

CLASSICAL QUOTIENT RINGS OF TRIVIAL EXTENSIONS

By

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Let R be a ring with identity. The right quotient ring of R , if it exists, is a ring Q which satisfies the following conditions:

- (i) R is a subring of Q .
- (ii) Every regular element of R is a unit of Q .
- (iii) Every element q of Q is of the form ac^{-1} for some elements a and c of R with c regular.

An (R, R) -bimodule M is called to satisfy the right Ore condition with respect to a multiplicatively closed subset D of R if, given $m \in M$ and $d \in D$, there exist $m' \in M$ and $d' \in D$ such that $md' = dm'$. It is well-known that R has the classical right quotient ring if and only if R satisfies the right Ore condition with respect to D when D is the set of all regular elements of R .

Let M be an (R, R) -bimodule. The trivial extension $A = R \ltimes M$ of R by an (R, R) -bimodule M is the Cartesian product $R \ltimes M$ with addition componentwise and multiplication given by $(r, m)(r', m') = (rr', mr' + rm')$. In general, it is difficult to determine the form of regular elements of A . If c is a regular element of R , (c, m) is not always a regular element of A and vice versa. Let $\Gamma = \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}$ be a generalized triangular matrix ring. In [3], Chatters remarked that H. Attarchi determined the form of a regular element of Γ under the assumption that cR (Rc) is an essential right (left) ideal of R for each right (left) regular element c of R . In this case, $\begin{pmatrix} c_1 & 0 \\ x & c_2 \end{pmatrix}$ is regular in Γ if and only if both c_1 and c_2 are regular in R . For example, if both R_R and ${}_R R$ have finite Goldie dimension, the above assumption is satisfied. So, if we can find a suitable description of regular elements of A by those of R , we can investigate whether A has the classical right quotient ring. The main purpose of this paper is to give a necessary and sufficient condition for A to have the classical right quotient ring under the condition that ${}_R M_R$ is faithful or both ${}_R M$ and M_R have finite Goldie dimension.

In Section 1, we show that every regular element of A has the form of (c, m) with c regular in R . Let $C(R)$ denote the set of all regular elements of R and $D = \{c \in C(R) \mid cm \neq 0 \text{ and } mc \neq 0 \text{ for every } 0 \neq m \in M\}$. In Section 2, we show that A

has the classical right quotient ring if and only if both R and M satisfy the right Ore condition with respect to D . When the above equivalent condition holds, D becomes a right denominator set and the classical right quotient ring of A has the form of $R[D^{-1}]\times M[D^{-1}]$. So, we see that the classical right quotient ring of a trivial extension is also given by a trivial extension. As by-products of results in Sections 1 and 2, we exhibit some corollaries concerning generalized triangular matrix rings in the final Section 3.

Throughout this paper, unless otherwise specified, A denotes the trivial extension of R by an (R, R) -bimodule M . For a subset I of R , $\mathfrak{l}_R(I)$ ($\mathfrak{r}_R(I)$) denotes the left (right) annihilator of I in R . Furthermore, let $C(R)$ denote the set of all regular elements of R and $D = \{c \in C(R) \mid cm \neq 0 \text{ and } mc \neq 0 \text{ for every } 0 \neq m \in M\}$. "The right quotient ring of R " means the classical right quotient ring of R .

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1. Regular elements in A .

LEMMA 1.1. *Assume that ${}_R M_R$ is faithful. Then $(c_0, m_0) \in C(A)$, $m_0 \in M$ if and only if $c_0 \in D$.*

PROOF. (\implies). Let $(c_0, m_0) \in C(A)$, $r' \in \mathfrak{r}_R(c_0)$ and $m \in M$. Since $(c_0, m_0)(0, r'm) = (0, c_0 r'm) = 0$ and $(c_0, m_0) \in C(A)$, we have $r'm = 0$ for every $m \in M$. Since ${}_R M$ is faithful, we have $r' = 0$. By the similar manner as above, we can prove that $\mathfrak{l}_R(c_0) = 0$. Thus we have $c_0 \in C(R)$. Moreover, since $(c_0, m_0) \in C(A)$, we have $(c_0, m_0)(0, m) = (0, c_0 m) \neq 0$ and $(0, m)(c_0, m_0) = (0, mc_0) \neq 0$ for every $0 \neq m \in M$. Hence $c_0 \in D$.

(\impliedby). Let $c_0 \in D$ and $0 \neq (r, m) \in A$. Since $c_0 \in D$, we have $(c_0, m_0)(r, m) = (c_0 r, c_0 m + m_0 r) \neq 0$ and $(r, m)(c_0, m_0) = (rc_0, mc_0 + rm_0) \neq 0$. Thus $(c_0, m_0) \in C(A)$.

Recall that a right R -module X is called to have finite Goldie dimension if X_R contains no infinite independent families of non-zero submodules.

LEMMA 1.2. *Let r be an element of R such that $rm \neq 0$ for every $0 \neq m \in M$. If M_R has finite Goldie dimension, then rM_R is an essential submodule of M_R .*

PROOF. Let M'_R be a submodule of M_R with $rM \cap M' = 0$. Since $rM \cap M' = 0$, $M' + rM' + r^2M' + \cdots + r^nM' + \cdots$ is a direct sum. Since M_R has finite Goldie dimension, $r^nM' = 0$ for some n . Therefore, we obtain $M' = 0$ by assumption on R .

LEMMA 1.3. *Assume that both M_R and ${}_R M$ have finite Goldie dimension. Then*

$(c_0, m_0) \in C(A)$, $m_0 \in M$ if and only if $c_0 \in D$.

PROOF. This proof is a slight modification of [3, Remarks (2), p.189].

(\implies). Let $(c_0, m_0) \in C(A)$. We put $K = \{r \in R \mid m_0 r \in c_0 M\}$. Let $r_1 \in r_R(c_0) \cap K$. Then there exists $m_1 \in M$ such that $m_0 r_1 = -c_0 m_1$. Moreover, since $(c_0, m_0)(r_1, m_1) = 0$ and $(c_0, m_0) \in C(A)$, we have $(r_1, m_1) = 0$. Therefore, we get $r_R(c_0) \cap K = 0$. Since $(c_0, m_0)(0, m) = (0, c_0 m) \neq 0$ for every $0 \neq m \in M$, and M_R has finite Goldie dimension, $c_0 M$ is an essential submodule of M_R by Lemma 1.2. Furthermore, it is easily verified that K is an essential right ideal of R . Therefore, we have $r_R(c_0) = 0$. By the similar argument as above, we can show that $l_R(c_0) = 0$. Thus $c_0 \in D$.

(\impliedby). This can be proved by the similar manner as in the proof (\impliedby) of Lemma 1.1.

By a slight modification of the proof of [4, (1, 36)], we have the following.

LEMMA 1.4. Assume that R has the right ring of fractions $R[D^{-1}]$. If $c_1, \dots, c_k \in D$, then there exist $d, d_1, \dots, d_k \in D$ such that $c_i^{-1} = d_i c^{-1}$ ($i=1, \dots, k$).

2. Quotient rings of A .

In this section, we assume that ${}_R M_R$ is faithful or both M_R and ${}_R M$ have finite Goldie dimension.

THEOREM 2.1. The following conditions are equivalent.

- (1) A has the right quotient ring.
- (2) R and M satisfy the right Ore condition with respect to D .

PROOF. In this case, we note that $(c, m) \in C(A)$, $m \in M$ if and only if $c \in D$ in view of Lemmas 1.1 and 1.3.

(2) \implies (1). It suffices to show that A satisfies the right Ore condition with respect to $C(A)$. Let $(r, m) \in A$ and $(c_0, m_0) \in C(A)$. Since R satisfies the right Ore condition with respect to D , there exist $r_1 \in R$ and $c_1 \in D$ such that $rc_1 = c_0 r_1$. Moreover, since M satisfies the right Ore condition with respect to D , there exist $c'_1 \in D$ and $m_1 \in M$ such that $(mc_1 - m_0 r_1)c'_1 = c_0 m_1$. Thus we have $(r, m)(c_1 c'_1, 0) = (c_0, m_0)(r_1 c'_1, m_1)$ with $(c_1 c'_1, 0) \in C(A)$. Hence A has the right quotient ring.

(1) \implies (2). Let $c \in D$, $(0, m) \in A$ and $(r, 0) \in A$. Then $(c, 0) \in C(A)$. Since A satisfies the right Ore condition with respect to $C(A)$, there exist $(r_i, m_i) \in A$ ($i=1, 2$) and $(c_i, m'_i) \in C(A)$ ($i=1, 2$) such that $(r, 0)(c_1, m'_1) = (c, 0)(r_1, m_1)$ and $(0, m)(c_2, m'_2) = (c, 0)(r_2, m_2)$, from which it follows that $rc_1 = cr_1$ and $mc_2 = cm_2$ with $c_1, c_2 \in D$. Hence both R and M satisfy the right Ore condition with respect to D .

REMARK. The following example indicates that the equivalence of (1) and (2) in Theorem 2.1 does not hold in general, if we do not suppose the standing assumption.

EXAMPLE 2.2 [3, Example 2.1]. Let T be a right Noetherian domain which is not left Ore. Let u be an indeterminate which commutes with the elements of T and C denotes the set of all elements of the polynomial ring $T[u]$ which have non-zero constant term. Let $V = T[u][C^{-1}]$ and W the right quotient division ring of T . We can make W into a right V -module by identifying W with V/uV , i.e. by setting $Wu=0$. We set

$$S = \begin{pmatrix} T[u] & 0 \\ T & T \end{pmatrix}, \quad Q = \begin{pmatrix} V & 0 \\ W & W \end{pmatrix}.$$

Then Q is the right quotient ring of S . Since ${}_rT$ does not have finite Goldie dimension, ${}_sS$ does not have finite Goldie dimension. Let

$$\Gamma = \begin{pmatrix} S & 0 \\ S & S \end{pmatrix}$$

be a 2×2 lower triangular matrix ring over S . Then Γ does not have the right quotient ring, but S has the right quotient ring Q . We put $R = S \oplus S$. Since S is regarded as an (R, R) -bimodule in the natural way, Γ is isomorphic to $R \bowtie S$. Note that, in this case, ${}_R S_R$ is not faithful and ${}_R S$ does not have finite Goldie dimension.

A right R -module X is called D -torsion-free if $xd \neq 0$ for every $0 \neq x \in X$ and $d \in D$.

If R satisfies the right Ore condition with respect to D , then $R[D^{-1}]$, the right of fractions and $M[D^{-1}]$, the right module of fractions exist.

THEOREM 2.3. *If A has the right quotient ring, then the following (1) and (2) hold.*

(1) $M[D^{-1}]$ has an $(R[D^{-1}], R[D^{-1}])$ -bimodule structure.

(2) The canonical embedding $A \longrightarrow R[D^{-1}] \bowtie M[D^{-1}]$ gives the right quotient ring $Q(A)$ of A .

PROOF. It is to be noted that $(c, m) \in C(A)$, $m \in M$ if and only if $c \in D$ in view of Lemmas 1.1 and 1.3.

(1) Let $c \in D$ and $m \in M$. Since M satisfies the right Ore condition with respect to D , there exist $m_1 \in M$ and $c_1 \in D$ such that $mc_1 = cm_1$. We define a left multiplication on $M[D^{-1}]$ by an element of $R[D^{-1}]$ via $c^{-1} \cdot m = m_1 c_1^{-1}$. If there exist other

element $c_2 \in D$ and $m_2 \in M$ satisfying $mc_2 = cm_2$, then we have $m = cm_1c_1^{-1} = cm_2c_2^{-1}$, from which it follows that $m_1c_1^{-1} = m_2c_2^{-1}$, for ${}_R M$ is D -torsion-free. Therefore, this multiplication is well-defined. Moreover, it is easily seen that $M[D^{-1}]$ has an $(R[D^{-1}], R[D^{-1}])$ -bimodule structure.

(2) Since M_R is D -torsion-free, M can be considered as a submodule of $M[D^{-1}]$. Therefore, we can suppose that $A \subseteq R[D^{-1}] \times M[D^{-1}]$. Let $(c_0, m_0) \in C(A)$. Since $(c_0, m_0)(c_0^{-1}, -c_0^{-1}m_0c_0^{-1}) = (1, 0)$ and $(c_0^{-1}, -c_0^{-1}m_0c_0^{-1})(c_0, m_0) = (1, 0)$ with $(c_0^{-1}, -c_0^{-1}m_0c_0^{-1}) \in R[D^{-1}] \times M[D^{-1}]$, (c_0, m_0) is a unit of $R[D^{-1}] \times M[D^{-1}]$. Let $(rc_1^{-1}, mc_2^{-1}) \in R[D^{-1}] \times M[D^{-1}]$. Since there exist $c, d_1, d_2 \in D$ such that $c_i^{-1} = d_i c^{-1}$ ($i=1,2$) by Lemma 1.4, we have $(rc_1^{-1}, mc_2^{-1})(c, 0) = (rd_1c^{-1}, md_2c^{-1})(c, 0) = (rd_1, md_2) \in A$ with $(c, 0) \in C(A)$. Hence we conclude that the canonical embedding $A \longrightarrow R[D^{-1}] \times M[D^{-1}]$ gives the right quotient ring $Q(A)$ of A .

COROLLARY 2.4. *Let $A = R \times R$. Then the following are equivalent.*

- (1) A has the right quotient ring $Q(A)$.
- (2) R has the right quotient ring $Q(R)$.

In this case, the canonical embedding $A \longrightarrow Q(R) \times Q(R)$ gives the right quotient ring $Q(A)$ of A .

PROOF. This directly follows from Theorems 2.1 and 2.3.

We exhibit the following example for which R does not satisfy the right Ore condition with respect to D , but R satisfies the right Ore condition with respect to $C(R)$.

EXAMPLE 2.5 [5, Example 5.5]. Let

$$R = \begin{pmatrix} \mathbf{Z} & 2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix} \supseteq {}_R I_R = \begin{pmatrix} 2\mathbf{Z} & 2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}$$

and $A = R \times R/I$. Then both $(R/I)_R$ and ${}_R(R/I)$ have finite Goldie dimension and $D = \left\{ \begin{pmatrix} z_1 & 2z_3 \\ z_2 & z_4 \end{pmatrix} \in C(A) \mid z_1 \notin 2\mathbf{Z} \right\}$. Since R does not satisfy the right Ore condition with respect to D , A does not have the right quotient ring in view of Theorem 2.1.

3. Generalized triangular matrix rings.

Let

$$\Gamma = \begin{pmatrix} S & 0 \\ U & T \end{pmatrix}$$

be a generalized triangular matrix ring, where S and T are rings with identity

and U a (T, S) -bimodule. We put $R=S\oplus T$. Since U is regarded as an (R, R) -bimodule in the natural way, Γ is isomorphic to $R\ltimes U$. Let $D_1=\{d_1\in C(S)\mid ud_1\neq 0 \text{ for every } 0\neq u\in U\}$ and $D_2=\{d_2\in C(T)\mid d_2u\neq 0 \text{ for every } 0\neq u\in U\}$. It is clear that ${}_R U_R$ satisfies the right Ore condition with respect to a subset (D_1, D_2) of R if, given $u\in U$ and $d_2\in D_2$, there exist $u'\in U$ and $d_1\in D_1$ such that $ud_1=d_2u'$. (In this case, ${}_T U_S$ is called to satisfy the right Ore condition with respect to D_1-D_2). It is to be noted that ${}_R U_R$ is unfaithful, whenever S or T is non-zero. So, we consider only in case both U_R and ${}_R U$ have finite Goldie dimension. If we apply Theorems 2.1 and 2.3 to Γ , then we have the following.

COROLLARY 3.1. *Assume that both U_S and ${}_T U$ have finite Goldie dimension. Then the following are equivalent.*

- (1) Γ has the right quotient ring.
- (2) S, T and ${}_T U_S$ satisfy the right Ore condition with respect to D_1, D_2 , and D_1-D_2 , respectively.

COROLLARY 3.2. *In the same situation as in the preceding corollary, the right quotient ring of Γ has the form of*

$$\begin{pmatrix} S[D_1^{-1}] & 0 \\ U[D_1^{-1}] & T[D_2^{-1}] \end{pmatrix}$$

COROLLARY 3.3. *Let $T_n(R)$ be the ring of $n\times n$ lower triangular matrices over R . Assume that both R_R and ${}_R R$ have finite Goldie dimension and that R has the right quotient ring $Q(R)$. Then $T_n(R)$ has the right quotient ring isomorphic to $T_n(Q(R))$.*

PROOF. We prove by induction on n . If $n=1$, then it is obvious. We suppose that $T_{n-1}(R)$ has the right quotient ring isomorphic to $T_{n-1}(Q(R))$. Since $T_n(R)$ can be considered as

$$\begin{pmatrix} & & 0 \\ T_{n-1}(R) & & \vdots \\ & & 0 \\ R \cdots R & R & \end{pmatrix},$$

${}_R(R\cdots R)_{T_{n-1}(R)}$ satisfies the right Ore condition with respect to $C(R)-C(T_{n-1}(R))$ and both ${}_R(R\cdots R)$ and $(R\cdots R)_{T_{n-1}(R)}$ have finite Goldie dimension, $T_n(R)$ has the right quotient ring isomorphic to $T_n(Q(R))$ by Corollary 3.2.

REMARK. It is well-known that, if R has the right quotient ring $Q(R)$ and

$Q(R)_{Q(R)}$ has finite Goldie dimension, then R_R has finite Goldie dimension. Therefore, Corollary 3.3 holds under weaker conditions than in [1, Corollary 3.6].

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