \leqq n$ *and $w(Z) \leqq w(X)$ onto X such that $H^*(f^{-1}(x); G)=0$ for all $x \in X$.*

As its consequence, we have both nonmetrizable and noncompact versions of Theorems 1.1 and 1.2. We may call such a mapping f an acyclic resolution of X (with respect to G), specially, in the case of $G=\boldsymbol{Z}_p$, an acyclic resolution of X modulo p. Finally we will note that there exists a compact metric space X of $c\text{-dim}_{\boldsymbol{Q}} X=1$ which does not admit an acyclic resolution with respect to $\boldsymbol{Q}$. Thereby we can see that approximable dimension is different from cohomological dimension and Theorem 1.3 is a good property obtained from approximable dimension.

In this paper, we mean the definition of cohomological dimension as follows: the *cohomological dimension of a space X with respect to a coefficient group G is less than and equal to n*, denoted by $c\text{-dim}_G X \leqq n$, provided that every map $f: A \to K(G, n)$ of a closed subset A of X into an Eilenberg-MacLane space $K(G, n)$ of type (G, n) admits a continuous extension over X (c. f. [10]). The dimension of a space X means the *covering dimension* of X and denotes by $\dim X$. $\boldsymbol{Z}$ is the additive group of all integers and for each prime number p, $\boldsymbol{Z}_p$ is the cyclic group of order p.

By a polyhedron we mean the space $|K|$ of a simplicial complex K with the *Whitehead topology*. In section 6, the topology of $|K|$ may be generated by a uniformity [Appendix, 22].

If v is a vertex of a simplicial complex K, let $\mathrm{st}(v, K)$ be the open star of v in $|K|$ and $\overline{\mathrm{st}}(v, K)$ be the closed star of v in $|K|$. If $A \subseteqq |K|$, then we define $\mathrm{st}(A, K)=\bigcup\{\mathrm{Int}\,\sigma : \sigma \in K,\ \sigma \cap A \neq \emptyset\}$ and $\overline{\mathrm{st}}(A, K)=\bigcup\{\sigma : \sigma \in K,\ \sigma \cap A \neq \emptyset\}$. The symbol $\mathrm{Sd}_j K$ means the j-th barycentric subdivision of K. We define the symbols $\mathcal{S}_i$ and $\bar{\mathcal{S}}_i$ for a simplicial complex K_i with an index to be the cover $\{\mathrm{st}(v, K_i) : v \in K_i^{(0)}\}$ and the cover $\{\overline{\mathrm{st}}(v, K_i) : v \in K_i^{(0)}\}$, respectively.

We use the symbol $\prec$ both to mean 'refine' for covers and 'subdivides' for subdivisions of a complex. The symbol $\prec^*$ is used for star refines.

Let $\mathcal{U}$ be an open cover of a space X. Then for $U \in \mathcal{U}$,

$$\mathrm{st}(U, \mathcal{U}) = \mathrm{st}^1(U, \mathcal{U}) = \bigcup\{U' : U' \in \mathcal{U},\ U' \cap U \neq \emptyset\},$$

$$\mathrm{st}^{j+1}(U, \mathcal{U}) = \bigcup\{U' : U' \in \mathcal{U},\ U' \cap \mathrm{st}^j(U, \mathcal{U}) \neq \emptyset\}.$$

By $\mathrm{st}^j(\mathcal{U})$ we mean the cover $\{\mathrm{st}^j(U, \mathcal{U}) : U \in \mathcal{U}\}$. If f and g are maps from a space Z to a space X, $(f, g) \leqq \mathcal{U}$ means that for each $z \in Z$, there exists $U \in \mathcal{U}$ with $f(z), g(z) \in U$. If X is a metric space with a metric d, we write $(f, g) \leqq \varepsilon$ instead of $(f, g) \leqq \mathcal{U}_\varepsilon$, where $\mathcal{U}_\varepsilon$ is the cover whose consists of all $\varepsilon/2$-neighborhoods in X. By the symbol $\mathcal{N}(\mathcal{U})$ we mean the nerve of the cover $\mathcal{U}$. For covers $\mathcal{U}$, $\mathcal{V}$, the symbol $\mathcal{U} \wedge \mathcal{V}$ is used for the following cover $\{U \wedge V, U, V : U \in \mathcal{U}, V \in \mathcal{V}\}$.

2. Edwards-Walsh complexes

In the latter section, we need Edwards-Walsh complexes for arbitrary simplicial complexes.

2.1. LEMMA. *Let $|L|$ be a simplicial complex with the Whitehead topology, p be a prime number and n be a natural number. Then there exists a combinatorial map (i.e. $\pi_L^{-1}(L')$ is a subcomplex of $\mathrm{EW}_{Z_p}(L, n)$ if L' is a subcomplex of L) $\phi_L : \mathrm{EW}_{Z_p}(L, n) \to |L|$ such that*

(i) *for $\sigma \in L$ with $\dim \sigma \geqq n+1$, $\phi_L^{-1}(\sigma) \in K(\bigoplus_1^{r_\sigma} \boldsymbol{Z}_p, n)$, where $r_\sigma = \mathrm{rank}\, \pi_n(\sigma^{(n)})$,*

(ii) *for $\sigma \in L$ with $\dim \sigma \leqq n$, $\phi_L^{-1}(\sigma) = \sigma$,*

(iii) *$\mathrm{EW}_{Z_p}(L, n)$ is a CW-complex,*

(iv) *$\phi_L^{-1}(\sigma)$ is a subcomplex of $\mathrm{EW}_{Z_p}(L, n)$ with respect to the triangulation in* (iii),

(v) *$\phi_L^{-1}(\sigma)^{(k)}$ is a finite CW-complex for $k \geqq n$,*

(vi) *for any subcomplex L' of L and map $f : |L'| \to K(\boldsymbol{Z}_p, n)$, there exists an extension of $f \circ \phi_L|_{\phi_L^{-1}(|L'|)}$.*

PROOF. We shall construct a sequence $K_1(L) \subseteqq K_2(L) \subseteqq \cdots$ of CW-complexes as follows. To produce $K_1(L)$, we shall construct a sequence $L(1, 0) \subseteqq L(1, 1) \subseteqq \cdots$ of CW-complexes as follows. If $\sigma \in L$ and $\dim \sigma \leqq n$, let $K_1(\sigma) \equiv \sigma$ and put $L(1, 0) \equiv \bigcup\{K_1(\sigma) : \sigma \in L, \dim \sigma \leqq n\}$.

We shall produce $L(1, 1)$ with $L(1, 0) \subseteqq L(1, 1)$. Suppose $\sigma \in L$ with $\dim \sigma =$

$n+1$. Let $K_1(\sigma)$ be a complex obtained from $\partial\sigma$ by attaching an $(n+1)$-cell by a map of degree p. Hence we have

$$K_1(\sigma)=\partial\sigma\cup_\alpha B^{n+1}, \qquad \text{where } \alpha: \partial B^{n+1}\to\partial\sigma \text{ is a map of degree } p.$$

Put $L(1, 1)\equiv\bigcup\{K_1(\sigma): \sigma\in L,\ \dim\sigma\leqq n+1\}$.

Next we shall construct $L(1, 2)$ with $L(1, 1)\subseteqq L(1, 2)$. Suppose $\sigma\in L$ with $\dim\sigma=n+2$. Let

$$K_1(\sigma)\equiv\begin{cases}\bigcup\{K_1(\tau): \tau\lneqq\sigma\} & n\geqq 2\\ A(\bigcup\{K_1(\tau): \tau\lneqq\sigma\}) & n=1,\end{cases}$$

where for a complex K, $A(K)$ means a complex obtained by attaching finite collection of 2-cells abelizing the fundamental group $\pi_1(K)$. Define $L(1, 2)$ to be $\bigcup\{K_1(\sigma): \sigma\in L,\ \dim\sigma\leqq n+2\}$. This process continues in an obvious way producing $L(1, 0)\subseteqq L(1, 1)\subseteqq\cdots$. Let $K_1(L)$ be $\bigcup\{L(1, i): 0\leqq i<\infty\}$. Then $K_1(L)$ has the natural structure of CW-complex in such a way that each $L(1, i)$ is a subcomplex as is each $K_1(\sigma)$. Further, it is clear that $K_1(\sigma)\cap K_1(\tau)=K_1(\sigma\cap\tau)$ for σ, $\tau\in L$ and $\pi_q(K_1(\sigma))=0$ $(q<n)$, $\bigoplus_1^{r_\sigma}\boldsymbol{Z}_p$ $(q=n)$, where $r_\sigma=\operatorname{rank}\pi_n(\sigma^{(n)})$.

To produce $K_2(L)$ we are going to attach $(n+2)$-cells to $K_1(L)$. To this end, we shall construct a sequence $L(2, 0)\subseteqq L(2, 1)\subseteqq\cdots$ of CW-complexes as follows. If $\sigma\in L$ and $\dim\sigma\leqq n$, let $K_2(\sigma)\equiv\sigma$ and put $L(2, 0)\equiv\bigcup\{K_2(\sigma): \sigma\in L,\ \dim\sigma\leqq n\}$. If $\sigma\in L$ and $\dim\sigma=n+1$, then $\pi_{n+1}(K_1(\sigma))$ is a finitely generated abelian group. Kill this generating set by attaching finitely many $(n+2)$-cells to form $K_2(\sigma)$. Let $L(2, 1)\equiv\bigcup\{K_2(\sigma): \sigma\in L,\ \dim\sigma\leqq n+1\}$. Next let us produce $L(2, 2)$. Suppose $\sigma\in L$ and $\dim\sigma=n+2$. Let $K_2(\partial\sigma)\equiv\bigcup\{K_2(\tau): \tau\lneqq\sigma\}\cup K_1(\sigma)$. Then it is clear that $\pi_q(K_2(\partial\sigma))=0$ $(q<n)$, $\bigoplus_1^{r_\sigma}\boldsymbol{Z}_p$ $(q=n)$, where $r_\sigma=\operatorname{rank}\pi_n(\sigma^{(n)})$ and $\pi_{n+1}(K_2(\partial\sigma))$ is a finitely generated abelian group. Kill this generating set by attaching finitely many $(n+2)$-cells to form $K_2(\sigma)$. Let $L(2, 2)\equiv\bigcup\{K_2(\sigma): \sigma\in L,\ \dim\sigma\leqq n+2\}$. This process continues in an obvious way producing $L(2, 0)\subseteqq L(2, 1)\subseteqq\cdots$. Let $K_2(L)$ be $\bigcup\{L(2, i): 0\leqq i<\infty\}$. Then $K_2(L)$ has the natural structure of CW-complex in such a way that each $L(2, i)$ is a subcomplex as is each $K_2(\sigma)$. Further, it is clear that $K_2(\sigma)\cap K_2(\tau)=K_2(\sigma\cap\tau)$ for σ, $\tau\in L$ and $\pi_q(K_2(\sigma))=0$ $(q<n$ or $q=n+1)$, $\bigoplus_1^{r_\sigma}\boldsymbol{Z}_p$ $(q=n)$, where $r_\sigma=\operatorname{rank}\pi_n(\sigma^{(n)})$.

The construction of $K_1(L)$, $K_2(L)$ with $K_1(L)\subseteqq K_2(L)$ given above indicates how one may recursively constructed a sequence $K_1(L)\subseteqq K_2(L)\subseteqq\cdots$. For each $\sigma\in L$, let $K(\sigma)=\bigcup\{K_i(\sigma): i\in\boldsymbol{N}\}$. Then by induction of the dimension of the skeleton we can construct a combinatorial map $\psi_L: \mathrm{EW}_{\boldsymbol{Z}_p}(L, n)\to|L|$ with the properties (i)-(vi) as

(1) $\psi_L^{-1}(L^{(n)})=L^{(n)}$ and $\psi_L|_{|L^{(n)}|}=id_{|L^{(n)}|}$,

(2) $\psi_L^{-1}(\sigma)$ is the mapping cylinder M_σ of the embedding $j_\sigma: \psi_L^{-1}(\partial\sigma)\hookrightarrow K(\sigma)$,

(3) $\psi_L|_{M_\sigma}$ is the cone of $\psi_L|_{\psi_L^{-1}(\partial\sigma)}$ such that $\psi_L(K(\sigma))$ is the barycenter of σ.

Hence for each simplex σ of $\dim\sigma\geqq n+1$, we have the property:

(4) if $n\geqq 2$,

$$\psi_L^{-1}(\sigma)^{(n+1)}=\sigma^{(n)}\times[0, 1]\cup_{\alpha_1} B^{n+1}\cup_{\alpha_2}\cdots\cup_{\alpha_{r_\sigma}} B^{n+1},$$

where for each $(n+1)$-dimensional face τ_i of σ, $\alpha_i: \partial B^{n+1}\to\partial\tau_i\times\{1\}$ is a map of degree p,

(5) if $n=1$,

$$\psi_L^{-1}(\sigma)^{(2)}=\sigma^{(1)}\times[0, 1]\cup_{\alpha_1} B^2\cup_{\alpha_2}\cdots\cup_{\alpha_{r_\sigma}} B^2\cup_{\beta_1} B^2\cup_{\beta_2}\cdots\cup_{\beta_{k_\sigma}} B^2,$$

where for each 2-dimensional face τ_i of σ, $\alpha_i: \partial B^2\to\partial\tau_i\times\{1\}$ is a map of degree p and the collection $\{[\beta_1], \cdots, [\beta_{k_\sigma}]\}$ generates the commutator subgroup of $\pi_1(\sigma^{(1)}\times[0, 1]\cup_{\alpha_1} B^2\cup_{\alpha_2}\cdots\cup_{\alpha_{r_\sigma}} B^2)$. □

3. Characterizations for compact spaces

3.1. Definition. Let G be an abelian group, n be a natural number and ε be a positive number. A map $\psi: Q\to P$ between compact polyhedra is (G, n, ε)-*approximable* provided that there exists a triangulation L of P such that for any triangulation M of Q there is a map $\psi': |M^{(n)}|\to|L^{(n)}|$ satisfying the following conditions:

(i) $(\psi', \psi|_{|M^{(n)}|})\leqq\varepsilon$,

(ii) for any map $\alpha: |L^{(n)}|\to K(G, n)$, there exists a map $\beta: Q\to K(G, n)$ such that $\beta|_{|M^{(n)}|}=\alpha\circ\psi'$.

Here the map ψ' is called a (G, n, ε)-*approximation* of ψ.

Note that it suffices for the condition (ii) to see that the map $\alpha\circ\psi'$ admits a continuous extension over $|M^{(n+1)}|$.

3.2. Definition. A map $f: X\to P$ from a compact space to a compact polyhedron is (G, n)-*cohomological* provided that for every positive number $\varepsilon>0$, there exists a compact polyhedron Q and maps $\varphi: X\to Q$, $\psi: Q\to P$ such that

(i) $(\psi\circ\varphi, f)\leqq\varepsilon$,

(ii) ψ is (G, n, ε)-approximable.

3.3. Theorem. *Let X be a compact space, p be a prime number and n be a natural number. Then X has cohomological dimension with respect to $\boldsymbol{Z}_p$ of* *less than and equal to n if and only if every map f of X to a compact polyhedron P is* $(\boldsymbol{Z}_p, n)$*-cohomological.*

PROOF. We establish the reverse implication first. Let A be a closed subset of X and let $h: A \to K(\boldsymbol{Z}_p, n)$ be a map. Because of the compactness of A, there is a compact subpolyhedron K of $K(\boldsymbol{Z}_p, n)$ such that $h(A) \subseteqq K$. Let P be the cone over K. Then there exists a continuous extension $f: X \to P$ of h, and there is a closed polyhedral neighborhood N of K and a retraction $r: N \to K$. Let us take a positive number $\delta > 0$ such that

(1) $O_\delta(K) = \{x \in P: d_P(x, K) < \delta\} \subseteqq N$,

(2) any two δ-near maps of a space into N are homotopic in N,

where d_P is a metric for P. By the condition, there exists a polyhedron Q and maps $\varphi: X \to Q$, $\psi: Q \to P$ such that

(3) $(\psi \circ \varphi, f) \leqq \delta/3$,

(4) ψ is $(\boldsymbol{Z}_p, n, \delta/3)$-approximable.

By (1) and (3), we have $\psi(\varphi(A)) \subseteqq O_{\delta/3}(h(A)) \subseteqq N$. Hence, there is a closed polyhedral neighborhood G of $\varphi(A)$ in Q such that

$$\psi(G) \subseteqq O_{\delta/2}(f(A)) \subseteqq N. \tag{5}$$

Let take a triangulation M of Q such that G is the carrier of a subcomplex M_1 of M. Then, by (4), there exists a triangulation L of P and a map $\psi': |M^{(n)}| \to |L^{(n)}|$ satisfying the following conditions:

(6) $(\psi', \psi|_{|M^{(n)}|}) \leqq \delta/3$,

(7) for any map $\alpha: |L^{(n)}| \to K(\boldsymbol{Z}_p, n)$ and every $(n+1)$-simplex σ of M, there exists a continuous extension $\alpha_\sigma: \sigma \to K(\boldsymbol{Z}_p, n)$ of $\alpha \circ \psi'|_{\partial\sigma}$.

Then by (6), (5) and (2), we see that $\psi'(|M_1 \cap M^{(n)}|) \subseteqq O_{\delta/2}(\psi(|M_1^{(n)}|)) \subseteqq N$, and

$$\psi'|_{|M_1 \cap M^{(n)}|} \simeq \psi|_{|M_1 \cap M^{(n)}|} \quad \text{in } N. \tag{8}$$

Since $\psi|_{|M_1 \cap M^{(n)}|}$ has a continuous extension $\psi|_G: G \to N$, by (8), we have a continuous extension $\psi^*: G \cup |M^{(n)}| \to N \cup |L^{(n)}| \subseteqq P$ of ψ' such that

$$\psi^*|_G \simeq \psi|_G \quad \text{in } N. \tag{9}$$

Considering r as a map into $K(\boldsymbol{Z}_p, n)$, take a continuous extension $r^*: N \cup |L^{(n)}| \to K(\boldsymbol{Z}_p, n)$ of r. For each $(n+1)$-simplex σ of M, by (7), there exists a map $\alpha_\sigma: \sigma \to K(\boldsymbol{Z}_p, n)$ such that

$$\alpha|_{\partial\sigma} = r^* \circ \psi^*|_{\partial\sigma}. \tag{10}$$

Hence we have a continuous extension $\theta: G \cup |M^{(n+1)}| \to K(\boldsymbol{Z}_p, n)$ of $r^* \circ \psi^*$ given by

(11) $\theta|_G = r^*\circ\psi^*$ and $\theta|_\sigma = \alpha_\sigma$ for each $(n+1)$-simplex σ of M.

Therefore we can find a continuous extension $\theta^*: Q\to K(\boldsymbol{Z}_p, n)$ of θ. Then by (9), (2) and (3), we see that

(12) $\theta^*\circ\varphi|_A = r^*\circ\psi^*\circ\varphi|_A \simeq r^*\circ\psi\circ\varphi|_A \simeq r^*\circ f|_A = h$ in $K(\boldsymbol{Z}_p, n)$.

Hence, by the homotopy extension theorem, h has a continuous extension $h^*: X\to K(\boldsymbol{Z}_p, n)$. Thus, $c\text{-}\dim_{\boldsymbol{Z}_p} X \leqq n$.

Conversely, suppose $c\text{-}\dim_{\boldsymbol{Z}_p} X \leqq n$. Let us take a map $f: X\to P$ of X to a compact polyhedron and a positive number $\varepsilon>0$. Then take a triangulation L of P such that

(13) $$mesh(L)\leqq \varepsilon\,,$$

and let $\psi_L: \mathrm{EW}_{\boldsymbol{Z}_p}(L, n)\to P$ be the map constructed in Lemma 2.1.

First, we show that there exists a map $g: X\to \mathrm{EW}_{\boldsymbol{Z}_p}(L, n)$ such that

(14) $\psi_L\circ g|_{f^{-1}(|L^{(n)}|)} = f|_{f^{-1}(|L^{(n)}|)}$,

(15) $g(f^{-1}(\sigma))\subseteqq \psi_L^{-1}(\sigma)$ for every simplex σ of L with $\dim\sigma \geqq n+1$.

Write L as the form

$$L = L^{(n)}\cup\sigma_1\cup\cdots\cup\sigma_s\,,\qquad \text{where } n+1\leqq \dim\sigma_1\leqq\cdots\leqq\dim\sigma_s\,.$$

By the property (1) in Lemma 2.1, we can define the map

$$f_0 \equiv f|_{f^{-1}(|L^{(n)}|)}: f^{-1}(|L^{(n)}|) \longrightarrow |L^{(n)}| \subseteqq \mathrm{EW}_{\boldsymbol{Z}_p}(L, n).$$

By $c\text{-}\dim_{\boldsymbol{Z}_p} f^{-1}(\sigma_1) \leqq c\text{-}\dim_{\boldsymbol{Z}_p} X\leqq n$ and the property (i) in Lemma 2.1, the map $f_0|_{f^{-1}(\partial\sigma_1)}: f^{-1}(\partial\sigma_1)\to f_0(f^{-1}(\partial\sigma_1)) = \partial\sigma_1 \subseteqq \psi_L^{-1}(\sigma_1)$ has a continuous extension $f_{\sigma_1}: f^{-1}(\sigma_1)\to\psi_L^{-1}(\sigma_1)$.

For each $i\geqq 2$, since $\partial\sigma_i \subseteqq L^{(n)}\cup\sigma_1\cup\cdots\cup\sigma_{i-1}$, we can similarly obtain a map $f_i: f^{-1}(|L^{(n)}|\cup f^{-1}(\sigma_1)\cup\cdots\cup f^{-1}(\sigma_i))\to\psi^{-1}(|L^{(n)}|\cup\sigma_1\cup\cdots\cup\sigma_i)$ such that

(16) $f_i|_{f^{-1}(|L^{(n)}|\cup\sigma_1\cup\dots\cup\sigma_{i-1})} = f_{i-1}$,

(17) $f_i(f^{-1}(\sigma_i))\subseteqq\psi_L^{-1}(\sigma_i)$.

Therefore the map f_s is a desired one.

By the compactness of $g(X)$, there exists a compact subpolyhedron K of $\mathrm{EW}_{\boldsymbol{Z}_p}(L, n)$ containing $g(X)$. Then by the same as in [15], we can find a map $\varphi: X\to K$ such that

(18) $\varphi(X)$ is a subpolyhedron Q of $\mathrm{EW}_{\boldsymbol{Z}_p}(L, n)$.

Moreover, by the construction and the property (iv) in Lemma 2.1, we may assume that

(19) $\psi_L\circ\varphi|_{f^{-1}(|L^{(n)}|)} = f|_{f^{-1}(|L^{(n)}|)}$,

(20) $\varphi(f^{-1}(\sigma))\subseteqq\psi_L^{-1}(\sigma)$ for every simplex σ of L with $\dim\sigma\geqq n+1$.

Thus, by (18), (20), (19) and (13), we have a compact polyhedron Q and maps $\varphi: X \to Q$, $\psi = \psi_L|_Q: Q \to P$ such that

(21) $\varphi(X) = Q$,

(22) $(\psi \circ \varphi, f) \leqq \varepsilon$.

Hence, it suffices to show the following:

CLAIM. *ψ is $(\boldsymbol{Z}_p, n, \varepsilon)$-approximable.*

PROOF OF CLAIM. Let M be a triangulation of Q. First, we show that there exists a map $\theta: |M^{(n+1)}| \to \mathrm{EW}_{Z_p}(L, n)^{(n+1)}$ satisfying the followings:

(23) $\theta|_{Q \cap \mathrm{EW}_{Z_p}(L, n)^{(n+1)}} = id_{Q \cap \mathrm{EW}_{Z_p}(L, n)^{(n+1)}}$,

(24) $\theta(Q \cap \psi_L^{-1}(\sigma)) \subseteqq \psi_L^{-1}(\sigma)^{(n+1)}$ for every simplex σ of L with $\dim \sigma \geqq n+1$.

Since $|M^{(n+1)}|$ is compact, there is a finite collection of cells $\{\tau_1, \cdots, \tau_k\}$ in $\mathrm{EW}_{Z_p}(L, n)$, $\dim \tau_1 \geqq \cdots \geqq \dim \tau_k \geqq n+2$, such that

(25) $|M^{(n+1)}| \cap \tau_i \neq \emptyset$ for each $i = 1, \cdots, k$,

(26) $|M^{(n+1)}| \subseteqq \mathrm{EW}_{Z_p}(L, n)^{(n+1)} \cup \tau_1 \cup \cdots \cup \tau_k$.

We take a small PL-ball $B \subseteqq \tau_1 \setminus \partial \tau_1$ such that $\dim B = \dim \tau_1$, and consider the inclusion $i_1: \partial B \cap |M^{(n+1)}| \to \partial B$. By $\dim (B \cap |M^{(n+1)}|) \leqq n+1 < \dim B$, i_1 has a continuous extension $\bar{i}_1: B \cap |M^{(n+1)}| \to \partial B$. Considering the map $\bar{i}_1$ and a retraction from $\mathrm{EW}_{Z_p}(L, n)^{(n+1)} \cup (\tau_1 \setminus \mathrm{Int}\, B) \cup \tau_2 \cup \cdots \cup \tau_k$ onto $\mathrm{EW}_{Z_p}(L, n)^{(n+1)} \cup \tau_2 \cup \cdots \cup \tau_k$, we have a map $\theta_1: |M^{(n+1)}| \to \mathrm{EW}_{Z_p}(L, n)^{(n+1)} \cup \tau_2 \cup \cdots \cup \tau_k$ such that

(27) $\theta_1|_{Q \cap \mathrm{EW}_{Z_p}(L, n)^{(n+1)}} = id_{Q \cap \mathrm{EW}_{Z_p}(L, n)^{(n+1)}}$,

(28) $\theta_1(|M^{(n+1)}| \cap \psi_L^{-1}(\sigma)) \subseteqq \psi_L^{-1}(\sigma)$ for every simplex σ of L with $\dim \sigma \geqq n+1$.

Inductively, for $i = 1, \cdots, k$, we can construct a map $\theta_i: |M^{(n+1)}| \to \mathrm{EW}_{Z_p}(L, n)^{(n+1)} \cup \tau_{i+1} \cup \cdots \cup \tau_k$ satisfying the corresponding to (27) and (28). Therefore θ_k is a required one.

Moreover, taking suitable subdivisions if necessary, we may assume that θ is simplicial.

CASE 1. $n \geqq 2$.

By the properties (1), (4) in Lemma 2.1, we see that

$$\mathrm{EW}_{Z_p}(L, n)^{(n+1)} = |L^{(n)}| \cup \bigcup \{\partial \sigma \times [0, 1] \cup_{\alpha_\sigma} B_\sigma^{n+1} : \sigma \in L,\ \dim \sigma = n+1\},$$

where $\alpha_\sigma: S^n \to \partial \sigma$ is a map of degree p. For each $(n+1)$-simplex σ of L, choose a point $z_\sigma \in B_\sigma^{n+1} \setminus (S^n \cup \theta(|M^{(n)}|))$, and take the retraction

$$r: \mathrm{EW}_{Z_p}(L, n)^{(n+1)} \setminus \{z_\sigma : \sigma \in L,\ \dim \sigma = n+1\} \longrightarrow |L^{(n)}|$$

induced by the compositions of the radial projection of $B_\sigma^{n+1} \setminus \{z_\sigma\}$ onto $\partial \sigma \times \{1\}$ and the natural projection of $\partial \sigma \times [0, 1]$ onto $\partial \sigma \times \{0\} \subseteqq |L^{(n)}|$. Now we define a map $\psi' : |M^{(n)}| \to |L^{(n)}|$ by $\psi' = r\circ\theta|_{|M^{(n)}|}$.

Let τ be an $(n+1)$-simplex of M. If $\psi'(\tau) \subseteqq \mathrm{EW}_{Z_p}(l, n)^{(n)} = |L^{(n)}|$, then

(29) $$\psi'|_{\partial\tau} = \theta|_{\partial\tau} \simeq 0 \quad \text{in } |L^{(n)}|.$$

Otherwise, there is finite PL $(n+1)$-balls $D_1, \cdots, D_m$ in $\tau\setminus\partial\tau$ such that

(30) $\bigcup_{i=1}^{m} \mathrm{Int}\, D_i \supseteqq \theta^{-1}(\{z_\sigma : \dim \sigma = n+1\}) \cap \tau$,

(31) $\theta(D_i) \subseteqq B_{\sigma_i} \setminus B_{\partial\sigma_i}$ for some $(n+1)$-simplex σ_i of L.

Then we have that

(32) $$[\psi'|_{\partial\tau}] = [r\circ\theta|_{\partial D_1}] + \cdots + [r\circ\theta|_{\partial D_m}] \quad \text{in } \pi_n(|L^{(n)}|).$$

Since each $r\circ\theta|_{\partial D_i}$ can be factorized through the attaching map α_{σ_i}, $[r\circ\theta|_{\partial D_i}] = p\cdot a_i$ for some $a_i \in \pi_n(|L^{(n)}|)$. Hence, by (32), we have

(33) $$[\psi'|_{\partial\tau}] = p\cdot(a_1 + \cdots + a_m) \quad \text{in } \pi_n(|L^{(n)}|).$$

Therefore, for any map $\xi : |L^{(n)}| \to K(Z_p, n)$, $\xi\circ\psi'|_{\partial\tau}$ can be extended over τ.

CASE 2. $n=1$.

For every simplex σ of $\dim \sigma \geqq 2$, $\psi_L^{-1}(\sigma^{(2)})$ may be represented as the form (5) in Lemma 2.1:

$$\psi_L^{-1}(\sigma)^{(2)} = \sigma^{(1)} \times [0, 1] \cup_{\alpha_1} B^2 \cup_{\alpha_2} \cdots \cup_{\alpha_{r_\sigma}} B^2 \cup_{\beta_1} B^2 \cup_{\beta_2} \cdots \cup_{\beta_{k_\sigma}} B^2.$$

Then choose points $u_1^\sigma, \cdots, u_{r_\sigma}^\sigma, v_1^\sigma, \cdots, v_{k_\sigma}^\sigma$ of $\psi_L^{-1}(\sigma^{(1)}) \setminus (\sigma^{(1)} \times [0, 1] \cup \theta(|M^{(1)}|))$ and the retraction $r : \mathrm{EW}_{Z_p}(L, n)^{(2)} \setminus \{u_1^\sigma, \cdots, u_{r_\sigma}^\sigma, v_1^\sigma, \cdots, v_{k_\sigma}^\sigma : \sigma \in L, \dim \sigma \geqq 2\} \to |L^{(1)}|$ induced by the compositions of the radial projections of $B^2 \setminus \{u_i^\sigma\}$ or $B^2 \setminus \{v_j^\sigma\}$ onto S^1 and the natural projection of $\sigma^{(1)} \times [0, 1]$ onto $\sigma^{(1)} \times \{0\} \subseteqq |L^{(1)}|$. Now we define a map $\psi' : |M^{(2)}| \to |L^{(1)}|$ by $\psi' \equiv r\circ\theta$.

Let τ be a 2-simplex of M and let $\xi : |L^{(1)}| \to K(Z_p, 1)$ be a map. If $\psi'(\tau) \subseteqq \mathrm{EW}_{Z_p}(L, n)^{(2)} = |L^{(1)}|$, then we have the map $\xi\circ\psi'|_\tau$ as an extension of $\xi\circ\psi'|_{\partial\tau}$.

Otherwise, we choose finite PL 2-balls $D_1, \cdots, D_m$ in $\tau\setminus\partial\tau$ such that

(34) $\bigcup_{i=1}^{m} \mathrm{Int}\, D_i \supseteqq \theta^{-1}(\{u_1^\sigma, \cdots, u_{r_\sigma}^\sigma, v_1^\sigma, \cdots, v_{k_\sigma}^\sigma : \sigma \in L, \dim \sigma \geqq 2\}) \cap \tau$,

(35) $\theta(D_i) \subseteqq B^2 \setminus \partial\sigma \times [0, 1]$ for some simplex σ of $\dim \sigma \geqq 2$.

Considering the map $\theta|_{\tau\setminus\bigcup_{i=1}^{m}(D_i\setminus\partial D_i)}$ as a homotopy, we have that

(36) $$\begin{aligned}[\theta|_{\partial\tau}] &= [\theta|_{\bigcup_{i=1}^{m}\partial D_i}] \\ &= [\theta|_{\partial D_1}] * \cdots * [\theta|_{\partial D_m}] \\ &= [r\circ\theta|_{\partial D_1}] * \cdots * [r\circ\theta|_{\partial D_m}] \\ &= [r\circ\theta|_{\bigcup_{i=1}^{m}\partial D_i}] \\ &= [r\circ\theta|_{\partial\tau}] \\ &= [\psi'|_{\partial\tau}] \quad \text{in } \pi_1(\mathrm{EW}_{Z_p}(L, n)^{(2)}).\end{aligned}$$

Moreover, by the property (5) in Lemma 2.1, for every $i=1, \cdots, m$,

(37) $[r\circ\theta|_{\partial D_i}]$ is the p-th power of an element of $\pi_1(|L^{(1)}|)$, or

(38) $[r\circ\theta|_{\partial D_i}]$ is a commutator of $\pi_1(\sigma^{(1)})$ for some simplex σ.

On the other hand, by the property (vi) of Lemma 2.1, there exists a continuous extension $\bar{\xi}: \mathrm{EW}_{Z_p}(L, n)^{(2)}\to K(\boldsymbol{Z}_p, 1)$ of ξ. Since $\pi_1(K(\boldsymbol{Z}_p, 1))=\boldsymbol{Z}_p$ is abelian, by (36), (37) and (38), we have

$$\begin{aligned}[\xi\circ\psi'|_{\partial\tau}]&=[\bar{\xi}\circ\theta|_{\partial\tau}]\\ &=[\bar{\xi}\circ r\circ\theta|_{\partial D_1}]+\cdots+[\bar{\xi}\circ r\circ\theta|_{\partial D_m}] \qquad (39)\\ &=0 \quad \text{in } \pi_1(K(\boldsymbol{Z}_p, 1)).\end{aligned}$$

Thus, $\xi\circ\psi'|_{\partial\tau}$ can be extended over τ.

Therefore, in any cases, we have the map $\psi': |M^{(n)}|\to|L^{(n)}|$ such that

(40) for any map $\xi: |L^{(n)}|\to K(\boldsymbol{Z}_p, n)$, $\xi\circ\psi'$ admits a continuous extension over $|M^{(n+1)}|$.

Now, for any point $y\in|M^{(n)}|$, let take a simplex σ of L such that

$$y\in\psi_L^{-1}(\sigma). \tag{41}$$

Then, by (23) and (24), we see

$$\theta(y)\in\psi_L^{-1}(\sigma)^{(n+1)}. \tag{42}$$

Moreover, by the construction in any cases, we have

$$\psi'(y)=r\circ\theta(y)\in\sigma^{(n)}\subseteq\sigma. \tag{43}$$

Hence, by (13), we obtain that

$$d(\psi(y), \psi'(y))\leqq\operatorname{diam}(\sigma)\leqq\varepsilon. \tag{44}$$

Therefore ψ' is a $(\boldsymbol{Z}_p, n, \varepsilon)$-approximation of ψ. It completes the proof of Claim and it follows the implication of the *only if*. □

4. Characterizations for metrizable spaces

Let us establish definitions. Let K be a simplicial complex and $f, g: X\to |K|$ be maps. We say that g is a *K-manification* of f if for each $x\in X$ and $\sigma\in K$, $f(x)\in\sigma$ implies $g(x)\in\sigma$. Let $\mathcal{U}$ be an open cover of X. Then a map $b: X\to|\mathcal{N}(\mathcal{U})|$ is called *$\mathcal{U}$-normal* map if $b^{-1}(\mathrm{st}(\langle U\rangle, \mathcal{N}(\mathcal{U})))=U$ for each $U\in\mathcal{U}$ and b is essential on each simplex of $\mathcal{N}(\mathcal{U})$ (i.e. $b|_{b^{-1}(\sigma)}: b^{-1}(\sigma)\to\sigma$ is a essential map for each $\sigma\in\mathcal{N}(\mathcal{U})$). Note that if $\mathcal{U}$ is a locally finite, then $\mathcal{U}$-normal map exists.

4.1. DEFINITION. Let Q, P be polyhedra, G be an abelian group, $\mathcal{U}$ be an open cover of P and n be a natural number. We say that a map $\psi: Q \to P$ is *$(G, n, \mathcal{U})$-approximable* if there exists a triangulation L of P such that for any triangulation M of Q there is a PL-map $\psi': |M^{(n)}| \to |L^{(n)}|$ satisfying the following conditions:

(i) $(\psi', \psi|_{|M^{(n)}|}) \leqq \mathcal{U}$,

(ii) for any map $\alpha: |L^{(n)}| \to K(G, n)$, there exists an extension $\beta: |M^{(n+1)}| \to K(G, n)$ of $\alpha \circ \psi'$.

4.2. DEFINITION. Let G be an abelian group and n be a natural number. A map $f: X \to P$ of a metrizable space X to a polyhedron P is called *(G, n)-cohomological* if for any open cover $\mathcal{U}$ of P there exist a polyhedron Q and maps $\varphi: X \to Q$, $\psi: Q \to P$ such that

(i) $(\psi \circ \varphi, f) \leqq \mathcal{U}$,

(ii) ψ is $(G, n, \mathcal{U})$-approximable.

4.3. THEOREM. *Let X be a metrizable space, p be a prime number and n be a natural number. Then X has cohomological dimension with respect to $\boldsymbol{Z}_p$ of less than and equal to n if and only if every map f of X to a polyhedron P is $(\boldsymbol{Z}_p, n)$-cohomological.*

PROOF OF NECESSITY. Suppose that $c\text{-}\dim_{Z_p} X \leqq n$. Let $f: X \to P$ be a map of X to a polyhedron P and $\mathcal{U}$ be an open cover of P. Then take a star refinement $\mathcal{U}_0$ of $\mathcal{U}$.

First, we show that there exist a simplicial complex K and maps $\varphi: X \to |K|$, $\psi: |K| \to P$ such that

(1) if $\sigma \in K$, there exists $U \in \mathcal{U}_0$ with $\psi(\sigma) \subseteqq U$,

(2) for each $x \in X$ if $\varphi(x) \in \operatorname{Int} \sigma$, $\sigma \in K$, there exists $U \in \mathcal{U}_0$ with $\psi(\sigma) \cup \{f(x)\} \subseteqq U$,

(3) there exist a triangulation L of P and a PL-map $\psi': |K^{(n)}| \to |L^{(n)}|$ such that

(i) $(\psi', \psi|_{|K^{(n)}|}) \leqq \mathcal{U}_0$

(ii) for any map $\alpha: |L^{(n)}| \to K(\boldsymbol{Z}_p, n)$ there is an extension $\beta: |K^{(n+1)}| \to K(\boldsymbol{Z}_p, n)$ of $\alpha \circ \psi'$.

By J. H. C. Whitehead's theorem [25], take a triangulation L of P such that

(4) $\operatorname{st}\{\overline{\operatorname{st}}(v, L): v \in L^{(0)}\} \prec \mathcal{U}_0$.

We will construct a map $c: X \to \mathrm{EW}_{Z_p}(L, n)$ such that

(5) $c|_{f^{-1}(|L^{(n)}|)} = f|_{f^{-1}(|L^{(n)}|)}$,

(6) $c(f^{-1}(\sigma))\subseteqq\psi_L^{-1}(\sigma)$ for $\sigma\in L$, where $\psi_L: \mathrm{EW}_{Z_p}(L, n)\to L$ is the map constructed in Lemma 2.1.

We define the map $c_n\equiv f|_{f^{-1}(|L^{(n)}|)}: f^{-1}(|L^{(n)}|)\to|L^{(n)}|\subseteqq\mathrm{EW}_{Z_p}(L, n)$. Inductively, suppose that for $n\leqq k$ we have defined the function $c_k: f^{-1}(|L^{(k)}|)\to \mathrm{EW}_{Z_p}(L, n)$ such that $c_k|_{f^{-1}(\sigma)}: f^{-1}(\sigma)\to\psi_L^{-1}(\sigma)\subseteqq\mathrm{EW}_{Z_p}(L, n)$ is continuous and $c_k|_{f^{-1}(\sigma)}=c_k|_{f^{-1}(\tau)}$ on $f^{-1}(\sigma)\cap f^{-1}(\tau)$ for $\sigma, \tau\in L^{(k)}$. Now, let $\sigma\in L$ with $\dim\sigma=k+1$. By the construction of c_k and $\mathrm{EW}_{Z_p}(L, n)$, $c_k|_{f^{-1}(\partial\sigma)}: \partial\sigma\to\psi_L^{-1}(\sigma)$ is continuous. Hence by $c\text{-}\dim_{Z_p} f^{-1}(\sigma)\leqq c\text{-}\dim_{Z_p} X\leqq n$ and (i) in Lemma 2.1, we we have an continuous extension $c_\sigma: f^{-1}(\sigma)\to\psi_L^{-1}(\sigma)$ of $c_k|_{f^{-1}(\partial\sigma)}$. Define c_{k+1} to be c_σ on $f^{-1}(\sigma)$ for $\sigma\in L$ with $\dim\sigma=k+1$. Finally, we define c to be $\bigcup_{k=n}^\infty c_k$. Then since X is compactly generated, the function c is continuous.

We define an open cover $\mathcal{B}=\{B_\sigma: \sigma\in L\}$ in the following way:

$$B_\sigma\equiv\mathrm{EW}_{Z_p}(L, n)\setminus\bigcup\{\psi_L^{-1}(\tau): \sigma\cap\tau=\emptyset\}.$$

Then note that we have

(7) $\psi_L^{-1}(\sigma)\subseteqq B_\sigma$

(8) if $x\in B_\sigma$ and $x\in\psi_L^{-1}(\tau)$, then $\sigma\cap\tau\neq\emptyset$.

Since $\mathrm{EW}_{Z_p}(L, n)$ is LC^n, for a star refinement $\mathcal{B}_1$ of $\mathcal{B}$, there exists an open refinement $\mathcal{B}_2$ of $\mathcal{B}_1$ such that if K is a simplicial complex of $\dim K\leqq n+1$, then every partial realization of K in $\mathrm{EW}_{Z_p}(L, n)$ relative to $\mathcal{B}_2$ extended to a full realization relative to $\mathcal{B}_1$ [2]. Select a star refinement $\mathcal{B}_3$ of $\mathcal{B}_2$.

Then by [21, Lemma 9.6], there exist an open cover $\mathcal{V}$ of X refining $f^{-1}(\mathcal{U}_0)\wedge c^{-1}(\mathcal{B}_3)$ and maps $\varphi: X\to|\mathcal{N}(\mathcal{V})|$, $\psi: |\mathcal{N}(\mathcal{V})|\to P$ such that

(9) φ is $\mathcal{V}$-normal,

(10) $\psi\circ\varphi$ is L-modification of f,

(11) if $\sigma\in\mathcal{N}(\mathcal{V})$, the exists $U\in\mathcal{U}_0$ with $f(\varphi^{-1}(\sigma))\cup\psi(\sigma)\subseteqq U$.

Then these $\mathcal{N}(\mathcal{V})$, φ and ψ satisfy the conditions (1)-(3).

It is easily seen that (11) implies (1) and (2). It remain to prove that (3) holds.

We shall construct a map $\psi_0: |\mathcal{N}(\mathcal{V})^{(n+1)}|\to\mathrm{EW}_{Z_p}(L, n)$ in the following way: note that if $\langle U\rangle\in\mathcal{N}(\mathcal{V})^{(n+1)}$, there exists $B_U\in\mathcal{B}_3$ with $U\subseteqq c^{-1}(B_U)$. ψ_0 on $|\mathcal{N}(\mathcal{V})^{(0)}|$ is defined by an element $\psi_0(\langle U\rangle)\in B_U$ for each $\langle U\rangle\in\mathcal{N}(\mathcal{V})^{(0)}$. Let $\langle U_0, \cdots, U_m\rangle\in\mathcal{N}(\mathcal{V})^{(n+1)}$. Then by $\emptyset\neq U_0\cap\cdots\cap U_m\subseteqq c^{-1}(B_{U_0})\cap\cdots\cap c^{-1}(B_{U_m})$, we have

$$\psi_0(\{\langle U_0\rangle, \cdots, \langle U_m\rangle\})\subseteqq\mathrm{st}(B_{U_0}, \mathcal{B})\subseteqq B \qquad \text{for some } B\in\mathcal{B}_2.$$

It show that ψ_0 is a partial realization of $\mathcal{N}(\mathcal{V})^{(n+1)}$ in $\mathrm{EW}_{Z_p}(L, n)$ relative to $\mathcal{B}_2$. Therefore, by the construction of $\mathcal{B}_2$, we may define ψ_0 to be a *full* *realization relative to* $\mathcal{B}_1$. Then by the same way in [21, p. 245 (8)] we can show that

(12) if $t\in|\mathcal{N}(\mathcal{V})^{(n+1)}|$ with $\psi(t)\in\operatorname{Int}\delta$ and $\psi_0(t)\in\psi_L^{-1}(\tau)$ for $\delta, \tau\in L$, then there exist $\sigma, \lambda\in L$ such that $\delta<\sigma$ and $\sigma\cap\lambda\neq\emptyset\neq\lambda\cap\tau$.

Now, by the property (v) in Lemma 2.1, we can choose

(13) a cellular map $\psi_1: |\mathcal{N}(\mathcal{V})^{(n+1)}|\to \mathrm{EW}_{Z_p}(L, n)^{(n+1)}$ such that for each $t\in|\mathcal{N}(\mathcal{V})^{(n+1)}|$, if $\psi_0(t)\in\psi_L^{-1}(\tau)$, then $\psi_1(t)\in\psi_L^{-1}(\tau)^{(n+1)}$.

By the simplicial approximation theorem, we assume that ψ_1 is PL.

If $n\geqq 2$, by the properties (4) and (1) in Lemma 2.1, we have

$$\mathrm{EW}_{Z_p}(L, n)^{(n+1)}=|L^{(n)}|\cup\bigcup\{\partial\sigma\times[0, 1]\cup_{\alpha_\sigma} B_\sigma^{n+1}: \sigma\in L,\ \dim\sigma=n+1\},$$

where $\alpha_\sigma: \partial B_\sigma^{n+1}\to\partial\sigma$ is a map of degree p. For each $(n+1)$-simplex σ of L, choose a point $z_\sigma\in B_\sigma^{n+1}\setminus\partial B_\sigma^{n+1}$, and take the retraction

$$r: \mathrm{EW}_{Z_p}(L, n)^{(n+1)}\setminus\{z_\sigma: \sigma\in L,\ \dim\sigma=n+1\}\longrightarrow|L^{(n)}|$$

induced by the compositions of the radial projection of $B_\sigma^{n+1}\setminus\{z_\sigma\}$ onto $\partial\sigma\times\{1\}$ and the natural projection of $\partial\sigma\times[0, 1]$ onto $\partial\sigma\times\{0\}\subseteqq|L^{(n)}|$.

If $n=1$, for every simplex σ of $\dim\sigma\geqq 2$, $\psi_L^{-1}(\sigma^{(2)})$ may be represented as the form (5) in Lemma 2.1:

$$\psi_L^{-1}(\sigma)^{(2)}=\sigma^{(1)}\times[0, 1]\cup_{\alpha_1} B^2\cup_{\alpha_2}\cdots\cup_{\alpha_{r_\sigma}} B^2\cup_{\beta_1} B^2\cup_{\beta_2}\cdots\cup_{\beta_{k_\sigma}} B^2.$$

Then choose points $u_1^\sigma, \cdots, u_{r_\sigma}^\sigma, v_1^\sigma, \cdots, v_{k_\sigma}^\sigma$ of $\psi_L^{-1}(\sigma^{(1)})^{(2)}\setminus\sigma^{(1)}\times[0, 1]$ for each B^2 and the retraction $r: \mathrm{EW}_{Z_p}(L, n)^{(2)}\setminus\{u_1^\sigma, \cdots, u_{r_\sigma}^\sigma, v_1^\sigma, \cdots, v_{k_\sigma}^\sigma: \sigma\in L,\ \dim\sigma\geqq 2\}\to|L^{(1)}|$ induced by the compositions of the radial projections of $B^2\setminus\{u_j^\sigma\}$ or $B^2\setminus\{v_j^\sigma\}$ onto S^1 and the natural projection of $\sigma^{(1)}\times[0, 1]$ onto $\sigma^{(1)}\times\{0\}\subseteqq|L^{(1)}|$.

In both cases, we put

$$\psi'\equiv r\circ\psi_1|_{|\mathcal{N}(\mathcal{V})^{(n)}|}: |\mathcal{N}(\mathcal{V})^{(n)}|\longrightarrow|L^{(n)}|.$$

Then the map ψ' holds the conditions (i), (ii). First, we show the condition (i). Let $t\in|\mathcal{N}(\mathcal{V})^{(n)}|$. By (12), there exist $\sigma, \lambda, \tau\in L$ such that $\sigma\cap\lambda\neq\emptyset\neq\lambda\cap\tau$ and $\psi(t)\in\sigma$, $\psi_0(t)\in\psi_L^{-1}(\tau)$. Then since $\psi_1(t)$ is an element of $\psi_L^{-1}(\tau)^{(n)}$, we have $\psi'(t)\in\tau$. Hence, we have $\psi(t), \psi'(t)\in\overline{\mathrm{st}}(\lambda, L)\subseteqq U$ for some $U\in\mathcal{V}_0$ (see (4)). Next, we must show the condition (ii). But, this is similar to the proof of Theorem 3.3.3. Hence, we omitted it here.

Now, we shall show that f is $(\boldsymbol{Z}_p, n)$-cohomological. By (2), we can easily see that $(\psi\circ\varphi, f)\leqq\mathcal{V}$. So, we show that ψ is $(\boldsymbol{Z}_p, n, \mathcal{V})$-approximable.

Let M be a triangulation of $|K|$. Note that for a simplicial approximation j of $id_{|M|}: |M|=|K|\to|K|$ with respect to K, we have that

$$j(|M^{(n+1)}|)\subseteqq|K^{(n+1)}| \quad \text{and} \quad j(|M^{(n)}|)\subseteqq|K^{(n)}|.$$

Then by (1) and (3), we can easily see that the map

$$\psi''\equiv\psi'\circ j: |M^{(n)}|\longrightarrow|L^{(n)}|$$

holds the conditions. □

The reverse implication is proved by the standard way [21]. First, we need some notations.

We may assume that the Eilenberg-MacLane space $K(\boldsymbol{Z}_p, n)$ is a metrizable, locally compact separable space. Then by the Kuratowski-Wojdyslawski's theorem, we can consider that $K(\boldsymbol{Z}_p, n)$ is a closed subset of a convex subset C of a normed linear space E. Note that C is AR(metrizable spaces). Since $K(\boldsymbol{Z}_p, n)$ is ANR, there exist a closed neighborhood F in C and a retraction $r: F\to K(\boldsymbol{Z}_p, n)$. Further, we can choose an open cover $\mathcal{W}_0$ of $\mathrm{Int}_C F$ such that

(1) for any space Z and any maps $\alpha, \beta: Z\to F$ with $(\alpha, \beta)\leqq\mathcal{W}_0$, the maps $r\circ\alpha, r\circ\beta: Z\to K(\boldsymbol{Z}_p, n)$ are homotopic in $K(\boldsymbol{Z}_p, n)$.

Then we take an open, *convex* cover $\mathcal{W}$ of C such that

(2) if $W\in\mathcal{W}$ with $W\cap K(\boldsymbol{Z}_p, n)\neq\emptyset$, there exists $U\in\mathcal{W}_0$ with $\mathrm{st}(W, \mathcal{W})\subseteqq U$.

Select a star refinement $\mathcal{V}$ of $\mathcal{W}$.

Let $h_0: C\to|\mathcal{N}(\mathcal{V})|$ be a Kuratowski's map wiɜh respect to $\mathcal{V}$ and define a map $h_1: |\mathcal{N}(\mathcal{V})|\to C$ in the following way: a map h_1 on $|\mathcal{N}(\mathcal{V})^{(0)}|$ is defined by an element $h_1(\langle V\rangle)\in V$ for each $\langle V\rangle\in|\mathcal{N}(\mathcal{V})^{(0)}|$. Next, by using the convexity of C, we extend h_1 *linearly* on each simplex $|\mathcal{N}(\mathcal{V})|$. Let $\sigma=\langle V_0, \cdots, V_m\rangle\in|\mathcal{N}(\mathcal{V})|$. Then by $V_0\cap\cdots\cap V_m\neq\emptyset$.

$$h_1(\{\langle V_0\rangle, \cdots, \langle V_m\rangle\})\subseteqq\mathrm{st}(V_0, \mathcal{V})\subseteqq W_\sigma \qquad \text{for some } W_\sigma\in\mathcal{W}.$$

Thus, by the construction of h_1, we have $h_1(\sigma)\subseteqq W_\sigma$.

Let $\mathcal{N}_1$ be a subcomplex $\mathcal{N}(\{V\in\mathcal{V}: V\cap K(\boldsymbol{Z}_p, n)\neq\emptyset\})$ of $\mathcal{N}(\mathcal{V})$. Let $\mathcal{N}_0$ be a simplicial neighborhood of $\mathcal{N}_1$ in $\mathcal{N}(\mathcal{V})$ such that if $\langle V_0\rangle\in\mathcal{N}_0$, there exists $\langle V_1\rangle\in\mathcal{N}_1$ with $V_0\cap V_1\neq\emptyset$. Then we can easily see the followings:

(3) for each $x\in K(\boldsymbol{Z}_p, n)$, there exists $W\in\mathcal{W}$ with $x, h_1\circ h_0(x)\in W$,

(4) $h_1(|\mathcal{N}_0|)\subseteqq\mathrm{st}(K(\boldsymbol{Z}_p, n), \mathcal{W})\subseteqq F$,

(5) $h_0(K(\boldsymbol{Z}_p, n))\subseteqq|\mathcal{N}_1|\subseteqq|\mathcal{N}_0|$.

PROOF OF SUFFICIENCY. Let A be a closed subset of X and $h: A\to K(\boldsymbol{Z}_p, n)$ be a map. We consider the above-mentioned nerve $\mathcal{N}(\mathcal{V})$ and maps h_0, h_1. We take an open cover $\mathcal{U}$ of $|\mathcal{N}(\mathcal{V})|$ such that

(6) $\mathrm{st}^3(|\mathcal{N}_1|, \mathcal{U})\subseteqq|\mathcal{N}_0|$,

(7) $\mathrm{st}^3(\mathcal{U})<h_1^{-1}(\mathcal{W})$,

and choose a subdivision $\mathfrak{N}$ of $\mathfrak{N}(\mathcal{V})$ such that if $\sigma\in\mathfrak{N}$ there exists $U\in\mathcal{U}$ with $\sigma\subseteqq U$.

Since C is AE, there is an extension $H: X\to C$ of h. Then by the assumption, the map $h_0\circ H: X\to|\mathfrak{N}(\mathcal{V})|$ is $(\boldsymbol{Z}_p, n)$-cohomological. Hence, there exist a polyhedron Q and maps $\varphi: X\to Q$, $\psi: Q\to|\mathfrak{N}(\mathcal{V})|$ such that

(8) $(\psi\circ\varphi, h_0\circ H)\leqq\mathcal{U}$,

(9) ψ is $(\boldsymbol{Z}_p, n, \mathcal{U})$-approximable.

By using the simplicial approximation theorem, we obtain a triangulation M of Q and a simplicial approximation $\psi^*: M\to\mathfrak{N}$ of ψ. Then by (8), (9), we have

(10) $(\psi^*\circ\varphi, h_0\circ H)\leqq\mathrm{st}\,\mathcal{U}$,

(11) ψ^* is $(\boldsymbol{Z}_p, n, \mathrm{st}\,\mathcal{U})$-approximable.

Now, by (11) with respect to M, there exist a triangulation L and a PL-map $\psi': |M^{(n)}|\to|L^{(n)}|$ such that

(12) $(\psi', \psi^*|_{|M^{(n)}|})\leqq\mathrm{st}\,\mathcal{U}$,

(13) for any map $\alpha: |L^{(n)}|\to K(\boldsymbol{Z}_p, n)$, there exists an extension $\beta: |M^{(n+1)}|\to K(\boldsymbol{Z}_p, n)$ of $\alpha\circ\psi'$.

CLAIM. *There exists a map* $\xi: Q\to K(\boldsymbol{Z}_p, n)$ *such that* $\xi|_{\psi^{*-1}(|\mathfrak{N}_0|)}=r\circ h_1\circ\psi^*|_{\psi^{*-1}(|\mathfrak{N}_0|)}$.

Construction of ξ. First, we shall see that

(14) for each $x\in D\equiv\psi^{*-1}(|\mathfrak{N}_0|)\cap|M^{(n)}|$, there exists $U\in\mathcal{W}_0$ such that $h_1\circ\psi^*(x)$, $h_1\circ\psi'(x)\in U$.

By (12), there exist $U_1, U_2, U_3\in\mathcal{U}$ such that $U_1\cap U_2\neq\emptyset\neq U_2\cap U_3$ and $\psi^*(x)\in U_1$, $\psi'(x)\in U_3$. Then by (7), we have $W\in\mathcal{W}$ with $h_1(U_1\cup U_2\cup U_3)\subseteqq W$. Since $\psi^*(x)\in|\mathfrak{N}_0|$, by (4), there exists $W'\in\mathcal{W}$ such that $h_1\circ\psi^*(x)\in W$ and $W'\cap K(\boldsymbol{Z}_p, n)\neq\emptyset$. Hence by (2), we obtain $U\in\mathcal{W}_0$ such that $h_1\circ\psi^*(x)$, $h_1\circ\psi'(x)\in\mathrm{st}(W', \mathcal{W})\subseteqq U$.

Therefore by (14) and (1), we see the followings:

(15) $h_1\circ\psi'(D)\subseteqq F$,

(16) $r\circ h_1\circ\psi^*|_D\simeq r\circ h_1\circ\psi'|_D$ in $K(\boldsymbol{Z}_p, n)$.

Since D is a subpolyhedron of $|M^{(n)}|$ and ψ' is PL, $\psi'(D)$ is subpolyhedron of $|L^{(n)}|$. Hence, from $\pi_q(K(\boldsymbol{Z}_p, n))=0$ for $q<n$ (if $n=1$, the path-connectedness of $K(\boldsymbol{Z}_p, n)$), there exists an extension

$$\alpha: |L^{(n)}|\longrightarrow K(\boldsymbol{Z}_p, n)$$

of $r\circ h_1|_{\psi'(D)}: \psi'(D)\to K(\boldsymbol{Z}_p, n)$.

Then by (13), we have an extension

$$\beta: |M^{(n+1)}|\longrightarrow K(\boldsymbol{Z}_p, n)$$

of $\alpha\circ\psi'$.

Now, put

$$R\equiv|M^{(n+1)}|\setminus\bigcup\{\mathrm{Int}\,\sigma:\sigma\in M,\ \dim\sigma=n+1,\ \sigma\subseteq\psi^{*-1}(|\mathcal{N}_0|)\}.$$

Then since for each $x\in D\subseteq R$ we have $\beta(x)=\alpha\circ\psi'(x)=r\circ h_1\circ\psi'(x)$,

(17) $$\beta_D\simeq r\circ h_1\circ\psi'(x)|_D\simeq r\circ h_1\circ\psi^*|_D \quad \text{in } K(\boldsymbol{Z}_p,\ n).$$

By the homotopy extension theorem, there exists an extension $\xi_R: R\to K(\boldsymbol{Z}_p, n)$ of $r\circ h_1\circ\psi^*|_D$.

Since for $\sigma\in M$ with $\dim\sigma=n+1$ and $\sigma\subseteq\psi^{*-1}(|\mathcal{N}_0|)$, we have $\xi_R|_{\partial\sigma}=r\circ h_1\circ\psi^*|_{\partial\sigma}$, there exists an extension $\xi_{n+1}: |M^{(n+1)}|\to K(\boldsymbol{Z}_p, n)$ of ξ_R such that $\xi_{n+1}|_{\psi^{*-1}(|\mathcal{N}_0|)\cap|M^{(n+1)}|}=r\circ h_1\circ\psi^*|_{\psi^{*-1}(|\mathcal{N}_0|)\cap|M^{(n+1)}|}$.

Hence, we can define a map $\xi': \psi^{*-1}(|\mathcal{N}_0|)\cup|M^{(n+1)}|\to K(\boldsymbol{Z}_p, n)$ by the following:

$$\xi'\equiv(r\circ h_1\circ\psi^*|_{\psi^{*-1}(|\mathcal{N}_0|)})\cup\xi_{n+1}.$$

Therefore from $\pi_q(K(\boldsymbol{Z}_p, n))=0$ for $q>n$, we obtain an extension $\xi: Q\to K(\boldsymbol{Z}_p, n)$ of ξ' such that $\xi|_{\psi^{*-1}(|\mathcal{N}_0|)}=r\circ h_1\circ\psi^*|_{\psi^{*-1}(|\mathcal{N}_0|)}$. It completes the construction.

Now, we put

$$h'\equiv\xi\circ\varphi: X\longrightarrow K(\boldsymbol{Z}_p, n).$$

Then to complete the proof it suffices to prove

(18) $$h'|_A\simeq h \quad \text{in } K(\boldsymbol{Z}_p,\ n).$$

First, we shall see that

$$\psi^*\circ\varphi(A)\subseteq|\mathcal{N}_0|.$$

Let $a\in A$. By (10), there exist $U_1, U_2, U_3\in\mathcal{U}$ such that

(19) $U_1\cap U_2\neq\emptyset\neq U_2\cap U_3$ and $\psi^*\circ\varphi(a)\in U_1$, $h_0\circ H(a)\in U_3$.

Then since $h_0\circ H(a)=h_0\circ h(a)\in h_0(K(\boldsymbol{Z}_p, n))\subseteq|\mathcal{N}_1|$, we have $\psi^*\circ\varphi(a)\in|\mathcal{N}_0|$ by (6).

Hence, by Claim, we have for each $a\in A$ $h'(a)=\xi\circ\varphi(a)=r\circ h_1\circ\psi^*\circ\varphi(a)$. Therefore, by (1), it suffices to see that

(20) there exists $U\in\mathcal{W}_0$ such that $h_1\circ\psi^*\circ\varphi(a)$, $h(a)\in U$.

Let $U_1, U_2, U_3\in\mathcal{U}$ with the property (19). By (7), there exists $W\in\mathcal{W}$ such that $U_1\cup U_2\cup U_3\subseteq h_1^{-1}(W)$. By (3) we choose $W'\in\mathcal{W}$ such that $h(a)$, $h_1\circ h_0\circ h(a)\in W'$. Therefore, since $h(a)\in K(\boldsymbol{Z}_p, n)$, there exists $U\in\mathcal{W}_0$ such that

$$h_1\circ\psi^*\circ\varphi(a),\ h(a)\in\mathrm{st}(W',\ \mathcal{W})\subseteq U.$$

It completes the proof. □

5. Approximable dimension

5.1. DEFINITION. A space X has *approximable dimension with respect to a coefficient group G of less than and equal to n (abbreviated, a-$\dim_G X \leqq n$)* provided that for every polyhedron P, map $f: X \to P$ and open cover $\mathcal{V}$ of P, there exist a polyhedron Q and maps $\varphi: X \to Q$, $\psi: Q \to P$ such that

(i) $(\psi \circ \varphi, f) \leqq \mathcal{V}$,

(ii) ψ is $(G, n, \mathcal{V})$-approximable.

If X is compact, we use compact polyhedron and positive number ε instead of above-mentioned polyhedron and open cover, respectively.

First, we state fundamental inequalities of a-$\dim_G$.

5.2. THEOREM. *For a compact Hausdorff or metrizable space X and an arbitrary abelian group G, we hold the following inequalities*:

$$c\text{-}\dim_G X \leqq a\text{-}\dim_G X \leqq \dim X .$$

PROOF. The second inequality is trivial. We can see the first inequality by the strategy similar to the proof of the sufficiency in Theorem 3.3, 4.3. □

As we will show in latter sections, our approach of a-$\dim_G$ gives useful applications. In general, a-$\dim_G$ is different from c-$\dim_G$ (see section 8). However, in special cases of coefficient group G, a-$\dim_G$ coincides with c-$\dim_G$.

5.3. THEOREM. *If $G = \boldsymbol{Z}$ or $\boldsymbol{Z}_p$, where p is a prime number, for every compact Hausdorff or metrizable space X,*

$$a\text{-}\dim_G X = c\text{-}\dim_G X .$$

PROOF. From Theorem 3.3, 4.3, 5.2, we see the fact. □

We will use the new notion, approximate (inverse) systems and their limits, instead of usual inverse systems and inverse limits. They were introduced by S. Mardešić and L. R. Rubin [17] and took an important role in [18]. We quote their basic definitions.

5.4. DEFINITION. An *approximate (inverse) system* of metric compacta $\mathcal{X} = (X_a, \varepsilon_a, p_{a,a'}, A)$ consists of the followings: A directed ordered set $(A, \leqq)$; a compact metric space $\mathcal{X}_a$ with a metric d and a real number $\varepsilon_a > 0$; for each pair $a \leqq a'$ from A, a map $p_{a,a'}: X_{a'} \to X_a$, satisfying the following conditions:

(A1) $d(p_{a_1, a_2} \circ p_{a_2 a_3}, p_{a_1 a_3}) \leqq \varepsilon_{a_1}$, $a_1 \leqq a_2 \leqq a_3$; $p_{aa} = id_{X_a}$, (A2) for every $a\in A$ and $\eta>0$, there exists $a'\geqq a$ such that $d(p_{a,a_1}\circ p_{a_1a_2}, p_{aa_2})\leqq\eta$ for every $a_2\geqq a_1\geqq a'$,

(A3) for every $a\in A$ and $\eta>0$, there exists $a'\geqq a$ such that for every $a''\geqq a'$ and every pair of points x, x' of $X_{a''}$, if $d(x, x')\leqq\varepsilon_{a''}$, then $d(p_{aa''}(x), p_{aa''}(x'))\leqq\eta$.

We refer to the number ε_a as the *meshs* of the approximate system $\mathcal{X}$.

If $\pi_a:\prod_{a\in A}X_a\to X_a$, $a\in A$, denote the projections, we define the *limit space* $X=\lim\mathcal{X}$ and the natural projections $p_a: X\to X_a$ as follows:

5.5. DEFINITION. A point $x=(x_a)\in\prod_{a\in A}X_a$ belongs to $X=\lim\mathcal{X}$ provided that for every $a\in A$,

$$x_a=\lim_{a_1} p_{aa_1}(x_{a_1}).$$

The projections $p_a: X\to X_a$ are given by $p_a=\pi_a|_X$.

Next we quote results from [17] and [18] needed in this note. The proofs may be found in them.

5.6. PROPOSITION. *Let $\mathcal{X}=(X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system. Then we have the following properties:*

(i) *if every X_a is non-empty, then $X=\lim\mathcal{X}$ is a non-empty compact Hausdorff space,*

(ii) *for each $a\in A$, $\lim_{a_1} d(p_a, p_{aa_1}\circ p_{a_1})=0$, where $d(f, g)=\sup\{d(f(x), g(x)): x\in X\}$,*

(iii) *for each open cover $\mathcal{U}$ of $X=\lim\mathcal{X}$, there is $a\in A$ such that for every $a_1\geqq a$, there exists an open cover $\mathcal{V}$ of X_{a_1} for which $p_{a_1}^{-1}(\mathcal{V})$ refines $\mathcal{U}$,*

(iii′) *if $\dim X_a\leqq n$ for all $a\in A$, then $\dim X\leqq n$,*

(iv) *for every $\varepsilon>0$, every compact ANR P and every map $h: X\to P$, there is $a\in A$ such that for every $a_1\geqq a$, there is a map $f: X_a\to P$ which satisfies $d(f\circ p_{a_1}, k)\leqq 2\varepsilon$.*

5.7. PROPOSITION. *Let $\mathcal{X}=(X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system. If for every $a_1\in A$, every compact ANR P, and every map $h: X_{a_1}\to P$, there ts $a_1'\geqq a_1$ such that for every $a_2\geqq a_1'$, there is $a_2'\geqq a_2$ such that for every $a_3\geqq a_2'$,*

$$h\circ p_{a_1a_2}\circ p_{a_2a_3}\simeq 0,$$

then every map from $X=\lim\mathcal{X}$ to P is null-homotopic.

Namely, under the above assumptson, the set $[X, P]$ is trivial.

In the proof of our main result we need the following characterization of a-$\dim_G$ by approximate systems.

5.8. THEOREM. *Let $\mathcal{X}=(X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system of compact polyhedra with the limit $X=\lim \mathcal{X}$ and p be a prime number. Then X has approximable dimension with respect to $G\leqq n$ if and only if for every $a\in A$ and every $\varepsilon>0$, there is $a'\geqq a$ such that for every $a''\geqq a'$, the map $p_{aa''}: X_{a''}\to X_a$ is (G, n, ε)-approximable.*

PROOF. Suppose that a-$\lim_G X\leqq n$. Take any $a\in A$ and any positive number $\varepsilon>0$. By (A2), there is $a_1\geqq a$ such that

$$d(p_{aa'}\circ p_{a'a''}, p_{aa''})\leqq \varepsilon/7\,, \qquad a_1\leqq a'\leqq a''\,. \tag{1}$$

Specially,

$$d(p_{aa'}\circ p_{a'a''}\circ p_{a''}, p_{aa''}\circ p_{a''})\leqq \varepsilon/7\,, \qquad a_1\leqq a'\leqq a''\,. \tag{1'}$$

Hence, by Definition 5.5, we have that

$$d(p_{aa'}\circ p_{a'}, p_a)\leqq \varepsilon/7\,, \qquad a_1\leqq a_{a'}\,. \tag{2}$$

By the assumption, there is a compact polyhedron Q and maps $\varphi: X\to Q$, $\psi: Q\to X_a$ such that

(3) $d(\psi\circ\varphi, p_a)\leqq\varepsilon/7$,

(4) ψ is $(G, n, \varepsilon/7)$-approximable.

Let take a positive number $\delta>0$ such that

$$\text{if } x, x'\in Q \text{ and } d(x, x')\leqq\delta, \quad \text{then } d(\psi(x), \psi(x'))\leqq\varepsilon/7. \tag{5}$$

By Proposition 5.6 (iv), there exists $a'\geqq a_1$ and a map $g: X_{a'}\to Q$ such that

$$d(\varphi, g\circ p_{a'})\leqq\delta\,. \tag{6}$$

Then, (6), (5), (3) and (2), we see

$$\begin{aligned} d(\psi\circ g\circ p_{a'}, p_{aa'}\circ p_{a'}) &\leqq d(\psi\circ g\circ p_{a'}, \psi\circ\varphi)+d(\psi\circ\varphi, p_a)+d(p_a, p_{aa'}\circ p_{a'}) \\ &\leqq 3\varepsilon/7\,. \end{aligned} \tag{7}$$

Hence we have a neighborhood U of $p_{a'}(X)$ in $X_{a'}$ such that

$$d(\psi\circ g|_U, p_{aa'}|_U)\leqq 4\varepsilon/7\,. \tag{8}$$

Then there exists $a_1'\geqq a'$ such that

$$p_{a'a''}(X_{a''})\subseteqq U \qquad \text{for every } a''\geqq a_1'\,. \tag{9}$$

By (8) and (1), we have that for every $a''\geqq a_1'$,

(10) $$d(\psi\circ g\circ p_{a'a''},\ p_{aa''})\leqq 5\varepsilon/7\,.$$

Now we show that $p_{aa''}$ is (G, n, ε)-approximable. By (4), take a triangulation T_a of X_a which realizes the $(G, n, \varepsilon/7)$-approximability of ψ. Let us take triangulations $T_{a''}$ of $X_{a''}$ and M of Q with $mesh(M)\leqq\delta$. Then we have a subdivision of $T'_{a''}$ of $T_{a''}$ and a simplicial approximation $h: |T'_{a''}|\to|M|$ of $g\circ p_{a'a''}$ such that

(11) $$d(h,\ g\circ p_{a'a''})\leqq mesh(M)\leqq\delta\,.$$

Hence, by (11), (5) and (10), we have

(12) $$d(\psi\circ h,\ p_{aa''})\leqq d(\psi\circ h,\ \psi\circ g\circ p_{a'a''})+d(\psi\circ g\circ p_{a'a''},\ p_{aa''})\leqq 6\varepsilon/7\,.$$

On the other hand, by the property of T_a, there exists a map $\psi': |M^{(n)}|\to |T_a^{(n)}|$ such that

(13) $d(\psi',\ \psi|_{|M^{(n)}|})\leqq\varepsilon/7$,

(14) for every map $\xi: |T_a^{(n)}|\to K(G, n)$, the map $\xi\circ\psi': |M^{(n)}|\to K(G, n)$ admits a continuous extension over Q.

By $h(|T_{a''}^{(n)}|)\subseteqq h(|(T'(_{a''}^{(n)}|)\subseteqq|M^{(n)}|$, we can define the composition $\psi'\circ h|_{|T_{a''}^{(n)}|}$: $|T_{a''}^{(n)}|\to|T_a^{(n)}|$. Then, by (12) and (13), we have that

(15) $$d(\psi'\circ h|_{|T_{a''}^{(n)}|},\ p_{aa''}|_{|T_{a''}^{(n)}|})\leqq\varepsilon\,.$$

Moreover, by (14), for every map $\xi: |T_a^{(n)}|\to K(G, n)$, the map $\xi\circ\psi'\circ h|_{|T_{a''}^{(n)}|}$: $|T_{a''}^{(n)}|\to K(G, n)$ admits a continuous extension over $X_{a''}$. That is, the map $p_{aa''}$ is (G, n, ε)-approximable.

Conversely, we assume that the condition of Theorem 5.8 is satisfied. Take a map $f: X\to P$ of X to a compact polyhedron P and a positive number $\varepsilon>0$. By Proposition 5.6 (iv), there exists $a\in A$ and a map $g: X_a\to P$ such that

(16) $$d(f,\ g\circ p_a)\leqq\varepsilon/2\,.$$

Let $\delta>0$ be a positive number such that

(17) if $x, x'\in X_a$ and $d(x, x')\leqq\delta$, then $d(g(x), g(x'))\leqq\varepsilon/2$.

By the same way in the first part of the proof, we can find $a'\geqq a$ such that

(18) $$d(p_{aa''}\circ p_{a''},\ p_a)\leqq\delta \quad \text{for every } a''\geqq a'\,.$$

Then we take $a''\geqq a'$ such that

(19) the map $p_{aa''}: X_{a''}\to X_a$ is (G, n, δ)-approximable

By (18), (17) and (16),

(20) $$d(f,\ g\circ p_{aa''}\circ p_{a''})\leqq d(f,\ g\circ p_a)+d(g\circ p_a,\ g\circ p_{aa''}\circ p_{a''})$$
$$\leqq \varepsilon/2+\varepsilon/2<\varepsilon\ .$$

Hence it suffices to show that $g\circ p_{aa''}$ is (G, n, ε)-approximable. Let M be a triangulation of P with $mesh(M)\leqq\varepsilon/2$. Let T_a be a triangulation of X_a which realizes the (G, n, δ)-approximability of $p_{aa''}$. Then for any triangulation of $T_{a''}$ of $X_{a''}$, there is a map $\varphi: |T_{a''}^{(n)}|\to|T_a^{(n)}|$ such that

(21) $d(\varphi,\ p_{aa''}|_{|T_{a''}^{(n)}|})\leqq\delta$,

(22) for any map $\xi: |T_a^{(n)}|\to K(G, n)$, the map $\xi\circ\varphi$ admits a continuous extension over $X_{a''}$.

On the other hand, we have a subdivision T'_a of T_a and a simplicial map $h: X_a\to P$ with respect to T'_a and M such that

(23) $$d(h,\ g)\leqq\varepsilon/2\ .$$

From $\varphi(|T_{a''}^{(n)}|)\subseteqq|T_a^{(n)}|\subseteqq|(T'_a)^{(n)}|$ and $h(|(T'_a)^{(n)}|)\subseteqq|M^{(n)}|$, we have the map $\psi: |T_{a''}^{(n)}|\to|M^{(n)}|$ defined by $\psi(z)=h\circ\varphi(z)$. Then by (21), (17) and (23),

(24) $$d(g\circ p_{aa''},\ \psi)\leqq d(g\circ p_{aa''},\ g\circ\varphi)+d(g\circ\varphi,\ h\circ\varphi)\leqq\varepsilon/2+\varepsilon/2=\varepsilon\ .$$

For any map $\xi: |M^{(n)}|\to K(G, n)$, consider the map $\xi\circ h_{|T_a^{(n)}|}: |T_a^{(n)}|\to K(G, n)$. Then, by (22), there is a map $\zeta: X_{a''}\to K(G, n)$ such that

(25) $$\zeta|_{|T_{a''}^{(n)}|}=\xi\circ(h|_{|T_a^{(n)}|})\circ\varphi|_{|T_{a''}^{(n)}|}\ .$$

Namely, the map $\xi\circ\psi$ has a continuous extension over $X_{a''}$. It follows that $g\circ p_{aa''}$ is (G, n, ε)-approximable. Therefore, we have a-$\dim_G X\leqq n$. □

5.9. COROLLARY. *Let $\mathcal{X}=(X_a, \varepsilon_a, p_{aa'}, A)$ be an approximate system of compact polyhedra with the limit $X=\lim\mathcal{X}$. Let $G=\boldsymbol{Z}$ or $\boldsymbol{Z}_p$. Then c-$\dim_G X\leqq n$ if and only if for every $a\in A$ and every $\varepsilon>0$, there exists $a'\geqq a$ such that for every $a''\geqq a'$, the map $p_{a'a''}: X_{a''}\to X_a$ is (G, n, ε)-approximable.*

In the latter we need the following property.

5.10. THEOREM. *Let X be a compact space of a-$\dim_G X\geqq n\geqq 1$. Then there is an approximate system $\mathcal{X}=(X_a, \varepsilon_a, p_{aa'}, A)$ with $\lim\mathcal{X}=X$ such that for every $a\in A$ and every pair $a\leqq a'$ from A,*

(i) *X_a is a compact polyhedron with a metric $d=d_a\leqq 1$,*

(ii) $\dim X_a\geqq n$,

(iii) *$p_{aa'}: X_{a'}\to X_a$ is a surjective PL-map, and*

(iv) $\mathrm{card}(A)\leqq\omega(X)$.

PROOF. By [18, Theorem 1, Proposition 12], it is known that every compact space X admits an approximate system $\mathscr{X}=(X_a, \varepsilon_a, p_{aa'}, A)$ of compact polyhedra with $\lim \mathscr{X}=X$ satisfying the conditions (i), (iii) and (iv). Suppose that the subset $A_0=\{a\in A: \dim X_a<n\}$ is cofinal in A. Then for any $a\in A$, let take $a'\in A_0$ with $a'\geqq a$. Then for any positive number $\varepsilon>0$, the map $p_{aa'}: X_{a'}\to X_a$ is $(G, n-1, \varepsilon)$-approximable. Hence for every $a''\geqq a'$, the map $p_{aa''}: X_{a''}\to X_a$ is $(G, n-1, \varepsilon)$-approximable. By Theorem 5.8, $a\text{-}\dim_G X\leqq n-1$. But it is a contradiction. Thus, tne subset A_0 is not cofinal. Therefore it suffices to consider the subsystem of $\mathscr{X}$ which is indexed by the set $A\setminus A_0$. □

6. Resolutions for compact spaces

We quote our main theorem as follows:

6.1. THEOREM. *Let X be a compact space having approximable dimension with respect to G of less than and equal to n. Then there exists a compact space Z of* $\dim Z\leqq n$ *and* $w(Z)\leqq w(X)$, *and a surjective* UV^{n-1}*-map* $f: Z\to X$ *such that for every* $x\in X$, *the set* $[f^{-1}(x), K(G, n)]$ *of homotopy classes is trivial.*

Our proof essentially depends on Mardešić-Rubin's way [18]. First, we introduce the notion of the n-dimensional core Z_L and the stacked n-dimensional core of a complex L from [18]. The detail is omitted here.

Let L be a finite complex and let n be a nonnegative integer. Let L, L', L'', $\cdots$, L^k, $\cdots$ be the iterated subdivisions of L. For each $k\geqq 0$, choose a simplicial approximation $q_{k\,k+1}: |L^{k+1}|\to|L^k|$ of the identity $id_L: |L|=|L^{k+1}|\to|L^k|$, and let $q_{k\,k+j}\equiv q_{k\,k+1}\circ\cdots\circ q_{k+j-1\,k+j}: |L^{k+j}|\to|L^k|$. Then $q_{k\,k+j}$ is also a simplicial approximation of id_L. Hence we have

(1) $d(q_{k\,k+j}, id_L)\leqq mesh(L^k)$ for $j\geqq 1$,

(2) $q_{k\,k+j}((L^{k+j})^{(n)})\subseteq(L^k)^{(n)}$ for $j\geqq 1$.

Therefore we have an inverse sequence of polyhedra

$$\mathcal{L}=(|(L^k)^{(n)}|, q_{k\,k+1}).$$

Then *n-dimensional core* of L is defined as the inverse limit

$$Z_L=\lim \mathcal{L}\,. \tag{3}$$

Clearly, we have

$$\dim Z_L\leqq n\,. \tag{4}$$

Let $q_k: Z_L\to|(L^k)^{(n)}|$ be the projections. They by the *Sperner's lemma,* each $q_{k\,k+1}$ is surjective, and thereby, all of $q_{k\,k+j}$ and q_k are surjective. Moreover, by (1),

(5) $$d(q_k, q_{k+j}) \leqq mesh(L^k) \quad \text{for } j \geqq 1 \text{ in } |L|.$$

Hence $\{q_k\}_{k\geqq 1}$ is a Cauchy sequence of map from Z_L to $|L|$, because of $\lim mesh(L^k)=0$. Therefore we have the map $f_L: Z_L \to |L|$ given by

(6) $$f_L = \lim q_k .$$

Then by (3), we see

(7) $$d(f_L, q_k) \leqq mesh(L^k).$$

Moreover, q_k is surjective and $\lim mesh(L^k)=0$. Hence $f_L(Z_L)$ is dense in $|L|$, and thereby f_L is surjective.

Next, in order to describe the stacked n-dimensional core of L, we define a new inverse sequence as follows: for each $k=0, 1, 2, \cdots$,

(8) $$L^{*k} = L^{(n)} \oplus (L')^{(n)} \oplus \cdots \oplus (L^k)^{(n)} .$$

Hence

(9) $$|L^{*k+1}| = |L^{*k}| \oplus |(L^{k+1})^{(n)}|.$$

The bonding maps $q^*_{kk+1}: |L^{*k+1}| \to |L^{*k}|$ are given by

(10) $$q_{k\,k+1}(x) = \begin{cases} x & \text{if } x \in |L^{*k}|, \\ q_{k\,k+1}(x) & \text{if } x \in |(L^{k+1})^{(n)}|. \end{cases}$$

We define the *stacked n-dimensional core* Z^*_L as the inverse limit of the inverse sequence $\mathcal{L}^* = (|L^{*k}|, q^*_{kk+1})$,

(11) $$Z^*_L = \lim \mathcal{L}^* = \Big(\bigoplus_{k\geqq 0} |(L^k)^{(n)}| \Big) \cup Z_L ,$$

and denote the natural projections by $q^*_k: Z^*_L \to |L^{*k}|$. Then

(12) $$\dim Z^*_L \leqq n .$$

Moreover we note the following properties:

(13) $Z_L \subseteqq Z^*_L$ and $|L^{*k}| \subseteqq Z^*_L$ for every $k \geqq 0$,

(14) $q^*_k|_{|(L^{k+j})^{(n)}|} = q_{k\,k+j}$ for $j \geqq 1$,

(15) $q^*_k|_{Z_L} = q_k$.

By (15), (5) and the definition of q^*_{kk+1},

(16) $$d(q^*_k, q^*_{k+j}) \leqq mesh(L^k) \quad \text{for } j \geqq 1 \text{ in } |L|.$$

Hence $\{q^*_k\}_{k\geqq 1}$ is a Cauchy sequence of maps from Z^*_L to $|L|$, and therefore we have the map $f_L^*: Z_L^* \to |L|$ defined by

(17) $$f_L^* = \lim q_k^* .$$

Then we know that

(18) $d(f_L^*, q_k^*) \leqq mesh(L^k)$,

(19) $f_L^*|_{|(L^k)^{(n)}|}$ is the inclusion of $|(L^k)^{(n)}|$ into $|L|$,

(20) $f_L^*|_{Z_L} = f_L$.

We note that if we have a metric d on $|L|$ such that $\mathrm{diam}(|L|) \leqq 1$, then we can choose metrics d^* on Z_L^* and d^k in $|L^{*k}|$ such that $\mathrm{diam}(Z_L^*) \leqq 1$, $\mathrm{diam}(|L^{*k}|) \leqq 1$ and

(21) $$d^k(q_k^*(x), q_k^*(x')) \leqq d^*(x, x') \quad \text{for } x, x' \in Z_L^*,\ k \geqq 0.$$

PROOF OF THEOREM 6.1. Let take an approximate system $\mathcal{X} = (X_a, \varepsilon_a, p_{aa'}, A)$ with the limit $\lim \mathcal{X} = X$ which satisfies the conditions (i)-(iv) in Theorem 5.10. Moreover, for each $a \in A$, we may choose a triangulation L_a of X_a such that

(v) $$6 \cdot mesh(L_a) \leqq \varepsilon_a .$$

As the proof as in [18], we will define a new ordering $<'$ in A. We consider the following three conditions for $a_1 < a_2$ and any integer $k \geqq 0$:

(1) $d(p_{a_1a'} \circ p_{a'a''}, p_{a_1a''}) \leqq mesh(L_{a_1}^k)$ for $a_2 \leqq a' \leqq a''$,

(2) if $d(x, x') \leqq \varepsilon_{a''}$ for $x, x' \in X_{a''}$, then for $a_2 \leqq a''$, $d(p_{a_1a''}(x), p_{a_1a''}(x')) \leqq mesh(L_{a_1}^k)$

(3) the map $p_{a_1a''}$ is $(G, n, mesh(L_{a_1}^k))$-approximable for $a_2 \leqq a''$.

Now we put $a_1 <' a_2$ provided that $a_1 < a_2$ and the conditions (1)-(3) hold for $k=0$. Then the ordering $<'$ on A satisfies the following conditions:

(4) if $a_1 <' a_2$, then $a_1 < a_2$,

(5) if $a_1 <' a_2$ and $a_2 \leqq a_3$, then $a_1 <' a_3$,

(6) for any $a \in A$, there is $a' \in A$ such that $a <' a'$.

Hence $A' = (A, <')$ is a directed set with no maximal element. We note that by Theorem 5.8, for any $a_1 \in A$ and integer $k \geqq 0$, there exists $a_2{}' > a_1$ such that the conditions (1)-(3) hold. Moreover,

(7) if $a_1 <' a_2$, then the set of all integers $k \geqq 0$, which satisfy the condition (2), is finite.

Hence, for each pair $a_1 <' a_2$, by (7), there is a maximal integer such that the conditions (1)-(3) hold. We denote the integer by $k(a_1, a_2)$. Clearly we have the following properties:

(8) if $a_1 <' a_2$, $d(p_{a_1a'} \circ p_{a'}, p_{a_1}) \leqq mesh(L_{a_1}^{k(a_1, a_2)})$ for $a' \geqq a_2$,

(9) if $a_1 <' a_2$ and $a_2 < a_3$, $k(a_1, a_2) \leqq k(a_1, a_3)$,

(10) for any $a_1 \in A$ and integer $k \geqq 0$, there is $a_2{}' > a_1$ such that $k(a_1, a_2) \geqq k$.

For each pair $a_1 <' a_2$, by (6) and the definition of $k(a_1, a_2)$, we have a map $g_{a_1 a_2}: |L_{a_2}^{(n)}| \to |(L_{a_1}^k)^{(n)}|$, where $k = k(a_1, a_2)$, such that

(11) $d(g_{a_1 a_2}, p_{a_1 a_2}|_{|(L_{a_1}^k)^{(n)}|}) \leqq 2 \cdot mesh(L_{a_1}^k)$,

(12) for any map $\xi: |(L_{a_1}^k)^{(n)}| \to K(G, n)$, the map $\xi \circ g_{a_1 a_2}$ admits a continuous extension over $|L_{a_2}| = X_{a_2}$.

Now, for each $a \in A'$, we define

(13) $$Z_a^* = Z_{L_a}^* .$$

For $a_1 < a_2$, the maps $r_{a_1 a_2}: Z_{a_2}^* \to Z_{a_1}^*$ are given by

(14) $$r_{a_1 a_2} = g_{a_1 a_2} \circ q_{0 a_2}^* ,$$

where $q_{0 a_2}^*: Z_{L_{a_2}}^* \to |L_{a_2}^{(n)}|$ is the map $q_0^*: Z_{L_{a_2}}^* \to |L_{a_2}^{(n)}|$. Note that

(15) $$r_{a_1 a_2}(Z_{a_2}^*) \subseteqq |(L_{a_1}^k)^{(n)}|, \qquad k = k(a_1, a_2).$$

By the same way as in [18, Lemma 7], we have that

(16) $\mathcal{Z} = (Z_a^*, \varepsilon_a, r_{aa'}, A')$ is approximate system of non-empty metric compacta Z_a^* of $\dim Z_a^* \leqq n$.

Therefore, by Proposition 5.6 (i), (iii'), the limit $Z = \lim \mathcal{Z}$ is a non-empty compact space of $\dim Z \leqq n$ and of $\omega(Z) \leqq \mathrm{card}(A) \leqq \omega(X)$. Let $r_a: Z \to Z_a^*$ be the projections.

For each $a \in A$, by f_a^*, we denote the map $f_{L_a}^*: Z_a^* = Z_{L_a}^* \to |L_a| = X_a$. Then by the same way as in [18], we can find the map $f: Z \to X$ such that

(17) $$f_a^* \circ r_a = p_a \circ f \qquad \text{for each } a \in A .$$

Next we show that the map f satisfies the required condition. Let take a given point $x \in X$. For each $a \in A$, put

(18) $x_a = p_a(x)$

(19) $N_a = N_a(x) = \{y \in X_a : d(x_a, y) \leqq \varepsilon_a\}$,

(20) $M_a = M_a(x) = f_a^{*-1}(N_a)$.

Then, by [18, Lemma 12 and 14], we can see that

(21) $\mathcal{N}(x) = (N_a, \varepsilon_a, p_{aa'}, A')$ is an approximate system of non-empty compact spaces with the limit $\{x\}$, and

(22) $\mathcal{M}(x) = (M_a, \varepsilon_a, r_{aa'}, A')$ is an approximate system of non-empty compact spaces with the limit $f^{-1}(x)$.

CLAIM 1. *f is a* UV^{n-1}*-map.*

PROOF OF CLAIM 1. For any a_1, let take $a_2{}' > a_1$. Since N_{a_2} is a neighborhood of x_{a_2} in the polyhedron X_{a_2}, there is a closed polyhedral neighborhood U of x_{a_2} in N_{a_2} such that

(23) $$U \text{ is contractible.}$$

Hence we may assume that

(24) $U=|T|$, where T is a subcomplex of the j-th barycentric subdivision $L^j_{a_2}$ of L_{a_2} for sufficiently large j.

Then, by the proof of [18, Lemma 17], there is $a_3{}'>a_2$ such that

(25) $$r_{a_2a_3}(M_{a_3})\subseteqq|T|.$$

By (9), taking a sufficiently large a_3 if necessary, we may assume that for some $l\geqq 0$, the l-th barycentric subdivision T^l of T is a subcomplex of $L^{k(a_2,a_3)}_{a_2}$. Hence,

(26) $$|Y^l|\cap|(L^{k(a_2,a_3)}_{a_2})^{(m)}|=|(T^l)^{(m)}| \quad \text{for every } m\geqq 0.$$

Moreover, by (23) and (24),

(27) $$\pi_m(|(T^l)^{(n)}|)=\pi_m(|T|)=0 \quad \text{if } m<n.$$

For any map $\alpha: S^m\to M_{a_3}$, $1\leqq m\leqq n-1$, by (25), (14) and (26),

(28) $$\alpha(S^m)\subseteqq|T|\cap|(L^{k(a_2,a_3)}_{a_2})^{(n)}|\subseteqq(T^l)^{(n)}|\subseteqq|T|\subseteqq N_{a_2}.$$

By (27),

(29) $$r_{a_2a_3}\circ\alpha\simeq 0 \quad \text{in } |(T^l)^{(n)}|.$$

Considering $|(T^l)^{(n)}|\subseteqq|(L^{k(a_2,a_3)}_{a_2})^{(n)}|\subseteqq Z^*_{a_2}$, by [18, Lemma 17],

(30) $$r_{a_1a_2}(|(T^l)^{(n)}|)\subseteqq M_{a_1}.$$

By (29) and (30), we have that

(31) $$r_{a_1a_2}\circ r_{a_2a_3}\circ\alpha\simeq 0 \quad \text{in } M_{a_1}.$$

It follows that $f^{-1}(x)$ is UV^m-connected for $m\leqq n-1$. We complete the proof of Claim 1.

CLAIM 2. *The set* $[f^{-1}(x), K(G, n)]$ *is trivial for every* $x\in X$.

PROOF OF CLAIM 2. By Proposition 5.7, it suffices to show that for every $a_1\in A'$ and every map $\xi: M_{a_1}\to K(G, n)$,

(32) $$\xi\circ r_{a_1a_2}\circ r_{a_2a_3}\simeq 0.$$

Here we use the same notation as in the proof of Claim 1, so indexes a_2 and a_3 are taken as in the proof of Claim 1.

By (12), we can find a continuous extension $\zeta: |L_{a_2}| \to K(G, n)$ of $\xi \circ g_{a_1 a_2}$. Since $q_{0a_2}|_{|(L_{a_2}^{k(a_2, a_3)})^{(n)}|}$ is the restriction of a simplicial approximation $q_{0k(a_2, a_3)}^{*}$: $|L_{a_2}^{k(a_2, a_3)}| \to |L_{a_2}|$ of $id_{|L_{a_2}|}$, by the homotopy extension theorem, the restriction $\xi \circ r_{a_1 a_2}|_{|(T^l)^{(n)}|} = \xi \circ g_{a_1 a_2} \circ q_{0a_2}^{*}|_{|(T^l)^{(n)}|}$ admits a continuous extension $\eta: |(T^l)| = U \to K(G, n)$. Then by (23), we have $\eta \simeq 0$. Particularly, since by the same way as in (28), we can see that $r_{a_2 a_3}(M_{a_3}) \subseteqq |(T^l)^{(n)}|$, by (34), we have that

$$\xi \circ r_{a_1 a_2} \circ r_{a_2 a_3} \simeq 0 . \tag{32}$$

It complete the proof of Claim 2 and it follows Theorem. □

7. Resolutions for metrizable spaces

By a polyhedron we mean the space $|K|$ of a simplicial complex K with the *Whitehead topology* (denoted by $|K|_w$). We may define a topology for $|K|$ by means of a uniformity in [Appendix, 22] (denoted by $|K|_u$).

7.1. THEOREM. *Let X be a metrizable space having approximable dimension with respect to an abelian group G of less than and equal to n. Then there exist an n-dimensional metrizable space Z and a perfect* UV^{n-1} *-surjection $\pi: Z \to X$ such that for $x \in X$, the set $[\pi^{-1}(x), K(G, n)]$ of homotopy classes is trivial.*

PROOF. The strategy is like the construction of Walsh-Rubin-Schapiro [24, 22].

Let d be a metric for X and let $\{\mathcal{U}_i : i \in N \cup \{0\}\}$ be a sequence of open covers of X, where each $\mathcal{U}_i$ consists of all $1/(i+1)$-neighborhoods.

First, we shall construct the followings:

Open covers $\mathcal{V}_i$ of X whose nerves $\mathcal{N}(\mathcal{V}_i)$ are locally finite dimensional, maps $b_i: X \to |\mathcal{N}(\mathcal{V}_i)|$ for $i \geqq 0$, f_i^*, $f_i |\mathcal{N}(\mathcal{V}_i)| \to |\mathcal{N}(\mathcal{V}_{i-1})|$ for $i \geqq 1$ and sequences $\mathcal{N}_i^j$, $j \in N \cup \{0\}$ of subdivisions of $\mathcal{N}(\mathcal{V}_i)$ for $i \geqq 0$ such that

(1) $\bar{\mathcal{S}}_i^{j+1} <^* \mathcal{S}_i^j$ for $j \geqq 0$,

(2) b_i is normal with respect to $b_i^{-1}(\mathcal{S}_i^j)$ and $\mathcal{N}_i^j$ for $j \geqq 0$,

(3) $f_i: \mathcal{N}_i^0 \to \mathcal{N}_{i-1}^3$ is simplicial for $i \geqq 1$,

(4) $f_i \circ b_i$ is $\mathcal{N}_{i-1}^j$-modification of b_{i-1}, $0 \leqq j \leqq 3$ for $i \geqq 1$,

(5) f_i maps each compact set in $|\mathcal{N}_i|_u$ onto a compact set in $|\mathcal{N}_{i-1}|_u$ which is contained in a finite union of simplexes of $\mathcal{N}_{i-1}$,

(6) $\mathcal{S}_i^0 < f_i^{-1}(\mathcal{S}_{i-1}^3)$ for $i \geqq 1$,

(7) $\bar{\mathcal{S}}_i^k < f_i^{-1}(\mathcal{S}_{i-1}^{k+3})$ for $k \geqq 1$ and $\bar{\mathcal{S}}_i^k < f_i^{*-1}(\mathcal{S}_{i-1}^{k+3})$ for $k \geqq 4$,

(8) $\mathcal{V}_i < \mathcal{U}_i \wedge b_{i-1}^{-1}(\mathcal{S}_{i-1}^3) \wedge b_{i-2}^{-1}(\mathcal{S}_{i-2}^6) \wedge \cdots \wedge b_0^{-1}(\mathcal{S}_0^{3i})$,

where we regard $|\mathcal{N}_i|_u$ as the uniform space with the uniform topology induced by the uniform base $\{\mathcal{S}_i^j\}_{j=0}^{\infty}$.

Further, we shall construct continuous (w. r. t. the Whitehead topology), uniformly continuous (w. r. t. the uniform topology) PL-maps $g_i: |(\mathcal{N}_i^3)^{(n)}| \to |(\mathcal{N}_{i-1}^3)^{(n)}|$ such that

(9) for each $t \in |(\mathcal{N}_i^3)^{(n)}|$, there exist $\sigma, \tau \in \mathcal{N}_{i-1}^2$ such that $f_i(t) \in \sigma$, $g_i(t) \in \tau$ and $\sigma \cap \tau \neq \emptyset$,

(10) for any map $\alpha: |(\mathcal{N}_{i-1}^3)^{(n)}|_w \to K(G, n)$, there exists an extension $\beta: |(\mathcal{N}_i^3)^{(n+1)}|_w \to K(G, n)$ of $\alpha \circ g_i: |(\mathcal{N}_i^3)^{(n)}|_w \to |(\mathcal{N}_{i-1}^3)^{(n)}|_w \to K(G, n)$,

(11) for each $x \in |\mathcal{N}_i|$, $g_i(\mathrm{st}(x, \mathcal{S}_i^2) \cap |(\mathcal{N}_i^3)^{(n)}|)$ is a Whitehead (i.e. finite) compact polyhedral subset of $|\mathcal{N}_{i-1}|$.

Let us start the construction. We take an open refinement $\mathcal{V}_0$ of $\mathcal{U}_0$ in X whose nerve $\mathcal{N}(\mathcal{V}_0)$ is locally finite dimensional and $\mathcal{V}_0$-normal map $b_0: X \to |\mathcal{N}(\mathcal{V}_0)|$. We define $\mathcal{N}_0^j$ to be a subdivision of $\mathrm{Sd}_{2j}\mathcal{N}(\mathcal{V}_0)$ for $j=0, 1, 2$ with $\mathcal{S}_0^j < \mathcal{S}_0^{j-1}$. By using [22, Proposition A.3], for the cover $\mathcal{E}_0 \equiv \{\mathrm{st}(x, \mathcal{S}_0^2): x \in |\mathcal{N}(\mathcal{V}_0)|\}$, we obtain an open cover $\mathcal{B}_0$ of $|\mathcal{N}(\mathcal{V}_0)|$ and a PL, $\mathcal{N}_0^2$-modification $r_0: |\mathcal{N}_0^2| \to |\mathcal{N}_0^2|$ of the identity such that

$(12)_0$ $r_0(\mathrm{Cl}\, B)$ is compact for $B \in \mathcal{B}_0$,

$(13)_0$ $\mathrm{Cl}\, B \cup r_0(\mathrm{Cl}\, B) \subseteq E$ for some $E \in \mathcal{E}_0$.

Since b_0 is (G, n)-cohomological, from the similar argument to the proof of the necessity in Theorem 4.3 we can take the followings:

Subdivision $\mathcal{N}_0^3$ of $\mathrm{Sd}_2\mathcal{N}_0^2$, locally finite open cover $\mathcal{V}_1$ of X and maps $b_1: X \to |\mathcal{N}(\mathcal{V}_1)|$, $f_1^*: |\mathcal{N}(\mathcal{V}_1)| \to |\mathcal{N}_0^3|$ such that

$(14)_1$ $\mathcal{S}_0^3 <^* \mathcal{S}_0^2 \wedge \mathcal{B}_0$,

$(15)_1$ $\mathcal{V}_1 <^* \mathcal{U}_1 \wedge b_0^{-1}(\mathcal{S}_0^3)$,

$(16)_1$ b_1 is $\mathcal{V}_1$-normal,

$(17)_1$ $f_1^* \circ b_1$ is $\mathcal{N}_0^3$-modification of b_0,

$(18)_1$ for each $\sigma \in \mathcal{N}(\mathcal{V}_1)$, there exists $U \in \mathrm{st}\, \mathcal{S}_0^3$ such that $b_0(b_1^{-1}(\sigma)) \cup f_1^*(\sigma) \subseteq U$,

$(19)_1$ for any triangulation M of $|\mathcal{N}(\mathcal{V}_1)|$, there exists a PL-map $p': |M^{(n)}|$ $|(\mathcal{N}_0^3)^{(n)}|$ such that

(i) $(p', f_1^*|_{|M^{(n)}|}) \leqq \{\overline{\mathrm{st}}(\lambda, \mathcal{N}_0^3): \lambda \in \mathcal{N}_0^3\}$,

(ii) for any map $\alpha: |(\mathcal{N}_0^3)^{(n)}| \to K(G, n)$, there exists an extension $\beta: |M^{(n+1)}| \to K(G, n)$ of $\alpha \circ p'$.

Let $\mathcal{N}_0^{j+1}$ denote a subdivision of $\mathrm{Sd}_2\mathcal{N}_0^j$ with $\mathcal{S}_0^{j+1} <^* \mathcal{S}_0^j$ for $j \geqq 3$.

Now, let $|\mathcal{N}_0^3|_m$ denote $|\mathcal{N}_0^3|$ with the metric topology [19, p. 301]. Then there is a $\mathcal{N}_0^3$-modification $j_0: |\mathcal{N}_0^3|_m \to |\mathcal{N}_0^3|_w$ of the identity *function* [19, p. 302]. By the simplicial approximation theorem, we obtain a subdivision $\mathcal{N}_1$ of $\mathcal{N}(\mathcal{V}_1)$ and a simplicial approximation $f_1: \mathcal{N}_1 \to \mathcal{N}_0^3$ of $j_0 \circ f_1^*$. Let $\mathcal{N}_1^0$ denote $\mathcal{N}_1$. Then by the simpliciality of f_1 and $(17)_1$, we have

(20) $\mathcal{S}_1^0 < f_1^{-1}(\mathcal{S}_0^3)$,

(21) $f_1 \circ b_1$ is $\mathcal{N}_0^3$-modification of b_0.

We take a subdivisions $\mathcal{N}_1^{j+1}$ of $\mathcal{N}_1^j$ for $j=0, 1$ such that

(22) $\mathcal{S}_1^{j+1} <^* \mathcal{S}_1^j$ for $j=0, 1$,

(23) $\mathcal{S}_1^j < f_1^{-1}(\mathcal{S}_0^{j+3})$ for $j=1, 2$,

(24) $\mathcal{N}_1^j < \mathrm{Sd}_{2j}\, \mathcal{N}_1^0$ for $j=1, 2$.

By using Lemma [22, Proposition A.3], for the cover $\mathcal{E}_1 \equiv \{\mathrm{st}(x, \mathcal{S}_1^2) : x \in |\mathcal{N}_1|\}$, we obtain an open cover $\mathcal{B}_1$ of $|\mathcal{N}(\mathcal{V}_0)|$ and a PL, $\mathcal{N}_1^2$-modification $r_1 : |\mathcal{N}_1^2| \to |\mathcal{N}_1^2|$ of the identity map such that

$(12)_1$ $r_1(\mathrm{Cl}\, B)$ is compact for $B \in \mathcal{B}_1$,

$(13)_1$ $\mathrm{Cl}\, B \cup r_1(\mathrm{Cl}\, B) \subseteqq E$ for some $E \in \mathcal{E}_1$.

Since b_1 is (G, n)-cohomological, from the similar argument to the proof of the necessity in Theorem 4.3 we can take the followings:

Subdivision $\mathcal{N}_1^3$ of $\mathrm{Sd}_2\, \mathcal{N}_1^2$, locally finite open cover $\mathcal{V}_2$ of X and maps $b_2 : X \to |\mathcal{N}(\mathcal{V}_2)|$, $f_2^* : |\mathcal{N}(\mathcal{V}_2)| \to |\mathcal{N}_1^3|$ such that

$(14)_2$ $\mathcal{S}_1^3 <^* \mathcal{S}_1^2 \wedge \mathcal{B}_1 \wedge f_1^{-1}(\mathcal{S}_0^6)$,

$(15)_2$ $\mathcal{V}_2 <^* \mathcal{U}_2 \wedge b_1^{-1}(\mathcal{S}_1^3) \wedge b_0^{-1}(\mathcal{S}_0^6)$,

$(16)_2$ b_2 is $\mathcal{V}_2$-normal,

$(17)_2$ $f_2^* \circ b_2$ is $\mathcal{N}_1^3$-modification of b_1,

$(18)_2$ for each $\sigma \in \mathcal{N}(\mathcal{V}_2)$, there exists $U \in \mathrm{st}\, \mathcal{S}_1^3$ such that $b_1(b_2^{-1}(\sigma)) \cup f_2^*(\sigma) \subseteqq U$,

$(19)_2$ for any triangulation M of $|\mathcal{N}(\mathcal{V}_2)|$, there exists a PL-map $p' : |M^{(n)}| \to |(\mathcal{N}_1^3)^{(n)}|$ such that

(i) $(p', f_2^*|_{|M^{(n)}|}) \leqq \{\overline{\mathrm{st}}(\lambda, \mathcal{N}_1^3) : \lambda \in \mathcal{N}_0^3\}$,

(ii) for any map $\alpha : |(\mathcal{N}_1^3)^{(n)}| \to K(G, n)$, there exists an extension $\beta : |M^{(n+1)}| \to K(G, n)$ of $\alpha \circ p'$.

Now, by using $(19)_1$ about the triangulation $\mathcal{N}_1^3$ of $|\mathcal{N}(\mathcal{V}_1)|$, we obtain a PL-map $g_1^* : |(\mathcal{N}_1^3)^{(n)}| \to |(\mathcal{N}_0^3)^{(n)}|$ such that

$(25)_1$ $(g_1^*, f_1^*|_{|(\mathcal{N}_1^3)^{(n)}|}) \leqq \{\overline{\mathrm{st}}(\lambda, \mathcal{N}_0^3) : \lambda \in \mathcal{N}_0^3\}$,

$(26)_1$ for any map $\alpha : |(\mathcal{N}_0^3)^{(n)}| \to K(G, n)$, there exists an extension $\beta : |(\mathcal{N}_1^3)^{(n+1)}| \to K(G, n)$ of $\alpha \circ g_1^*$.

Consider the inclusion map $i_0 : (\mathcal{N}_0^3)^{(n)}| \hookrightarrow |\mathcal{N}_0^3|$ and the composition

$$r_0 \circ i_0 \circ g_1^* : |(\mathcal{N}_1^3)^{(n)}| \longrightarrow |(\mathcal{N}_0^3)^{(n)}| \hookrightarrow |\mathcal{N}_0^3| = |\mathcal{N}(\mathcal{V}_0)| \longrightarrow |\mathcal{N}(\mathcal{V}_0)|.$$

The image A of the PL-map $r_0 \circ i_0 \circ g_1^*$ has dimension $\leqq n$. Then we can take a $\mathcal{N}_0^3$-modification $s_0 : A \to |(\mathcal{N}_0^3)^{(n)}|$ of the inclusion map $A \hookrightarrow |\mathcal{N}_0^3|$. Let $g_1 : |(\mathcal{N}_1^3)^{(n)}| \to |(\mathcal{N}_0^3)^{(n)}|$ denote the composition map $s_0 \circ r_0 \circ i_0 \circ g_1^*$.

Then this has the following properties:

CLAIM 1.

(9)$_1$ *for each* $t\in|(\mathcal{N}_1^3)^{(n)}|$, *there exist* $\sigma, \tau\in\mathcal{N}_0^2$ *such that* $f_1(t)\in\sigma$, $g_1(t)\in\tau$ *aud* $\sigma\cap\tau\neq\emptyset$,

(10)$_1$ *for any map* $\alpha: |(\mathcal{N}_0^3)^{(n)}|\rightarrow K(G, n)$ *there exist an extension* $\beta: |(\mathcal{N}_1^3)^{(n+1)}|\rightarrow K(G, n)$ *of* $\alpha\circ g_1$,

(11)$_1$ *for each* $x\in|\mathcal{N}_1|$, $g_1(\mathrm{st}(x, \bar{S}_1^2)\cap|(\mathcal{N}_1^3)^{(n)}|)$ *is a Whitehead (i.e. finite) compact polyhedral subset of* $|\mathcal{N}_0|$.

PROOF OF CLAIM 1. We show the property (9)$_1$. Let $t\in|(\mathcal{N}_1^3)^{(n)}|$. By (25)$_1$, there exist $\sigma, \lambda, \tau\in\mathcal{N}_0^3$ such that $f_1^*(t)\in\sigma$, $g_1^*(t)\in\tau$ and $\sigma\cap\lambda\neq\emptyset\neq\lambda\cap\tau$. We may assume that $\lambda=|v_0, v_1|$, $v_0\in\sigma$ and $v_1\in\tau$.

Since j_0 is $\mathcal{N}_0^3$-modification of the identity function, we have $j_0\circ f_1^*(t)\in\sigma$. Since f_1 is simplicial approximation of $j_0\circ f_1^*$, we have $f_1(t)\in\sigma$.

Select $\tilde{\tau}\in\mathcal{N}_0^2$ with $\tau\subseteq\tilde{\tau}$. Since r_0 is $\mathcal{N}_0^3$-modification of the identity map, we have $r_0\circ i_0\circ g_1^*(t)\in\tilde{\tau}$. Further since s_0 is $\mathcal{N}_0^3$-modification of $A\hookrightarrow|\mathcal{N}_0^3|$ and $\mathcal{N}_0^3\prec\mathcal{N}_0^2$, we have $g_1(t)=s_0\circ r_0\circ i_0\circ g_1^*(t)\in\tilde{\tau}$.

CASE 1. $v_1\in(\mathcal{N}_0^2)^{(0)}$ (i.e. $v_1\in\tilde{\tau}^{(0)}$).

By $\mathcal{N}_0^3\prec\mathrm{Sd}_2\,\mathcal{N}_0^2$, we have $v_0\notin(\mathcal{N}_0^2)^{(0)}$. Hence, there exists $\gamma\in\mathcal{N}_0^2$ such that $|v_0, v_1|\subseteq\gamma$ and $v_0\in\mathrm{Int}\,\gamma$. Then if $\tilde{\sigma}\in\mathcal{N}_0^2$ with $\sigma\subseteq\tilde{\sigma}$, we have $\gamma\prec\tilde{\sigma}$. Therefore we have $\tilde{\sigma}\cap\tilde{\tau}\neq\emptyset$, $f_1(t)\in\tilde{\sigma}$ and $g_1(t)\in\tilde{\tau}$.

CASE 2. $v_1\notin(\mathcal{N}_0^2)^{(0)}$.

If $v_0\in(\mathcal{N}_0^2)^{(0)}$, the proof is similar to Case 1. Let $v_0\notin(\mathcal{N}_0^2)^{(0)}$. By $\mathcal{N}_0^3\prec\mathrm{Sd}_2\,\mathcal{N}_0^2$, there exist $\gamma_0, \gamma_1\in\mathcal{N}_0^2$ such that $v_0\in\mathrm{Int}\,\gamma_0$, $v_1\in\mathrm{Int}\,\gamma_1$ and $\gamma_0\prec\gamma_1$ or $\gamma_1\prec\gamma_0$. Then if $\tilde{\sigma}\in\mathcal{N}_0^2$ with $\sigma\subseteq\tilde{\sigma}$, we have $\gamma_0\prec\tilde{\sigma}$. Similarly, we have $\gamma_1\prec\tilde{\tau}$. Therefore we have $\tilde{\sigma}\cap\tilde{\tau}\neq\emptyset$, $f_1(t)\in\tilde{\sigma}$ and $g_1(t)\in\tilde{\tau}$.

By $g_1^*\simeq g_1$, we can see the property (10)$_1$ by the homotopy extension theorem and (26)$_1$.

We show the property (11)$_1$. First, we shall see that

$$g_1^*(\mathrm{st}(x, \bar{S}_1^2)\cap|(\mathcal{N}_1^3)^{(n)}|)\subseteq B \quad \text{for some } B\in\mathfrak{B}_0\,. \tag{27}$$

Let $\mathrm{st}(x, \bar{S}_1^2)$ be represented by $\bigcup\{\overline{\mathrm{st}}(v_\alpha, \mathcal{N}_1^2): \alpha\in A\}$. There exists $\sigma_x\in\mathcal{N}_1^2$ with $x\in\mathrm{Int}\,\sigma_x$.

For each $\alpha\in A$, we choose $\sigma_\alpha\in\mathcal{N}_1^2$ such that $\sigma_x\leqslant\sigma_\alpha$ and $v_\alpha\in\sigma_\alpha$. Further we select minimum and maximal dimensional simplexes $\tau_x, \tau_\alpha\in\mathcal{N}_1^0$ with $\tau_x\leqslant\tau_\alpha$ respectively such that $\sigma_x\subseteq\tau_x$ and $\sigma_\alpha\subseteq\tau_\alpha$.

If $\sigma_x\subseteq\mathrm{Int}\,\tau_x$, we have $\overline{\mathrm{st}}(v_\alpha, \mathcal{N}_1^2)\subseteq\tau_\alpha$ from $v_\alpha\in\mathrm{Int}\,\tau_\alpha$. Then there exists a vertex $v\in\mathcal{N}_1^2$ such that $\bigcup_\alpha \tau_\alpha\subseteqq\overline{\mathrm{st}}(v, \mathcal{N}_1^0)$. Since f_1 is the simplicial map from $\mathcal{N}_1^0$ to $\mathcal{N}_0^3$, we have $f_1(\bigcup_\alpha \tau_\alpha)\subseteqq f_1(\overline{\mathrm{st}}(v, \mathcal{N}_1^0))\subseteqq\overline{\mathrm{st}}(f_1(v), \mathcal{N}_0^3)$. By the nearness between f_1 and g_1^* (see proof of $(9)_1$) and $(14)_1$, we obtain

$$(28)\qquad g_1^*(\mathrm{st}(x, \bar{\mathcal{S}}_1^2)\cap|(\mathcal{N}_1^3)^{(n)}|)\subseteqq\mathrm{st}(\overline{\mathrm{st}}(f_1(v), \mathcal{N}_0^3), \bar{\mathcal{S}}_0^3)\subseteqq B \quad \text{for some } B\in\mathcal{B}_0.$$

If $\sigma_x\cap\partial\tau_x\neq\emptyset$ and $\sigma_x\cap\mathrm{Int}\,\tau_x\neq\emptyset$, we choose a face $\tilde{\tau}_x$ with $\tilde{\tau}_x\lneqq\tau_x$ such that $\sigma_x\cap\partial\tau_x\subseteqq\tilde{\tau}_x$. Then there exists a vertex $v\in\tilde{\tau}_x$ such that $\bigcup_\alpha \overline{\mathrm{st}}(v_\alpha, \mathcal{N}_1^2)\subseteqq\overline{\mathrm{st}}(v, \mathcal{N}_1^0)$. Hence we have (28) in the same way.

Since $\mathrm{st}(x, \bar{\mathcal{S}}_1^2)\cap|(\mathcal{N}_1^3)^{(n)}|$ is a subpolyhedron of $|\mathcal{N}_1|$ and g_1^* is a PL-map, we see that $g_1^*(\mathrm{st}(x, \bar{\mathcal{S}}_1^2)\cap|(\mathcal{N}_1^3)^{(n)}|)$ is a subpolyhedron of $|\mathcal{N}_0|$. Then by (27) and $(12)_0$, $r_0\circ i_0\circ g_1^*(\mathrm{st}(x, \bar{\mathcal{S}}_1^2)\cap|(\mathcal{N}_1^3)^{(n)}|)$ is a subpolyhedron of $|\mathcal{N}_0|$ and a compact set of $|\mathcal{N}_0|_w$. Since s_0 is a PL-map, we have see the property $(11)_1$.

Now, we shall take a base for a uniformity for $|\mathcal{N}_1|$. We choose a subdivisions $\mathcal{N}_1^j$ for $j\geqq 4$ of $\mathcal{N}_1$ such that

(29) $\mathcal{N}_1^{j+1}<\mathrm{Sd}_2\,\mathcal{N}_1^j$ for $j\geqq 3$,

(30) $\bar{\mathcal{S}}_1^{j+1}<^*\mathcal{S}_1^j$ for $j\geqq 3$,

(31) $\bar{\mathcal{S}}_1^{j+1}<f_1^{-1}(\mathcal{S}_0^{j+4})\wedge f_1^{*-1}(\mathcal{S}_0^{j+4})\wedge\mathcal{F}_1^{j+4}$ for $j\geqq 3$,

where $\mathcal{F}_1^{j+4}$ is defined as follows. $g_1^{-1}(\mathcal{S}_0^{j+4}\cap|(\mathcal{N}_0^3)^{(n)}|)$ is the open cover of $|(\mathcal{N}_1^3)^{(n)}|_w$. Extend it to an open cover $\mathcal{F}_1^{j+4}$ of $|\mathcal{N}_1|_w$. Then clearly the uniformity make f_1, f_1^* and g_1 uniformly continuous.

We shall show that f_1 holds the property (5). First, note that the composition

$$j_0\circ id\circ f_1^*\colon |\mathcal{N}_1|_u \longrightarrow |\mathcal{N}_0|_u \longrightarrow |\mathcal{N}_0|_m \longrightarrow |\mathcal{N}_0|_w\,,$$

where $id\colon |\mathcal{N}_0|_u\to|\mathcal{N}_0|_m$ is the identity map, is continuous.

Let K be a compact set of $|\mathcal{N}_1|_u$. There exist $\sigma_1, \cdots, \sigma_l\in\mathcal{N}_0$ such that $j_0\circ f_1^*(K)=j_0\circ id\circ f_1^*(K)\subseteqq\sigma_1\cup\cdots\cup\sigma_l$. Since f_1 is a simplicial approximation of $j_0\circ f_1^*$, we have $f_1(K)\subseteqq\sigma_1\cup\cdots\cup\sigma_l$. By the continuity of f_1, $f_1(K)$ is a compact set of $|\mathcal{N}_0|_u$.

As we proceed in this work, we have $\mathcal{V}_i$, f_i^*, f_i, $\mathcal{N}_i^j$ and g_i with the properties (1)-(11).

From now on, we consider X to be the uniform space with the uniformity generated by the sequence $\{\mathcal{V}_i\}_{i=0}^\infty$ of open covers of X and $|\mathcal{N}_i|$ to be the uniform space with the uniformity generated by the sequence $\{\mathcal{S}_i^j\}_{j=0}^\infty$. Then by the construction, the topology induced by $\{\mathcal{V}_i\}_{i=0}^\infty$ and the original metric topology are identical.

We shall construct the resolution of X. The construction essentially depends on Rubin-Schapiro's way [22]. Hence, the detail is omitted here.

For $j \geqq 0$, let $f_{j,j}$ denote the identity on $\mathfrak{N}_j$ and let $f_{i,j}$ denote the composition $f_{j+1} \circ \cdots \circ f_i : |\mathfrak{N}_i| \rightarrow |\mathfrak{N}_j|$ for $i > j$.

The functions

$$b_i : (X, \{\mathcal{V}_i\}_{i=0}^{\infty}) \longrightarrow (|\mathfrak{N}_i|, \{\mathcal{S}_i^j\}_{j=0}^{\infty})$$

and

$$f_{i+1,i} : (|\mathfrak{N}_{i+1}|, \{\mathcal{S}_{i+1}^j\}_{j=0}^{\infty}) \longrightarrow (|\mathfrak{N}_i|, \{\mathcal{S}_i^j\}_{j=0}^{\infty})$$

are uniformly continuous for $i \geqq 0$. Then since the sequence $\{f_{i,j} \circ b_i\}_{i=j}^{\infty}$ is Cauchy in the uniform space $C(X, |\mathfrak{N}_j|_u)$ with the uniformity of uniform convergence, we have a uniformly continuous, limit map

$$f_{\infty,j} \equiv \lim_{q \to \infty} f_{q,j} \circ b_q : (X, \{\mathcal{V}_i\}_{i=0}^{\infty}) \longrightarrow (|\mathfrak{N}_j|, \{\mathcal{S}_j^i\}_{i=0}^{\infty}),$$

such that

(32) $f_{\infty,j}$ is $\mathfrak{N}_j^3$-modification of b_j,

(33) $(f_{\infty,j}, b_j) \leqq \mathcal{S}_j^1$,

(34) $f_{\infty,j}$ is a topological irreducible (i. e. surjective) map relative to $\mathfrak{N}_j^3$,

(35) $f_{i+1,i} \circ f_{\infty,i+1} = f_{\infty,i}$ for $i \geqq 0$.

We consider $\prod_{i=0}^{\infty} |\mathfrak{N}_i|_u$ to be the uniform space by the product uniformity. Note that $\varprojlim \{|\mathfrak{N}_j|_u, f_{i+1,i}\}$ is a non-empty subspace by the property (34).

Then by (35), there exist a uniformly continuous map $f_w : X \rightarrow \varprojlim |\mathfrak{N}_i|_u$ with $f_{\infty,i} = pr_i \circ f_w$ and especially the map f_w is a uniformly embedding onto a dense subset $f_w(X)$ in $\varprojlim |\mathfrak{N}_i|_u$, where $pr_i : \prod_{j=0}^{\infty} |\mathfrak{N}_j|_u \rightarrow |\mathfrak{N}_i|_u$ is the natural projection.

Let Z denote the limit of the inverse sequence $\{|(\mathfrak{N}_i^3)^{(n)}|_u, g_{i+1,i}\}$. Then we consider Z to be the sub-uniform space of the uniform space $\prod_{i=0}^{\infty} |\mathfrak{N}_i|_u$. Note that Z has dimension $\leqq n$.

We begin with a description of the map π. For $j \geqq 0$, a uniformly continuous map $\pi_j : Z \rightarrow \prod_{i=0}^{\infty} |\mathfrak{N}_i|_u$ is defined by

$$\pi_j(\boldsymbol{z}) \equiv (f_{j,0}(z_j), f_{j,1}(z_j), \cdots, f_{j,j-1}(z_j), z_j, z_{j+1}, \cdots)$$

for $\boldsymbol{z} = (z_j) \in Z$ and let π_0 be the inclusion map. Then since the sequence $\{\pi_j\}_{j=0}^{\infty}$ is Cauchy in $C(Z, \prod_{i=0}^{\infty} |\mathfrak{N}_i|_u)$, there is a uniformly continuous, limit map $\pi : Z \rightarrow \prod_{i=0}^{\infty} |\mathfrak{N}_i|_u$. Then the map π is proper from Z into $\varprojlim \{|\mathfrak{N}_i|_u, f_{i+1,i}\}$ ([22, p. 239]). We must show that $\pi^{-1}(\boldsymbol{x})$ is a UV^{n-1}-set and the set $[\pi^{-1}(\boldsymbol{x}), K(G, n)]$ is trivial for $\boldsymbol{x} \in \varprojlim \{|\mathfrak{N}_i|_u, f_{i+1,i}\}$.

For $\boldsymbol{x} = (x_i) \in \varprojlim \{|\mathfrak{N}_i|_u, f_{i+1,i}\}$, let $\delta N(x_i)$ and $\varepsilon N(x_i)$ denote $\mathrm{st}(x_i, \mathcal{S}_i^0)$ and $\mathrm{st}(x_i, \mathcal{S}_i^2)$, respectively. Then we have the following properties [22]: for $\boldsymbol{x} = (x_i) \in \varprojlim \{|\mathfrak{N}_i|_u, f_{i+1,i}\}$,

(36) $g_{i,i-1}(\delta N(x_i) \cap |(\mathfrak{N}_i^3)^{(n)}|) \subseteq \varepsilon N(x_{i-1})$,

(37) $\varprojlim\{\varepsilon N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|,\ g_{i,i-1}|\ldots\}=\pi^{-1}(\boldsymbol{x})=\varprojlim\{\delta N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|,\ g_{i,i-1}|\ldots\}$.

By $\bar{\mathcal{S}}_i^2<^*\mathcal{S}_i^1$, there exists $F_i\in\mathcal{S}_i^1$ such that $\mathrm{st}(x_i,\ \bar{\mathcal{S}}_i^2)\subseteqq F_i$. Further, by $\mathcal{S}_i^1<\mathcal{S}_i^0$, there is a $S\in\mathcal{S}_i^0$ such that $F_i\subseteqq S$. Hence we have the contractible set F_i such that

(38) $$\varepsilon N(x_i)\subseteqq F_i\subseteqq\delta N(x_i).$$

CLAIM 2. *$\pi^{-1}(\boldsymbol{x})$ is a UV^{n-1}-set for for $\boldsymbol{x}=(x_i)\in\varprojlim\{|\mathcal{N}_i|_u,\ f_{i+1,i}\}$.*

PROOF OF CLAIM 2. It suffices to show that the map

$$g_{i+1,i}|\ldots:\ \delta N(\mathcal{N}_{i+1})\cap|(\mathcal{N}_{i+1}^3)^{(n)}|\longrightarrow\delta N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|$$

induces a zero homomorphism of homotopy group of dimension less than n. By (36) and (38), we have

$$g_{i+1,i}(\delta N(x_{+1})\cap|(\mathcal{N}_{i+1}^3)^{(n)}|)\subseteqq F_i\cap|(\mathcal{N}_i^3)^{(n)}|\subseteqq\delta N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|.$$

Since F_i is contractible, we have

$$\pi_k(F_i\cap|(\mathcal{N}_i^3)^{(n)}|)=0\qquad\text{for } k<n.$$

Therefore $g_{i+1,i}|\ldots$ induces a zero homomorphism of homotopy group of dimension less than n.

CLAIM 3. *$[\pi^{-1}(\boldsymbol{x}),K(G,n)]\approx\check{H}^n(\pi^{-1}(\boldsymbol{x});G)$ is trivial for $\boldsymbol{x}\in\varprojlim\{|\mathcal{N}_i|_u,\ f_{i+1,i}\}$.*

PROOF OF CLAIM 3. By (11), (36), (37) and the continuity of Čech cohomology, we have

$$\check{H}^n(\pi^{-1}(\boldsymbol{x});G)\approx\varinjlim\{H^n(g_{i,i-1}(\varepsilon N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|_u);G),\ g_{i,i-1}|_{\ldots}^*\}.$$

Hence it suffices to show that

$$g_{i,i-1}|_{\ldots}^*:\ H^n(g_{i,i-1}(\varepsilon N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|);G)\to H^n(g_{i+1,i}(\varepsilon N(x_{i+1})\cap|(\mathcal{N}_{i+1}^3)^{(n)}|);G)$$

is the zero homomorphism.

Let $G_{i,i-1}$ denotes $g_{i,i-1}(\varepsilon N(x_i)\cap|(\mathcal{N}_i^3)^{(n)}|_u)$. Then by (11) the subspace $G_{i,i-1}$ of $|(\mathcal{N}_{i-1}^3)^{(n)}|_u$ and the subspace $G_{i,i-1}$ of $|(\mathcal{N}_{i-1}^3)^{(n)}|_w$ is identical. Hence from now on, we may consider that $G_{i,i-1}$ is the subspace of $|(\mathcal{N}_{i-1}^3)^{(n)}|_w$.

Let $[\alpha]\in[G_{i,i-1},\ K(G,n)]$. Then from $\pi_q(K(G,n))=0$ for $q<n$, there exists an extension $\tilde{\alpha}:|(\mathcal{N}_{i-1}^3)^{(n)}|_w\to K(G,n)$ of α. By (10), we have an extension $\beta:\ |(\mathcal{N}_i^3)^{(n+1)}|_w\to K(G,n)$ of $\tilde{\alpha}\circ g_{i,i-1}|_{G_{i+1,i}}$.

Since F_i is the contractible set, $F_i\cap|(\mathcal{N}_i^3)^{(n)}|_w$ is contractible in $F_i\cap|(\mathcal{N}_i^3)^{(n+1)}|_w$. Hence, there exists a homotopy $H:(F_i\cap|(\mathcal{N}_i^3)^{(n)}|_w)\times I\to F_i\cap|(\mathcal{N}_i^3)^{(n+1)}|_w$ such that H_0 is the inclusion map and H_1 is a constant map. Since

$G_{i+1,i} \subseteqq \varepsilon N(x_i) \cap |(\mathcal{N}_i^3)^{(n)}|_w \subseteqq F_i \cap |(\mathcal{N}_i^3)^{(n)}|_w$, we can define the following compositions:

$$\tilde{H} \equiv \beta \circ i_2 \circ H \circ i_1 : G_{i+1,i} \times I \hookrightarrow (F_i \cap |(\mathcal{N}_i^3)^{(n)}|_w) \times I \longrightarrow F_i \cap |(\mathcal{N}_i^3)^{(n+1)}|_w$$
$$\hookrightarrow |(\mathcal{N}_i^3)^{(n+1)}|_w \longrightarrow K(G, n),$$

where i_1 and i_2 are the inclusion maps.

Then we have $\tilde{H}_0 = \beta|_{G_{i+1,i}} = \alpha \circ g_{i,i-1}|_{G_{i+1,i}}$ and $\tilde{H}_1 =$ a constant. It completes the proof of Claim 3. Then the map

$$\pi_X \equiv \pi|_{\pi^{-1}(X)} : \pi^{-1}(X) \twoheadrightarrow X$$

is a desired one for Theorem. □

8. Summary

From Theorem 6.1, 7.1, we have the following theorem.

8.1. THEOREM. *Let X be a compact Hausdorff or metrizable space and n be a natural number. Then the following conditions are equivalent, respectively*:

(i) *X has cohomological dimension with respect to $\boldsymbol{Z}_p$ of less than and equal to n,*

(ii) *X is a continuous or perfect image of an n-dimensional compact Hausdorff or metrizable space Z under an acyclic map π in the sense of cohomology with coefficient in $\boldsymbol{Z}_p$,*

(iii) *there exists an n-dimensional compact Hausdorff or metrizavle space Z and a continuous or perfect UV^{n-1}-surjection $\pi : Z \to X$ such that for $x \in X$, $\check{H}^n(\pi^{-1}(x); \boldsymbol{Z}_p)$ is trivial.*

PROOF. We can easily see the implication (iii)⇒(ii). The implication (ii)⇒(i) is a corollary to the classical Vietoris-Begle's theorem. We have the implication (i)⇒(iii) from Theorem 6.1, 7.1. □

Although cohomological dimension with respect to $\boldsymbol{Z}$ or $\boldsymbol{Z}_p$ is characterized by the existence of acyclic resolutions, we have an unexpected fact about cohomological dimension with respect to $\boldsymbol{Q}$.

8.2. THEOREM (SHCHEPIN). *Let X be a compact space of c-$\dim_{\boldsymbol{Q}} X \leqq 1$. If X admits an acyclic resolution, that is, there exists a compact space Z of $\dim Z \leqq 1$ and a map $f : Z \to X$ such that $\check{H}^*(f^{-1}(x); \boldsymbol{Q}) = 0$ for all $x \in X$, then $\dim X \leqq 1$.*

PROOF. By $\dim f^{-1}(x) \leqq \dim Z \leqq 1$, $\check{H}^1(f^{-1}(x); \boldsymbol{Z})$ is torsion free. Hence, by the universal coefficient theorem for Čech cohomology groups, we have that $\check{H}^1(f^{-1}(x); \boldsymbol{Z})=0$. Therefore we have $\check{H}^*(f^{-1}(x); \boldsymbol{Z})=0$ for all $x \in X$. It follows that $c\text{-}\dim_{\boldsymbol{Z}} X \leqq c\text{-}\dim_{\boldsymbol{Z}} Z = \dim Z = 1$. Particularly, we have $\dim X = 1$. □

8.3. COROLLARY. *For each $n=2, 3, \cdots, \infty$, there exists an n-dimensional compact metric space $X(n)$ such that*

$$1 = c\text{-}\dim_{\boldsymbol{Q}} X(n) < a\text{-}\dim_{\boldsymbol{Q}} X(n).$$

PROOF. For each $n=2, 3, \cdots, \infty$, by [6, Theorem 2.1], there exists an n-dimensional compact metric space $X(n)$ of $c\text{-}\dim_{\boldsymbol{Q}} X(n)=1$. If $a\text{-}\dim_{\boldsymbol{Q}} X(n) \leqq 1$, by Theorem 6.1, there exists a compact metric space Z of $\dim Z \leqq 1$ and a map $f: Z \to X$ such that $\check{H}^*(f^{-1}(x); \boldsymbol{Q})=0$ for all $x \in X$. Then by Theorem 8.2, we have $\dim X(n) \leqq 1$. But it is a contradiction. Therefore $a\text{-}\dim_{\boldsymbol{Q}} X(n) > 1$. □

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