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GRAPHS AND FINITE DISTRIBUTIVE PARTIAL LATTICES

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Abstract

The Hasse diagram graph of a finite distributive partial lattice is characterized by means of prime convexes.

Median graphs contitute a well known and widely studied class of graphs; see for example the papers [1] and [2] and the references therein. They constitute a subclass of the Hasse diagram graphs of distributive partial lattices. In this paper we give a characterization for the Hasse diagram graphs G of finite distributive partial lattices by means of prime convexes of G. This characterization generalizes that of Mulder and Schrijver for median graphs reprinted in [1, Theorem 2.2].

A meetsemilattice S is a partial lattice if for any two elements a, b having an upper bound in S also the element $a\vee b$ belongs to S. Clearly every finite meetsemilattice is a partial lattice. A partial lattice S is distributive if its every subset $(k]=\{s|s\leq k\}$ is a distributive lattice. A finite distributive partial lattice S can be embedded in the distributive lattice $I(S)$ of ideals of S, where the join of two ideals I and J is $I\vee J=\{s|\leq i\vee j, i\in I$ and $j\in J\}$. By using this lattice we see that one shortest path joining two points a and b of the Hasse diagram graph S contains the point $a\wedge b$, and if $a>b$, then every point c, $a\geq c\geq b$, is on some shortest a-b path.

The graphs $G=(V, X)$ considered here are finite, connected and undirected without loops and multiple lines. The points of G constitute the set V and its lines the set X. A pointset $A\subset V$ of G is called a *convex* if A contains all points of any shortest a-b path (of any a-b geodesic) for every two points $a, b\in A$. The intersection of two convexes is also a convex and thus the least convex containing a given pointset B of G is $\cap\{C|C$ is a convex and $B\subset C\}$. This set is briefly denoted by $\langle B\rangle$. A convex $A\neq V$ is called *prime* if the set $V\backslash A$ is also a convex. The sets ϕ and V are trivial prime convexes. A graph G has the prime convex intersection property (is a *prime convex intersection graph*) if its every

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convex A is the intersection of all prime convexes containing A. By [1, Theorem 2.2], every median graph is a prime convex intersection graph. The class of prime convex intersection graphs is rather wide: for example every complete graph belongs to this class.

Let $a, b, c \in V$. A point t satisfying the distance conditions $d(a, b)=d(a, t)+d(t, b)$, $d(b, c)=d(b, t)+d(t, c)$ and $d(a, c)=d(a, t)+d(t, c)$ is a *median* of the points a, b and c. A graph is a median graph if its all three points have exactly one median.

If A is a subset of a set U, then $\bar{A}=U \backslash A$ is its complement in U.

When proving the main theorem of this note we need two auxiliary results which we prove first.

LEMMA 1. *A connected graph G is a prime convex intersection graph if and only if for any noempty convex A and any point x, $x \notin A$, there is a prime convex P separating A and x, i.e. $A \subset P$ and $x \in \bar{P}$.*

PROOF. If G is a prime convex intersection graph, A its nonempty convex and x its point such that $x \notin A$, there is a prime convex P separating A and x, because otherwise A connot be represented as an intersection of prime convexes of G. Conversely, if there is a prime convex separating any convex A and any point x of the lemma then G is a prime convex intersection graph. Indeed, if there is a nonempty convex A which cannot be expressed as the intersection of prime convexes, then the intersection contains a point x not belonging to A. By assumption there is a prime convex P searating A and x, and thus the intersection cannot contain the point x, and the lemma follows.

LEMMA 2. *The convex $\langle a, b\rangle$ of a prime convex intersection graph G consists of points on a–b geodesics for every pair $a, b \in V$.*

PROOF. Let a and b be a pair of points such that the convex $\langle a, b\rangle$ contains at least one point v which is not on any a–b geodesic. This implies the existence of two points x and z, x is on an a–b geodesic and z is on another a–b geodesic, such that no point $x_1, \cdots, x_m$ of an x–z geodesic $x=x_0, x_1, \cdots, x_m, x_{m+1}=z$ is on any a–b geodesic. Clearly a and b can be chosen such that every convex $\langle u, w\rangle$ with $d(u, w)<d(a, b)$ is the set of all points on u–w geodesics. We may assume further that $d(a, b) \geq d(x, b)$, $d(z, b) \geq d(x, b)$, and that x and z are as near to b as possible. Let us consider the point x_1. Because $d(a, x)<d(a, b)$, the convex $\langle a, x\rangle$ consists of points on a–x geodesics, and thus $x_1 \notin \langle a, x\rangle$. By Lemma 1, the prime convex intersection property of G implies now the existence of a prime

convex P separating $\langle a, x\rangle$ and x_1: $\langle a, x\rangle\subset P$ and $x_1\in\bar{P}$. Because $x_1\in\langle a, b\rangle$, we have $x_1, b\in\bar{P}$. Let $x=b_0, b_1, b_2, \cdots, b_{k-1}, b_k=b$ be the points of an x-b geodesic. Because x and z are as near to b as possible, $d(z, b)\geq d(x, b)$ and $d(x_1, z)\geq d(x_1, x)=1$, them a b_i-x_1 geodesic goes over $x, i=1,\cdots, k$. This implies that there is no prime convex separating $\langle a, x\rangle$ and x_1, which is a contradiction. Thus the assumption is false and the convex $\langle a, b\rangle$ consists of points on a-b geodesics for every pair $a, b\in V$, and the lemma follows.

Now we can present the characterization theorem of this note.

THEOREM. *A connected graph G is isomorphic to the Hasse diagram graph of a finite distributive partial lattice if and only if the following two conditions hold:*

(i) *G is a prime convex intersection graph;*

(ii) *$\cap\{\bar{P}|P\in\mathcal{K}\}\neq\phi$ or $\mathcal{K}=\phi$ for the collection $\mathcal{K}$ of all nontrivial prime convexes in G having the following property: if $P_1\in\mathcal{K}$, there are $P_2, P_3, \cdots, P_n\in\mathcal{K}(n\geq 3)$ such that $P_i\cap P_j\neq\phi$ and $P_1\cap P_2\cap\cdots\cap P_n=\phi$.*

PROOF. Mulder and Schrijver proved that a connected graph G is a median graph if and only if G is a prime convex intersection graph and its prime convexes satisfy the Helly property [1, Theorem 2.2]. The condition (ii) above is nothing but a weakened Helly property for prime convexes of G.

Assume first that G is the Hasse diagram graph of a finite distributive partial lattice S.

(i) Let $x\in S$. The element corresponding x in the ideal lattice $I(S)$ of S is $(x]$. Because I(S) is distributive, one $(z]$-$(x]$ geodesic goes over the element $(z]\wedge(x]=(z\wedge x]$. Thus, if the distance $d((z], (x])=n$ in $I(S)$, then $d(z, x)=n$ in S, because the z-$z\wedge x$-x path always belongs to S. In particular, if C is a convex of the Hasse diagram graph of $I(S)$, then the set $\{x|(x]\in C$ in $I(S)\}=C_s$ is a convex in S. Moreover, if C is a prime convex in $I(S)$, then C_s is a prime convex in S. Let A be a nonempty convex of G, x a point of G with $x\notin A$ and A^* the least convex of the graph $G(I(S))$ of $I(S)$ with the property: $(z]\in A^*$ in $G(I(S))$ if $z\in A$ in G. Clearly, $(x]\notin A^*$ in $G(I(S))$. Because $I(S)$ is a distributive lattice, the graph $G(I(S))$ is a median graph and has thus the prime convex intersection property. Hence there is a prime convex C in G(I(S)) separating A^* and $(x]$, which implies that the prime convex C_s separates A and x in G. By Lemma 1, this proves that G has the prime convex intersection property, and thus (i) holds for G.

(ii) Assume that the collection $\mathcal{K}$ of the theorem is nonempty. We prove that least element 0 of S belongs to $\cap\{\bar{P}|P\in\mathcal{K}\}$, from which the assertion follows. In fact, we prove the assertion for $n=3$; the proofs are the same for other values of n and hence they are omitted. Let $P_1, P_2, P_3\in\mathcal{K}$ be three prime convexes of G such that $P_i\cap P_j\neq\phi$ and $P_1\cap P_2\cap P_3=\phi$. The sets $P_1\cap P_2, P_1\cap P_3$ and $P_2\cap P_3$ are convexes of G, and because S is finite, every one of them has a least element, and let them be $a\in P_1\cap P_2, b\in P_1\cap P_3$ and $c\in P_2\cap P_3$. Assume that $0\notin\cap\{\bar{P}|P\in\mathcal{K}\}$, which means that 0 belongs to at least one set of $\mathcal{K}$, say to P_1. Because $0, a, b\in P_1$, then also $a\wedge b\wedge c\in P_1$. The relation $a, c\in P_2$ implies that $a\wedge c\in P_2$. On the other hand, $a\geq a\wedge c\geq a\wedge b\wedge c$, where $a, a\wedge b\wedge c\in P_1$, and thus $a\wedge c\in P_1$. Accordingly, $a\wedge c\in P_1\cap P_2$, and because a is the least element in this convex, $a=a\wedge c\geq c$. Similarly we see that $b\leq c$. Because there is an upper bound c for a and b, the element $a\vee b$ exists, and as well known, an a-b geodesic goes over $a\vee b$ in the Hasse diagram graph of a finite distributive lattice. Thus $a\vee b\in P_1$. Because $c, b\in P_3$ and $c\geq a\vee b$, the element $a\vee b$ belongs to P_3, and analogously we see that $a\vee b\in P_2$. Now, $a\vee b\in P_1\cap P_2\cap P_3$, which intersection should be empty, and hence the assumption $0\notin\cap\{\bar{P}|P\in\mathcal{K}\}$ must be false. This proves the property (ii).

Assume conversely that G is a graph satisfying the properties (i) and (ii) of the theorem. We choose an arbitrary point from the set $\cap\{\bar{P}|P\in\mathcal{K}\}$ and denote it by h. Let a and b be two arbitrary points in V and let us consider the intersection $\langle h, a\rangle\cap\langle h, b\rangle\cap\langle a, b\rangle$. Because the convexes $\langle h, a\rangle$, $\langle h, b\rangle$ and $\langle a, b\rangle$ are the intersections of corresponding prime convexes, we can substitute the intersection $\langle h, a\rangle\cap\langle h, b\rangle\cap\langle a, b\rangle$ by the expression

$$(\cap\{P_i|P_i \text{ is a prime convex and } \langle h,a\rangle\subset P_i\})\cap(\cap\{U_j|U_j \text{ is a prime convex and } \langle h,b\rangle\subset U_j\})\cap(\cap\{W_k|W_k \text{ is a prime convex and } \langle a,b\rangle\subset W_k\}).$$

Now, $P_i\cap W_k$, $P_i\cap U_j$, $U_j\cap W_k\neq\phi$, and if $\langle h, a\rangle\cap\langle h, b\rangle\cap\langle a, b\rangle=\phi$, then $h\notin\cap\{\bar{P}|P\in\mathcal{K}\}$, which is a contradiction. Thus $\langle h, a\rangle\cap\langle h, b\rangle\cap\langle a, b\rangle\neq\phi$. Moreover, this intersection contains exactly one element. This can be seen as follows: Every prime convex P of G (or its complement $\bar{P}$) contains at least two of the points a, b, h. If the intersection $\langle h, a\rangle\cap\langle h, a\rangle\cap\langle a, b\rangle$ contains two disjoint points x and y, then every P (or $\bar{P}$) contains both x and y, and the convex x cannot be separated from the point y, which contradicts (i) by Lemma 1. Thus $\langle h, a\rangle\cap\langle h, b\rangle\cap\langle a, b\rangle=\{d\}$. According to Lemma 2, a convex $\langle x, z\rangle$ consists of points on x-z geodesics. Thus the relation $\{d\}=\langle h, a\rangle\cap\langle h, b\rangle\cap\langle a, b\rangle$ shows that every triple h, a, b, where a and b are arbitrary points of G, has a unique median.

We order now the points of V as follows:

$$a \leq b \Longleftrightarrow a \text{ is on a } b\text{–}h \text{ geodesic} \Longleftrightarrow a \in \langle h, b \rangle.$$

This definition suggests us to define the meet $a \wedge b$ as the unique median d of the points a, b and h. Assume that c is a point such that $c \in \langle h, a \rangle \cap \langle h, a \rangle$ and $c \notin \langle h, d \rangle$. The intersection $\langle h, d \rangle \cap \langle c, b \rangle$ is empty, because if x belongs to this intersection, then the d–x–c–h path is a d–h geodesic and $c \in \langle d, h \rangle$, which is a contradiction. There is a prime convex P separating the convexes $\langle h, d \rangle$ and $\langle c, b \rangle : \langle h, d \rangle \subset \overline{P}$ and $\langle c, b \rangle \subset P$. Indeed, as seen above, the points h, d and c have a median u which is on a d–h geodesic and thus belongs to the convex $\langle d, h \rangle$. By the prime convex intersection property of G and Lemma 1, there is a prime convex P separating $\langle c, b \rangle$ and u ($\langle c, b \rangle \subset P$ and $u \in \overline{P}$). If now h or d belongs to P, then also u belongs to P because u is on a c–h geodesic as well as on a c–d geodesic. Thus $h, d \in \overline{P}$, whence also $\langle h, d \rangle \subset \overline{P}$. If $a \in \overline{P}$, then $c \in \overline{P}$ because it is on an a–h geodesic, and thus a must belong to P. Because d is on an a–b geodesic, the relation $a, b \in P$ implies a contradiction, and hence $c \in \langle h, d \rangle$. This proves that d is a maximum lower bound of a and b, and thus the order defined on V is a meetsemilattice order. Accordingly, V is a meetsemilattice with h as the least element. Because V is finite, it is a partial lattice. The Hasse diagram graph of V is isomorphic to G: When a line belongs to an x–h geodesic, there is nothing to prove, and hence we assume that the line (a, b) of G does not belong to any x–h geodesic. This is possible only if $d(a, h) = d(b, h)$. But then a, b and h have no median, which is absurd, and the ismorphism follows.

It remains to show that every set $(k] = \{v | v \in V \text{ and } v \leq k\}$ is a distributive lattice. By the order definition above, $\langle h, k \rangle = (k]$. Every convex A of a prime convex intersection graph induces a prime convex intersection graph. By Mulder and Schrijver [1, Theorem 2.2], a prime convex intersection graph $\langle h, k \rangle$ is a median graph (and then the Hasse diagram graph of a distributive lattice with h as the least element and k as the greatest element by [1, Theorem 3.1]) if its prime convexes needed to separate its convexes satisfy the Helly property. The prime convexes needed to separate the convexes of $\langle h, k \rangle$ are obtained from the prime convexes of $\mathscr{K}$ by intersecting them with $\langle h, k \rangle$. Let now $P_1, P_2 \cdots, P_m$ be prime convexes of $\mathscr{K}$ such that $P_i \cap P_j \cap \langle h, k \rangle \neq \phi$. We denote the corresponding prime convexes of $\langle h, k \rangle$ by $P_i^0 = P_i \cap \langle h, k \rangle$. By Lemma 2, the convex $\langle h, k \rangle$ consists of points on h–k geodesics in G. If $h, k \notin P_i^0$, then P_i^0 is not prime because its every point is on some h–k geodesic. Hence either h or k belongs to P_i^0. The relation $h \in P_i^0$ contradicts the property $h \in \cap \{\overline{P} | P \in \mathscr{K}\}$, and thus $k \in P_i^0$, and this relation holds for every $i, i = 1, \cdots, m$. Then $k \in P_1^0 \cap P_2^0 \cap \cdots \cap P_m^0$, and the Helly property of the prime convexes needed to separate the convexes of $\langle h,$

$k\rangle$ follows. This proves the distributivity of $\langle h, k\rangle = (k]$, and thus G is the Hasse diagram graph of a finite distributive partial lattice.

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