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ON CURVES AND SUBMANIFOLDS IN AN INDEFINITE-RIEMANNIAN MANIFOLD

By

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§ 0. Introduction.

In a Riemannian manifold, a curve is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve is a geodesic. If only the first curvature is a non-zero constant and others are all identically zero, then the curve is called a circle. If the first and second curvatures are non-zero constants and others are all identically zero, then the curve is called a helix. For the circle, the following theorem is well known [13].

THEOREM A. *Let M be a connected submanifold of a Riemannian manifold $\overline{M}$. Every circle in M is a circle in $\overline{M}$ if and only if M is totally umbilical and has the parallel mean curvature vector in $\overline{M}$.*

For curves and submanifolds in a Riemannian manifold, see also [15].

In this paper, we shall be concerned with curves in an indefinite-Riemannian manifold. If a manifold M has an indefinite metric g, there exist null vectors in M. This situation causes a difference in the Frenet formula of curves. The purpose of this paper is to study "circle" and "helix" in an indefinite-Riemannian (especially Lorentzian) manifold and prove results similar to Theorem A.

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§ 1. Preliminaries.

Let $R^n_{i,j}$ be an n dimensional affine space with an inner product g whose canonical form is

$$\begin{bmatrix} I_{n-i-j} & & \\ & -I_i & \\ & & 0_j \end{bmatrix},$$

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where I_i is the $i\times i$ identity matrix and 0_j is the $j\times j$ 0 matrix. We call (i, j) a *signature* of $R^n_{i,j}$. The metric g is non-degenerate if and only if $j=0$, in which case we denote by R^n_i and say that the signature of R^n_i is i.

Let M be an n-dimensional smooth manifold equipped with a metric g, where the metric g means a symmetric non-degenerate (0, 2)-tensor field on M with constant signature. A tangent space $T_p(M)$ at a point $p\in M$ is furnished with the canonical inner product. If the signature of the metric g is i, then we call M an *indefinite-Riemannian manifold* of signature i and denote by M_i. If g is positive definite, then M is a Riemannian manifold. Especially if $i=1$, then M is called a *Lorentzian manifold.* A tangent vector x of M_i is said to be *spacelike* if $g(x, x)>0$ or $x=0$, *timelike* if $g(x, x)<0$ and *null* if $g(x, x)=0$ and $x\neq 0$. In particular, on the Lorentzian manifold, null vectors are also said to be *lightlike.* This terminology derives from the relativity theory. Let $x_1, \cdots, x_i, x_{i+1}, \cdots, x_n$ be tangent vectors of $M_i(\dim M=n)$. Assume that they satisfy $g(x_A, x_B)=\varepsilon_A\delta_{AB}$, where $\varepsilon_A=g(x_A, x_A)=+1$ (resp. -1) if x_A is spacelike (resp. timelike) then $\{x_A, A\in[1, n]\}$ is called an orthonormal basis of M_i.

In a Lorentzian manifold M_1, timelike vectors and null vectors are called *causal vectors.* There are no non-zero cusal vectors orthogonal to a timelike vector. In a Lorentzian manifold, a null vector n_1 is orthogonal to a null vector n_2 if and only if n_1 is linearly dependent to n_2.

A *pseudosphere* S^n_i of radius 1 in R^{n+1}_i is defined by

$$S^n_i=\{x\in R^{n+1}_i:\ g(x, x)=1\};$$

then S^n_i is a complete n-dimensional indefinite-Riemannian manifold of signature i and of constant sectional curvature 1. Similarly we define a *pseudohyperbolic space* H^n_i of radius 1 in R^{n+1}_{i+1} by

$$H^n_i=\{x\in R^{n+1}_{i+1}:\ g(x, x)=-1\};$$

then H^n_i is a complete n-dimensional indefininte-Riemannian manifold of signature i and of constant sectional curvature -1. R^n_i is a complete n-dimensional indefinite-Riemannian manifold of signature i and of constant sectional curvature 0. By $\bar{N}^n_i$, we denote one of S^n_i, H^n_i or R^n_i to simplify the presentation. $\bar{N}^n_i$ are called an *indefinite-Riemannian space form.*

Next, we recall the general theory of indefinite-Riemannian submanifolds immersed into an indefinite-Riemannian manifold (cf. [9], [16]) and show some lemmas which are subsequently useful. Let $f: M_i\to\bar{M}_j$ be an isometric immersion of an n-dimensional indefinite-Riemannian manifold M_i of signature i into an $(n+p)$-dimensional indefinete-Riəmannian manifold $\bar{M}_j$ of signature j. For all local formulas we may consider f as an imbedding and thus identify $p\in M_i$ with $f(p)\in\overline{M}_j$. The tangent space $T_p(M_i)$ is identified with a subspace of $T_p(\overline{M}_j)$. Denote by $T(M_i)$ the tangent bundle. The normal space $T_p^{\perp}$ is the subspace of $T_p(\overline{M}_j)$ consisting of vectors which are orthogonal to $T_p(M_i)$ with respect to the metric g of $\overline{M}_j$. By ∇ (resp. $\overline{\nabla}$) we denote the covariant differentiation of M_i (resp. $\overline{M}_j$). Then we have the Gauss' formula

$$\overline{\nabla}_X Y=\nabla_X Y+B(X, Y), \tag{1.1}$$

where X and Y are tangent vector fields of M_i and $B(X, Y)$ is called the *second fundamental form* of the immersion. The formula of Weingarten is given by

$$\overline{\nabla}_X N=-A^N(X)+\nabla_X^{\perp}N, \tag{1.2}$$

where X (resp. N) is a tangent (resp. normal) vector field of M_i and and $\nabla^{\perp}$ is the covariant differentiation with respect to the induced connection in the normal bundle $N(M_i)$. A^N is called the *shape operator* of M_i and satisfies the relation

$$g(A^N(X), Y)=g(B(X, Y), N).$$

For an orthonormal basis $\{N_1, \cdots, N_p\}$ of $N(M_i)$ we write $A^{N_I}=A^I$, to simplify the notation.

We next define the covariant differentiation $\tilde{\nabla}$ induced on the Whitney sum $T(M_i)\oplus N(M_i)$ as follows: For any $N(M_i)$-valued tensor field T of type $(0, k)$. we define

$$(\tilde{\nabla}_X T)(Y_1, \cdots, Y_k):=\nabla_X^{\perp}(T(Y_1, \cdots, Y_k))-\sum_{r=1}^{k}T(Y_1, \cdots, \nabla_X Y_r, \cdots, Y_k)$$

and $\tilde{\nabla}T$ is also defined by $(\tilde{\nabla}T)(Y_1, \cdots, Y_k, X):=(\tilde{\nabla}_X T)(Y_1, \cdots, Y_k)$ which is an $N(M_i)$-valued tensor field of type $(0, k+1)$. We denote by $\tilde{\nabla}^2 T$ the covariant derivative of $\tilde{\nabla}T$. In particular, for the second fundamental form B, it follows that

$$(\tilde{\nabla}B)(X, Y, Z)=\nabla_Z^{\perp}(B(X, Y))-B(\nabla_Z X, Y)-B(X, \nabla_Z Y), \tag{1.3}$$

$$\begin{aligned}(\tilde{\nabla}^2 B)(X, Y, Z, W)=\nabla_W^{\perp}((\tilde{\nabla}B)(X, Y, Z))-(\tilde{\nabla}B)(\nabla_X X, Y, Z)\\ -(\tilde{\nabla}B)(X, \nabla_W Y, Z)-(\tilde{\nabla}B)(X, Y, \nabla_W Z).\end{aligned} \tag{1.4}$$

For the shape operator A^N we define its covariant differentiation by setting

$$(\tilde{\nabla}_X A^N)(Y):=\nabla_X(A^N(Y))-A^{\nabla_X^{\perp}N}(Y)-A^N(\nabla_X Y).$$

Then we have the relation $g((\tilde{\nabla}_X B)(Y, Z), N)=g((\tilde{\nabla}_X A^N)(Y), Z)$.

The *mean curvature vector field* H of the immersion is defined by

$$H:=(1/n)\sum_{j=1}^{n}\varepsilon_j B(E_j, E_j),$$

where $\{E_1, \cdots, E_n\}$ is a frame of M_i and $\varepsilon_i=\pm 1$. If the second fundamental form $B(X, Y)$ satisfies

$$B(X, Y)=g(X, Y)H$$

for all vector field X, Y of M_i, then M_i is called a *totally umbilical submanifold.* If the second fundamental form vanishes identically on M_i, then M_i is said to be *totally geodesic.* The mean curvature vector field H is said to be parallel if $\nabla_X^{\perp} H=0$.

Since the second fundamental form B is a bilinear symmetric function on $T_p(M_i)$, using results of [4], we have following lemmas.

LEMMA 1.1. *For any point p of M_1, we assume that B satisfies $B(t, s)=0$, where $t\in T_p(M_1)$ is a unit timelike vector and $s\in T_p(M_1)$ is a unit spacelike vector such that $g(t, s)=0$. Then M_1 is a totally umbilical submanifold.*

LEMMA 1.2. *Let B be the second fundamental form of a Lorentzian submanifold M_1. If B satisfies $B(n, n)=0$, for any null vector n at any point in M_1, then M_1 is a totally umbilical submanifold.*

LEMMA 1.3. *If B satisfies $B(n_1, n_2)=0$ for any null vectors n_1 and a_2 such that $g(n_1, n_2)=-1$ at any point of M_1, then M_1 is a totally geodesic submanifold.*

LEMMA 1.4. *For any point p of M_1, if B satisfies*

$$2B(t, t)=-B(s, s)$$

for any unit timelike vector t and unit spacelike vectors such that $g(t, s)=0$ then M_1 is a totally geodesic submanifold.

LEMMA 1.5. *Let H be the mean curvature vector field of a Lorentzian submanifold M_1. For any point p of M_1, we assume that H satisfies $\nabla_s^{\perp} H=0$, for any spacelike vector $s\in T_p(M_1)$. Then H is parallel.*

PROOF. A spacelike vector s can be put as $s=n_1-t$, where n_1 is a null vector and t a unit timelike vector, respectively. Hence we have

$$\nabla_s^{\perp} H=\nabla_{n_1}^{\perp} H-\nabla_t^{\perp} H=0. \tag{1.5}$$

On the other hand, we can put $s=-n_2+t$, where $g(n_1, n_2)=-1$. Then it follows that

$$\nabla_s^{\perp} H=-\nabla_{n_2}^{\perp} H+\nabla_t^{\perp} H=0. \tag{1.6}$$

Combining (1.5) and (1.6), we get

$$\nabla^{\perp}_{n_1}H=\nabla^{\perp}_{n_2}H. \tag{1.7}$$

Changing n_1 (resp. n_2) into $2n$ (resp. $n_2/2$) in (1.7), we obtain $4\nabla^{\perp}_{n_1}H=\nabla^{\perp}_{n_2}H$. From this equation and (1.7), it follows that $\nabla^{\perp}_{n_1}H=\nabla^{\perp}_{n_2}H=0$, which together with (1.6) implies $\nabla^{\perp}_{t}H=0$. Therefore we conclude that $\nabla^{\perp}_{x}H=0$, for any tangent vector x.

Similarily we have

LEMMA 1.6. *If H satisfies $\nabla^{\perp}_{t}H=0$ for any timelike vector, then H is parallel.*

§ 2. Curves.

A curve in an indefinite-Riemannian manifold M_i is a smooth mapping $c: I\rightarrow M_i$, where I is an open interval in the real line R^1. As an open submanifold of R^1, I has a coordinate system consisting of the identity map u of I. The velocity vector of c at $t\in I$ is

$$c'(t):=dc(d/du|_t)\in T_{c(t)}(M_i).$$

A curve $c(t)$ is said to be regular if $c'(t)$ is not equal to zero for any t. A curve $c(t)$ in an indefinite-Riemannian manifold M_i is said to be *spacelike* if all of its velocity vectors $c'(t)$ are spacelike; similarly for *timelike* and *null*. If $c(t)$ is a spacelike or timelike curve, we can reparameterize it such that $g(c'(t), c'(t))=\varepsilon$ (where $\varepsilon=+1$ if c is spacelike and $\varepsilon=-1$ if c is timelike, respectively). In this case $c(t)$ is said to be unit speed or arc lenght parametrization. Here and in the sequel, we assume that the spacelike or timelike curve $c(t)$ has an arc length parametrization.

We define here a circle and a helix in an indefinite-Riemannian manifold M_i (cf. [1], [5], [15], [18]). Let $c=c(t)$ be a timelike curve in M_i. By $k_j(t)$, we denote the j-th curvature of $c(t)$. If $k_j(t)\equiv 0$ for $j>2$ and if the principal vector field Y and the binormal vector field Z are spacelike, then we have the following Frenet formulas along $c(t)$:

$$\begin{cases} c'(t)=:X, \\ \nabla_X X=k_1(t)Y, \\ \nabla_X Y=k_1(t)X+k_2(t)Z, \\ \nabla_X Z=-k_2(t)Y, \end{cases} \tag{2.1}$$

where ∇ denotes the covariant differentiation in M_i. A curve $c=c(t)$ is called a *circle* if $k_2(t)\equiv 0$ and $k_1(t)$ is a positive constant along $c(t)$. If both $k_1(t)$ and $k_2(t)$ are positive constants along $c(t)$, then $c(t)$ is called a *helix*. Let $c(t)$ be a circle.

Then the components satisfy a system of differential equations, because of the Frenet formulas for $c(t)$. According to the fundamental theory of differential equations, we see that there exists a unique solution satisfying the given initial condition in a sufficiently small interval of $t=0$. Namely, for any point p of M_1 and any orthonormal vecters x and y at p (where x is timelike and y is spacelike, respectively), there exists locally a circle passing through p with a tangent vector x, which satisfies certain conditions. A similar phenomenon holds also on the helix.

We remark that if the principal vector field Y of a spacelike curve $c(t)$ is timelike and the binormal vector field Z is spacelike, then we have the following Frenet formula along $c(t)$:

$$(2.1)' \qquad \begin{cases} c'(t)=:X, \\ \nabla_X X=k_1(t)Y, \\ \nabla_X Y=k_1(t)X+k_2(t)Z, \\ \nabla_X Z=k_2(t)Y. \end{cases}$$

Next we consider a null curve in a Lorentzian manifold (cf. [1], [2], [5], [6], [7]). By a *Cartan frame* (X, Y, Z) of a null curve $c=c(t)$ we mean a family of vector fields $X=X(t)$, $Y=(t)$, $Z=Z(t)$ along the curve $c(t)$ satisfying the following conditions:

$$(2.2) \qquad \begin{cases} c'(t)=:X, & g(X, X)=g(Y, Y)=0, \\ g(X, Y)=-1, & g(X, Z)=g(Y, Z)=0, \quad g(Z, Z)=1 \\ \nabla_X X=k_1(t)Z, & \nabla_X Y=k_2(t)Z, \quad \nabla_X Z=k_2(t)X+k_1(t)Y, \end{cases}$$

where $k_1(t)$ and $k_2(t)$ are functions defined along the curve $c(t)$. Especially if $k_1(t)$ and $k_2(t)$ are positive constant along $c(t)$, then we call the curve $c=c(t)$ a *Cartan framed null curve with constant curvatures.* On the definition of the Cartan frame of a null curve $x(t)$, if $k_2(t)\equiv 0$ then $c=c(t)$ is called a *generalized null cubic.* Moreover if k is constant, then $c(t)$ is called a *generalized null cubic with constant curvature.* For any point p of a Lorentzian manifold, any constnats k_1 and k_2, and any Cartan frame (X, Y, Z) at p, there exists locally a Cartan framed null curve $c(t)$ with constant curvatures passing through p with velocity vector $c'(p)=X(p)$, which satisfies certain conditions. A similar situation holds also on the generalized null cubic with constant iurvature.

§3. Circles.

Let $c=c(t)$ be a regular timelike curve in a Lorentzian manifold M_1. In this section, we assume that $c(t)$ is a circle, that is, $c(t)$ satisfies

$$\begin{cases} c'(t)=:X \\ \nabla_X X=kY, \\ \nabla_X Y=kX \end{cases} \tag{3.1}$$

along the curve $c(t)$, where Y is a spacelike vector field and k a positive constant, respectively.

PROPOSITION 3.1 (cf. [15]). *Let $c(t)$ be a timelike curve in a Lorentzian manifold M_1. If $c(t)$ is a circle, then the velocity vector field X of $c(t)$ satisfies*

$$\nabla_X \nabla_X X - g(\nabla_X X, \nabla_X X)X = 0. \tag{3.2}$$

Conversely, if the velocity vector field of a timelike curve $c(t)$ satisfies (3.2), *then $c(t)$ is either a geodesic or a circle.*

PROOF. If $c(t)$ is a circle, we have (3.2) from (3.1). Conversely, we assume (3.2). Since $g(X, \nabla_X X)=0$, it follows that

$$\begin{aligned} d(g(\nabla_X X, \nabla_X X))/dt &= 2g(\nabla_X \nabla_X X, \nabla_X X) \\ &= 2g(\nabla_X X, \nabla_X X)g(X, \nabla_X X)=0, \end{aligned}$$

by virtue of (3.2). Hence $g(\nabla_X X, \nabla_X X)$ is constant along $c(t)$. If it is 0, $c(t)$ is a geodesic. We assume that $g(\nabla_X X, \nabla_X X)$ is non zero constant. Since M_1 is the Lorentzian manifold, there is no non-zero causal vector which is orthogonal to a timelike vector. Therefore from $g(X, \nabla_X X)=0$, we see that $\nabla_X X$ is a spacelike vector field along $c(t)$ and we can put

$$g(\nabla_X X, \nabla_X X)=k^2, \qquad \nabla_X X=kY,$$

where Y is a unit spacelike vector field along $c(t)$ and k is a positive constant. Then we have

$$\nabla_X Y=(1/k)\nabla_X \nabla_X X=(1/k)(k^2 X)=kX,$$

by virtue of (3.2). Thus $c(t)$ is a circle.

THEOREM 3.2. *Let M_1* ($\dim M_1>2$) *be a connected Lorentzian submanifold of anindefinite-Riemannian manifold $\overline{M}_i$. If, for some $k>0$, every timelike circle with curvature k in M_1 is a timelike circle in $\overline{M}_i$, then M_1 is totally umbilical and has parallel mean curvature vector in $\overline{M}_i$. Conversely, if M_1 is totally umbilical and has the parallel mean curvature vector, then every timelike circle in M_1 is a timelike circle in $\overline{M}_i$.*

PROOF. For an arbitrary point p of M_1, we consider orthonormal vectors x and y in $T_p(M_1)$ such that x is timelike and y is spacelike, respectively. Let $c(t)$ be a circle in M_1 such that

$$c(0)=p, \quad X(p)=x, \quad (\nabla_X X)(p)=ky,$$

where ∇ is the covariant differentiation on M_1 and X is the velocity vector field of $c(t)$. X satisfies

$$\nabla_X\nabla_X X-g(\nabla_X X, \nabla_X X)X=0, \tag{3.3}$$

on M_1. By assumption, $c(t)$ is a circle in $\bar{M}_i$. Then it follows that

$$\bar{\nabla}_X\bar{\nabla}_X X-g(\bar{\nabla}_X X, \bar{\nabla}_X X)X=0, \tag{3.4}$$

where $\bar{\nabla}$ is the covariant differentiation on $\bar{M}_i$. Substituting (1.1) and (1.2) into (3.4), and taking the normal part of it, we get

$$B(X, \nabla_X X)+\nabla_X^{\perp}B(X, X)=0, \tag{3.5}$$

by virtue of (3.3). Hence, by means of (1.3), we have

$$(\tilde{\nabla}B)(x, x, x)=-3kB(x, y),$$

at p. This shows that, given a unit timelike vector $x\in T_p(M_1)$, $B(x, y)$ is independent of a unit spacelike vector y provided y is orthogonal to x. Changing y into $-y$, we have $B(x, y)=0$, where x and y are orthonormal vectors such that x is timelike and y is spacelike, respectively. Since p is arbitrary, we have, from Lemma 1.1, that M_1 is totally umbilical. Henc it follows that $B(X, Y)=g(X, Y)H$, for any orthonormal vector fields X and Y. Substituting this equation into (3.5), we get $\nabla_X^{\perp}H=0$, for any timelike vector field X. From Lemma 1.6, it follows that the mean curvature vector is parallel.

Next we consider the converse. Let $c(t)$ be a timelike circle in M_1. So the velocity vector field X of $c(t)$ satisfies (3.3). Since M_1 is totally umbilical and has the parallel mean curvature vector, it follows that

$$B(X, X)=-H, \qquad B(\nabla_X X, X)=g(\nabla_X X, X)H=0,$$
$$A^{B(X,X)}(X)=-g(H, H)X, \qquad \nabla_X^{\perp}B(X, X)=\nabla_X^{\perp}H=0$$

for a timelike vector field. Substituting these equations into (1.1), we have

$$\bar{\nabla}_X\bar{\nabla}_X X=\nabla_X\nabla_X X-g(H, H)X. \tag{3.6}$$

On the other hand, using (1.1), $B(X, X)=g(X, X)H=-H$ yield

$$g(\bar{\nabla}_X X, \bar{\nabla}_X X)=g(\nabla_X X, \nabla_X X)+g(H, H). \tag{3.7}$$

From (3.3), (3.6) and (3.7) it follows that

$$\bar{\nabla}_X\bar{\nabla}_X X - g(\bar{\nabla}_X X, \bar{\nabla}_X X)X = 0.$$

Hence $c(t)$ is a timelike circle in $\bar{M}_i$.

If a spacelike circle $c(t)$ has a timelike principal vector, the velocity vector field $X := c'(t)$ satisfies

$$\nabla_X\nabla_X X + g(\nabla_X X, \nabla_X X)X = 0.$$

From this equation we have following

COROLLARY 3.3. *Let M_1 be a Lorentzaian submanifold in an indefinite-Riemannian manifold $\bar{M}_i$. If every spacelike circle with a timelike principal vector field in M_1 is a circle in $\bar{M}_i$, then M_1 is totally umbilical and has the parallel mean curvature vector. The converse is also true.*

§ 4. Helices.

Next we consider helices in a Lorentzian maifold M_1. Let $c = c(t)$ be a regular timelike helix in a Lorentzian manifold M_1. Then we have

$$(4.1)\qquad \begin{cases} c'(t) =: X, \\ \nabla_X X = k_1 Y, \\ \nabla_X Y = k_1 X + k_2 Z, \\ \nabla_X Z = -k_2 Y \end{cases}$$

along the curve $c(t)$, where Y, Z are spacelike vector fields and k_1, k_2 are positive constants, respectively.

PROPOSITION 4.1. *Let $c(t)$ be a timelike curve in a Lorentzian manifold M_1* (dim $M \geqq 3$). *If $c(t)$ is a helix, then the velocity vector field X of $c(t)$ satisfies*

$$(4.2)\qquad \nabla_X\nabla_X\nabla_X X - K\nabla_X X = 0,$$

where K is a constant. Conversely, if the velocity vector field of a timelik curve $c(t)$ satisfies (4.2), *then $c(t)$ is one of a geodesic, a circle and a helix.*

PROOF. Suppose that $c(t)$ is a timelike helix. Then, from (4.1), it is easily seen that the velocity vector field X satisfies (4.2) with $K = k_1^2 - k_2^2$.

Conversely, we assume that the timelike curve $c(t)$ satisfies (4.2). Differentiating $g(X, \nabla_X X) = 0$ in the direction of X, we have

$$g(\nabla_X\nabla_X X, X) + g(\nabla_X X, \nabla_X X) = 0.$$

Moreover, differentiating this equation in the direction of X, we obtain

(4.3) $$g(\nabla_X\nabla_X\nabla_X X,\ X)+3g(\nabla_X\nabla_X X,\ \nabla_X X)=0.$$

Substituting (4.2) into (4.3), we get $g(\nabla_X\nabla_X X, \nabla_X X)=0$. This implies that $g(\nabla_X X, \nabla_X X)$ is constant along $c(t)$. If it is 0, then $c(t)$ is a geodesic. If $g(\nabla_X X, \nabla_X X)\not\equiv 0$, then there exists a unit spacelike vector field Y along $c(t)$ and a positive constant k_1 such that

(4.4) $$\nabla_X X=k_1 Y.$$

Since M_1 ($\dim M_1\geqq 3$) is the Lorentzian manifold, we can put

(4.5) $$\nabla_X Y=k_1 X+bZ,$$

where Z is a unit spacelike vector field which is orthogonal to both X and Y, and b is a function, respectively. If $b\equiv 0$, then $c(t)$ is a circle. Hence we may assume that b is a positive function. By means of (4.2) we have

$$\begin{aligned} d(g(\nabla_X\nabla_X X,\ \nabla_X X))/dt=0&=g(\nabla_X\nabla_X\nabla_X X,\ \nabla_X X)+g(\nabla_X\nabla_X X,\ \nabla_X\nabla_X X)\\ &=Kg(\nabla_X X,\ \nabla_X X)+g(\nabla_X\nabla_X X,\ \nabla_X\nabla_X X). \end{aligned}$$

Substituting (4.4) and (4.5) into this equation, we get

$$k_1^2b^2=k_1^4-Kk_1^2.$$

Since $k_1\not\equiv 0$ and b is positive, it follows that $b=\sqrt{k_1^2-K}$, i.e., b is a positive constant. We put $b=k_2$. Hence (4.5) is reduced to

(4.6) $$\nabla_X Y=k_1 X+k_2 Z.$$

Differentiating (4.6) in the direction of X, we have

(4.7) $$\nabla_X\nabla_X Y=k_1^2 Y+k_2(\nabla_X Z).$$

On the other hand, it follows that

(4.8) $$\nabla_X\nabla_X Y=(1/k_1)\nabla_X\nabla_X\nabla_X X=(1/k_1)(k_1^2-k_2^2)\nabla_X X=(k_1^2-k_2^2)Y,$$

by virtue of (4.2) and (4.4). Making use of (4.7) and (4.8), we obtain

(4.9) $$\nabla_X Z=-k_2 Y.$$

From (4.4), (4.6) and (4.9), we conclude that $c(t)$ is a helix.

Next we prove the following

THEOREM 4.2. *Let* M_1 ($\dim M_1\geqq 3$) *be a connected Lorentzian submanifold of an indefinite-Riemannian manifold* $\overline{M}_i$. *If, for some* $k_1, k_2>0$, *every timelike helix with curvatures* k_1 *and* k_2 *in* M_1 *is a timelike helix in* $\overline{M}_i$, *then* M_1 *is a totally geodesic submanifold in* $\overline{M}_i$.

PROOF. For any point p of M_1, let x, y and z are three orthonormal vectors in $T_p(M_1)$ such that x is timelike, and y and z are spacelike, respectively. Let $c(t)$ be a helix in M_1 such that

$$c(0)=p, \quad c'(t)=:X, \quad X(p)=x, \quad Y(p)=y, \quad Z(p)=z,$$

$$(\nabla_X X)(p)=k_1 y, \quad (\nabla_X Y)(p)=k_1 x+k_2 z, \quad (\nabla_X Z)(p)=-k_2 y,$$

where Y (resp, Z) is the principal (resp. binormal) vector field of $c(t)$. From Proposition 4.1, X satisfies

$$\nabla_X\nabla_X\nabla_X X-k\nabla_X X=0, \qquad k=k_1^2-k_2^2. \tag{4.10}$$

Since $c(t)$ is a helix in $\overline{M}_i$, we have

$$\overline{\nabla}_X\overline{\nabla}_X\overline{\nabla}_X X-K\overline{\nabla}_X X=0,$$

where K is a constant. Substituting (1.1), (1.2) and (4.10) into this equation, we obtain for normal part of M_1

$$\begin{aligned} B(X, \nabla_X\nabla_X X)+\nabla_X^{\perp}B(X, \nabla_X X)-B(X, A^{B(X,X)}(X)) \\ +\nabla_X^{\perp}(\nabla_X^{\perp}B(X, X))-KB(X, X)=0, \end{aligned} \tag{4.11}$$

for tangent part of M_1

$$-A^{B(X,\nabla_X X)}(X)-\nabla_X(A^{B(X,X)}(X))-A^{\nabla_X^{\perp}B(X,X)}(X)+(k-K)\nabla_X X=0. \tag{4.12}$$

From (4.11) it follows that

$$\begin{aligned} 4k_1^2B(x, x)+4k_1k_2B(x, z)+5k_1(\tilde{\nabla}B(x, y, x) \\ +3k_1^2B(y, y)-B(x, A^{B(x,x)}(x))+(\tilde{\nabla}^2B)(x, x, x, x) \\ +k_1(\tilde{\nabla}B)(x, x, y)-KB(x, x)=0 \end{aligned} \tag{4.13}$$

at a point p, by virtue of (1.3) and (1.4). Changing z into $-z$ in (4.13) we have that $B(x, z)=0$, where x and z are orthonormal vectors of $T_p(M_1)$ such that x is timelike and y is spacelike, respectively. Since p is an arbitrary point of M_1, we see that M_1 is totally umbilical by virtue of Lemma 1.1. Changing y into $-y$ in (4.13) and using the fact that M is totally umbilical we obtain $\nabla_y^{\perp}H=0$, where y is a unit spacelike vector. Hence from Lemma 1.5, we see that the mean curvature vector field is parallel. Therefore it follows that $(\tilde{\nabla}B)(x, x, x)=0$ and $(\tilde{\nabla}^2B)(x, x, x, x)=0$ for a timelike vector x, which imply that (4.13) is reduced to

$$(-k_1^2+K-g(H, H))H=0. \tag{4.14}$$

On the other hand, the inner product of (4.12) with Y implies

$$g((\nabla_X A^{B(X,X)})(X), Y)+k_1g(B(X, X), B(Y, Y))-k_1(k_1^2-k_2^2-K)g(Y, Y)=0.$$

Since M_1 is totally umbilical with parallel mean curvature vector, this equation is reduced to

$$g(H, H)=-k_1^2+k_2^2+K.$$

Combining this equation together with (4.14), we have $H=0$. This means that M_1 is a totally geodesic submanifold of $\bar{M}_i$.

§ 5. Cartan framed null curves.

In this section we consider the Cartan framed null curves. Let M_1 ($\dim M_1 \geqq 3$) be a Lorentzian manifold. By $c=c(t)$ we denote a Cartan framed null curve with constant curvatures k_1 and k_2 in M_1. That is, there are vector fields X, Y and Z along the curve $c(t)$ and they satisfy

$$(5.1)\qquad \begin{cases} c'(t)=:X, \quad g(X, X)=g(Y, Y)=0, \quad g(X, Y)=-1, \\ g(X, Z)=g(Y, Z)=0, \quad g(Z, Z)=1, \\ \nabla_X X=k_1 Z, \quad \nabla_X Y=k_2 Z, \quad \nabla_X Z=k_2 X+k_1 Y, \end{cases}$$

where ∇ is the covariant differentiation in M_1.

By a straightforward calculation, we have the following

PROPOSITION 5.1. *A Cartan framed null curve $c(t)$ with constant curvatures k_1 and k_2 satisfies following equation*:

$$\nabla_X \nabla_X \nabla_X X=2k_1 k_2 \nabla_X X.$$

We consider the converse of this propositon.

PROPOSITION 5.2. *Let $c=c(t)$ be a null curve of a Lorentzian manifold M_1. Suppose the velocity vector field $X:=c(t)$ of the null curve $c(t)$ and a null vector field Y defined along $c(t)$ satisfy the followings*:

$$(5.2)\qquad \nabla_X \nabla_X \nabla_X X=2g(\nabla_X X, \nabla_X X)^{1/2} g(\nabla_X Y, \nabla_X Y)^{1/2} \nabla_X X,$$

$$g(\nabla_X X, \nabla_X X)>0, \quad g(\nabla_X Y, \nabla_X Y)>0, \quad g(X, Y)=-1.$$

Then $c=c(t)$ is a Cartan framed null curve with constant curvatures.

PROOF. Differentiating $g(X, X)=0$ in the direction of X, we have

$$(5.3)\qquad g(\nabla_X X, X)=0.$$

Differentiating (5.3) twice in the direction of X, we obtain

$$(5.4)\qquad g(\nabla_X \nabla_X \nabla_X X, X)+3g(\nabla_X \nabla_X X, \nabla_X X)=0.$$

Substituting (5.2) into (5.4) and making use of (5.3), we get

$$g(\nabla_X\nabla_X X,\ \nabla_X X)=0. \tag{5.5}$$

This equation shows that $g(\nabla_X X, \nabla_X X)$ is constant along the curve. Hence, by assumption, we may put

$$kZ:=\nabla_X X. \tag{5.6}$$

where Z is a unit spacelike vector field and k is a positive constant. From (5.3) it follows that

$$g(X, Z)=0. \tag{5.7}$$

Differentiating (5.5) in the direction of X, we have

$$2kg(\nabla_X Y,\ \nabla_X Y)^{1/2}+g(\nabla_X Z,\ \nabla_X Z)=0, \tag{5.8}$$

by virtue of (5.2). From this equation it follows that

$$4k^2 g(\nabla_X\nabla_X Y,\ \nabla_X Y)=g(\nabla_X\nabla_X Z,\ \nabla_X Z). \tag{5.9}$$

On the other hand, from $\nabla_X\nabla_X Z=(1/k)\nabla_X\nabla_X\nabla_X X$, we obtain

$$g(\nabla_X\nabla_X Z,\ \nabla_X Z)=g(\nabla_X\nabla_X X,\ \nabla_X X)=0$$

by virtue of (5.2) and (5.5). Hence (5.9), reduces to $g(\nabla_X\nabla_X Y, \nabla_X Y)=0$ and it implies that $g(\nabla_X Y, \nabla_X Y)$ is constant along the curve $c(t)$. Therefore we can put $g(\nabla_X Y, \nabla_X Y)=w^2$, where w is a positive constant along the curve. Substituting this equation into (5.8), we have

$$g(\nabla_X Z,\ \nabla_X Z)=-2kw. \tag{5.10}$$

This means that $\nabla_X Z$ is a timelike vector field. Since M_1 is the Lorentzian manifold, we may put

$$\nabla_X Z=aX+bY, \tag{5.11}$$

where a and b are functions. Hence we get

$$g(\nabla_X\nabla_X X,\ X)=-bk.$$

On the other hand, from (5.3) it follows that

$$g(\nabla_X\nabla_X X,\ X)=-g(\nabla_X X,\ \nabla_X X)=-k^2.$$

From these two equations, we obtain $b=k$ (=constant). Therefore (5.11) implies that $\nabla_X Z=aX+kY$, from which it follows that

$$g(\nabla_X Z,\ \nabla_X Z)=-2ak=-2kw,$$

by virtue of (5.10). Hence we have $a=w$ (=constant) and

(5.12) $$\nabla_X Z = wX + kY.$$

Differentiating (5.12) in the direction of X, we get

(5.13) $$\nabla_X \nabla_X Z = w\nabla_X X + k\nabla_X Y.$$

On the other hand, by virtue of (5.2) and (5.6), it follows that

(5.14) $$\nabla_X \nabla_X Z = (1/k)\nabla_X \nabla_X \nabla_X X = (1/k)2kw\nabla_X X = 2w\nabla_X X.$$

Combining (5.13) and (5.14), and using (5.6), we obtain

(5.15) $$\nabla_X Y = wZ.$$

Differentiating $g(X, Y) = -1$ in the direction of X, we have $g(\nabla_X X, Y) + g(X, \nabla_X Y) = 0$. Together with (5.15), it implies

(5.16) $$g(Z, Y) = 0.$$

From (5.6), (5.7), (5.12), (5.15) and (5.16), we obtain the conclusion.

Next we shall prove the following theorem.

THEOREM 5.3. *Let M_1 be a Lorentzian submanifold of an indefinite-Riemannian manifold M_i. If every Cartan framed null curve with constant curvatures in M_1 is also a Cartan framed null curve with constant curvatures in $\bar{M}_i$, then M_1 is a totally geodesic submanifold in $\bar{M}_i$.*

PROOF. For an arbitrary point p of M_1, let x, y and z be three vectors in $T_p(M_1)$ such that x and y are null vectors and z is a spacelike unit vector, respectively. We assume that they satisfy $g(x, y) = -1$ and $g(x, z) = g(y, z) = 0$. Let $c = c(t)$ be a Cartan framed null curve with a Cartan frame (X, Y, Z) and constant curvatures k_1, k_2, such that

(5.17) $$\begin{aligned} &c(0) = p, \quad c'(t) =: X, \quad X(p) = x, \quad Y(p) = y, \quad Z(p) = z, \\ &(\nabla_X X)(p) = k_1 z, \quad (\nabla_X Y)(p) = k_2 z, \quad (\nabla_X Z)(p) = k_2 x + k_1 y, \end{aligned}$$

where ∇ is the covariant differentiation on M_1. From Proposition 5.1, X satisfies

(5.18) $$\nabla_X \nabla_X \nabla_X X = 2k_1 k_2 \nabla_X X,$$

on M_1. By assumption, $c(t)$ is a Cartan framed null curve in $\bar{M}_i$. Hence, from Proposition 5.1, we have

(5.19) $$\bar{\nabla}_X \bar{\nabla}_X \bar{\nabla}_X X = 2K_1 K_2 \bar{\nabla}_X X,$$

where $\bar{\nabla}$ is the covariant derivative of $\bar{M}_i$ and K_1, K_2 are positive constants. Combining (1.1), (1.2), (5.16) and (5.17), and taking the normal part of it, we obtain

$$4B(X, \nabla_X\nabla_X X)+5(\tilde{\nabla}B)(X, \nabla_X X, X)+3B(\nabla_X X, \nabla_X X) \\ -B(X, A^{B(X,X)}(X))+(\tilde{\nabla}^2B)(X, X, X, X) \\ +(\tilde{\nabla}B)(X, X, \nabla_X X)-2K_1K_2B(X, X)=0.$$

Consequently, by virtue of (5.17), the above equation gives

$$(5.20)\qquad 4k_1k_2B(x, x)+4k_1^2B(x, y)+5k_1(\tilde{\nabla}B)(x, z, x) \\ +3k_1^2B(z, z)-B(x, A^{B(x,x)}(x))+(\tilde{\nabla}^2B)(x, x, x, x) \\ +k_1(\tilde{\nabla}B)(x, x, z)-2k_1k_2B(x, x)=0,$$

at p. Changing z into $-z$ in this equation we obtain

$$(5.21)\qquad 2(2k_1k_2-K_1K_2)B(x, x)+4k_1^2B(x, y)+3k_1^2B(z, z) \\ -B(x, A^{B(x,x)}(x))+(\tilde{\nabla}B)(x, x, x, x)=0.$$

We remark that x and y are null vectors such that $g(x, y)=-1$. Changing x into $2x$ and y into $(1/2)y$ in (5.21), we obtain

$$8(2k_1k_2-K_1K_2)B(x, x)+4k_1^2B(x, y)+3k_1^2B(z, z) \\ -16B(x, A^{B(x,x)}(x))+16(\tilde{\nabla}^2B)(x, x, x, x)=0.$$

From (5.21) and this equation, it follows that

$$2(2k_1k_2-K_1K_2)B(x, x)-5B(x, A^{B(x,x)}(x))+5(\tilde{\nabla}^2B)(x, x, x, x)=0.$$

Substituting this equation into (5.21), we have

$$(5.22)\qquad 4B(x, A^{B(x,x)}(x))-4(\tilde{\nabla}^2B)(x, x, x, x)+4k_1^2B(x, y)+3k_1^2B(z, z)=0.$$

Changing x into $2x$ and y into $(1/2)y$ in (5.22), we obtain

$$(5.23)\qquad 4B(x, y)=-3B(z, z),$$

by virtue of (5.22). Since z is a unit spacelike vector, and x and y are null vectors such that $g(x, z)=g(y, z)=0$ and $g(x, y)=0$, we can put $x=z+t$ and $y=(1/2)(t-z)$, where t is a unit timelike vector having the property that $g(z, t)=0$. Hence (5.23) is reduced to

$$4B(z+t, (t-z)/2)=-3B(z, z),$$

from which it follows that

$$2B(t, t)=-B(z, z).$$

Therefore from Lemma 1.3, we conclude that M_1 is a totally geodesic submanifold of $\overline{M}_i$.

For the generalized null cubic, we have the following results similar to the

Cartan framed null curve.

PROPOSITION 5.4. *The generalized null cubic* $c=c(t)$ *satisfies* $\nabla_X\nabla_X\nabla_X X=0$, *where* ∇ *is the covariant derivative along the curve.*

THEOREM 5.5. *If a null curve* $c=c(t)$ *satisfies*

$$X:=c'(t),\quad \nabla_X\nabla_X\nabla_X X=0,\quad g(\nabla_X X, \nabla_X X)>0,$$

then $c(t)$ *is a generalized null cubic with constant curvature.*

THEOREM 5.6. *Let* M_1 *be a Lorentzian submanifold of an indefinite-Riemannian manifold* $\overline{M}_i$. *If every generalized null cubic in* M_1 *is also a generalized null cubic in* $\overline{M}_i$, *then* M_1 *is totally geodesic in* $\overline{M}_i$.

§ 6. Examples.

In this section we give examples of curves mentioned in the previous sections.

Circles [11].

On two-dimensional flat spaces, we have circles as follows:

$$c(t)=(a\cos(t/a),\ a\sin(t/a)),$$

$$c(t)=(b\sinh(t/b),\ b\cosh(t/b)),$$

$$c(t)=(b\cosh(t/b),\ b\sinh(t/b)).$$

The first is on $S^1\subset R^2$ or $H_1^1\subset R_2^2$, the second on $S_1^1\subset R_1^2$ and the third on $H^1\subset R_1^2$.

Spacelike helix on H_1^3.

By $x=(x_1, x_2, x_3, x_4)$, we denote a point in R_2^4. In R_2^4 we define a surface $V^2(\alpha)$ by

$$x_1^2-x_3^2=-\cos^2\frac{\alpha}{2},\qquad x_2^2-x_4^2=-\sin^2\frac{\alpha}{2}.$$

Then $V^2(\alpha)$ can be expressed as an isometric immersion

$$f:\ V^2(\alpha)\longrightarrow H_1^3$$

as follows

(6.1) $$x_1=\lambda\sinh\theta,\quad x_2=\mu\sinh\phi,\quad x_3=\lambda\cosh\theta,\quad x_4=\mu\cosh\phi$$

where $\lambda=\cos\alpha/2$, $\mu=\sin\alpha/2$. Then we have

$$X:=f_*(\partial/\partial\theta)=(\lambda\cosh\theta,\ 0,\ \lambda\sinh\theta,\ 0)$$

$$Y:=f_*(\partial/\partial\phi)=(0,\ \mu\cosh\phi,\ 0,\ \mu\sinh\phi)$$

and the line element of $V^2(\alpha)$ is given by

$$ds^2=\lambda^2 d\theta^2+\mu^2 d\phi^2.$$

For the tangent vectors X and Y of $V^2(\alpha)$, we have the normal vector N of $V^2(\alpha)$ as follows

$$N=(\mu\sinh\theta,\ -\lambda\sinh\phi,\ \mu\cosh\theta,\ -\lambda\cosh\phi).$$

It follows that

$$\begin{aligned}\nabla_\theta N&=dN/d\theta-g(dN/d\theta,\ x)x\\&=(\mu\cosh\theta,\ 0,\ \mu\sinh\theta,\ 0),\\ \nabla_\phi N&=dN/d\phi-g(dN/d\phi,\ x)x\\&=(0,\ -\lambda\cosh\phi,\ 0,\ -\lambda\sinh\phi),\end{aligned}$$

where ∇ is the covariant derivatve on H_1^3. Hence the eigenvalues κ_1 and κ_2 of the shape operator A of this immersion satisfy

$$\kappa_1=\mu/\lambda,\qquad \kappa_2=-\lambda/\mu.$$

REMARK. If $\alpha=\pi/2$, then $\lambda=\mu=1$. Therefore the mean curvature vector of $V^2(\pi/2)$ is zero. This coresponds to the Clifford surface of the Riemannian space form (cf. [17]).

We construct a curve $c=c(\alpha, m)$ on $V^2(\alpha)$ as follows

$$\begin{aligned}&x_1=-\sinh(t/k),\qquad x_2=-\sinh(mt/k),\\&x_3=-\cosh(t/k),\qquad x_4=-\cosh(mt/k),\qquad k=(\lambda^2+\mu^2m^2)^{1/2},\end{aligned}\tag{6.2}$$

Then $c(t)$ is a helix on H_1^3 with curvatures

$$k_1=\lambda\mu(1-m^2)/k^2,\qquad k_2=m/k^2.$$

REMARK. We can construct a helix on H_1^n. It is a helix on H_1^3 in H_1^n. This result is given by the reduction of the normal bundle of submanifolds in an indefinite-Riemannian space form [5].

Timelike helix on H_1^3.

We construct a curve $c(t)$ on H_1^3 as follows

$$c(t)=(\mu\sin(mt/k),\ \ \mu\cos(mt/k),\ \ \lambda\sin(t/k),\ \ \lambda\cos(t/k)),$$
$$k=(\lambda^2-\mu^2m^2)^{1/2},$$

where λ and μ satisfy $-\lambda^2+\mu^2=-1$. Then $c(t)$ is a timelike helix on H_1^3 with curvatures

$$k_1=\lambda\mu(1-m^2)/k^2,\qquad k_2=m/k^2.$$

Spacelike helix on S_1^3.

We define a curve $c(t)$ on S_1^3 as follows

$$c(t)=(q\cos(t/k),\ q\sin(t/k),\ r\sinh(t/k),\ r\cosh(t/k)),$$
$$k=(1+2r^2)^{1/2},$$

where $q^2-r^2=1$. This curve $c(t)$ is a spacelike helix on S_1^3 with the curvatures

$$k_1=2r\sqrt{1+r^2}/k^2, \qquad k_2=1/k^2.$$

Timelike helix on S_1^3.

We give a curve $c(t)$ on S_1^3 as follows

$$c(t)=(\mu\cos(t/k),\ \mu\sin(t/k),\ \lambda\cosh(t/k),\ \lambda\sinh(t/k)),$$
$$k=\sqrt{2\lambda^2-1},$$

where $\lambda^2+\mu^2=1$. Then $c(t)$ is a timelike helix on S_1^3 with the curvatures

$$k_1=2\lambda\mu/k^2, \qquad k_2=1/k^2.$$

Cartan framed null curve on R_1^3.

We consider a curve $c(t)$ on R_1^3, as follows

$$c(t)=(a\cosh t,\ at,\ a\sinh t).$$

This curve is a Cartan framed null curve on R_1^3. We can easily see that the curvatures k_1 and k_2, and the triple (X, Y, Z) are given as follows

$$k_1=a, \qquad k_2=1/2a,$$
$$X=(a,\ a\sinh t,\ a\cosh t),$$
$$Y=(-1/2a,\ (\sinh t)/2a,\ (\cosh t)/2a),$$
$$Z=(0,\ \cosh t,\ \sinh t),$$

respectively.

Cartan framed null curve on H_1^3.

A Cartan framed null curve on H_1^3 is defined as follows

$$c(t)=(\cosh\sqrt{2}t,\ \sqrt{2}\sinh t,\ \sinh\sqrt{2}t,\ \sqrt{2}\cosh t).$$

The curvatures k_1 and k_2, and the triple (X, Y, Z) of $c(t)$ are given as follows

$$k_1=\sqrt{2}, \qquad k_2=3/2\sqrt{2},$$
$$X=(\sqrt{2}\sinh\sqrt{2}t,\ \sqrt{2}\cosh t,\ \sqrt{2}\cosh\sqrt{2}t,\ \sqrt{2}\sinh t),$$
$$Y=\frac{-1}{2\sqrt{2}}(-\sinh\sqrt{2}t,\ \cosh t,\ -\cosh\sqrt{2}t,\ \sinh t),$$

$$Z=(\sqrt{2}\cosh\sqrt{2}t,\ \sinh t,\ \sqrt{2}\sinh\sqrt{2}t,\ \cosh t),$$

respectively.

Generalized null cubic on R_1^3 [1], [7].

On R_1^3, the curve

$$c(t)=\left(\frac{4}{3}t^3-t,\ 2t^2,\ \frac{4}{3}t^3+t\right)$$

is an example of the generalized null cubic.

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