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REAL HYPERSURFACES WITH PARALLEL RICCI TENSOR OF A COMPLEX SPACE FORM

By

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Introduction.

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C_n or a complex hyperbolic space H_nC, according as $c>0$, $c=0$ or $c<0$. The induced almost contact metric structure of real hypersurfaces of $M_n(c)$ will be denoted by (J, g, P).

Many subjects for real hypersurfaces of a complex projective space have been studied by Cecil and Ryan [1], Kimura [8], [9], Kon [10], Maeda [13], Okumura [15], Takagi [16], [17], [18] and so on. One of those, done by Kimura, asserted the following interesting result.

THEOREM K ([9]). *There are no real hypersurfaces of P_nC with parallel Ricci tensor on which the structure vector P is principal.*

On the other hand, real hypersurfaces of a complex hyperbolic space H_nC have also been investigated from different points of view and there are some studies by Chen [2], Chen, Ludden and Montiel [3], Montiel [12] and Montiel and Romero [14]. In particular, it is proved in [12] the following fact:

THEOREM M. *There are no Einstein real hypersurfaces in H_nC.*

A Riemannian curvature tensor is said to be *harmonic* if the Ricci tensor S is of Codazzi type. Although the concept is closely related to a parallel Ricci tensor, it was shown by Derdziński [4] and Gray [5] that it is essentially weaker than the latter one. Nakagawa, Umehara and the present author [6] proved that there exist infinitely many hypersurfaces with harmonic curvature and non-Ricci parallel in a Riemannian space form.

Recently, some studies about the non-existance for real hypersurfaces with

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harmonic curvature of P_nC (resp. H_nC) have been made by Kwon and Nakagawa [11] (resp. Kim [7]). Their results are following:

THEOREM KNK. *There are no real hypersurfaces with harmonic curvature of $M_n(c)$, $c\neq 0$ on which the structure vector is principal.*

The main purpose of the present paper is to improve Theorem K and Theorem KNK, and study also real hypersurfaces with harmonic curvature of a complex space form $M_n(c)$, $c\neq 0$. We shall prove the followings:

THEOREM A. *There are no real hypersurfaces with parallel Ricci tensor of a complex space form $M_n(c)$, $c\neq 0$.*

THHOREM B. *There are no real hypersurfaces with harmonic curvature of $M_n(c)$, $c\neq 0$ satisfying one of the following conditions:*

(1) *P is an eigenvector corresponding to the Ricci tensor,* (2) *the number of Ricci curvatures does not exceed* 2.

1. Preliminaries.

We begin by recalling fundamental formulas on real hypersurfaces of a Kaehlerian manifold. Let N be a real $2n$-dimensional Kaehlerian manifold equipped with a parallel almost complex structure F and a Riemannian metric tensor G which is F-Hermitian, and covered by a system of coordinate neighborhoods $\{U; x^A\}$. Let M be a real hypersurface of N covered by a system of coordinate neighborhoods $\{V; y^h\}$ and immersed isometrically in N by the immersion $i: M\to N$. Throughout the present paper the following convention on the range of indices are used, unless otherwise stated:

$$A, B, \cdots=1, 2, \cdots, 2n;\ i, j, \cdots=1, 2, \cdots, 2n-1.$$

The summation convention will be used with respect to those system of indices. When the argument is local, M need not be distinguished from $i(M)$. Thus, for simplicity, a point p in M may be identified with the point $i(p)$ and a tangent vector X at p may also be identified with the tangent vector $i_*(X)$ at $i(p)$ via the differential i_* of i. We represent the immersion i locally by $x^A=x^A(y^h)$ and $B_j=(B_j^A)$ are also $(2n-1)$-linearly independent local tangent vectors of M, where $B_j^A=\partial_j x^A$ and $\partial_j=\partial/\partial y^j$. A unit normal C to M may then be chosen. The induced Riemannian metric g with components g_{ji} on M is given by $g_{ji}=G(B_j, B_i)$ because the immersion is isometric.

For the unit normal C to M, the following representations are obtained in each coordinate neighborhood:

$$FB_i=J_i^h B_h+p_i C, \quad FC=-p^i B_i, \tag{1.1}$$

where we have put $J_{ji}=G(FB_j, B_i)$ and $p_i=G(FB_i, C)$, p^h being components of a vector field P associated with P_i and $J_{ji}=J_j^r g_{ri}$. By the properties of the almost Hermitian structure F, it is clear that J_{ji} is skew-symmetric. A tensor field of type (1, 1) with components J_i^h will be denoted by J. By the properties of the almost complex structure F, the following relations are then given:

$$J_i^r J_r^h=-\delta_i^h+p_i p^h, \quad p^r J_r^h=0, \quad p_r J_i^r=0, \quad p_i p^i=1,$$

that is, the aggregate (J, g, P) defines an almost contact metric structure. Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation formed with g_{ji}, the equations of Gauss and Weingarten for M are respectively obtained:

$$\nabla_j B_i=h_{ji} C, \quad \nabla_j C=-h_j^r B_r, \tag{1.2}$$

where h_{ji} are components of a second fundamental form σ, $A=(h_j^k)$ which is related by $h_{ji}=h_j^r g_{ri}$ being the shape operator derived from C. We notice hear that h_{ji} is symmetric. By means of (1.1) and (1.2) the covariant derivatives of the structure tensors are yielded:

$$\nabla_j J_{ih}=-h_{ji} p_h+h_{jh} p_i, \quad \nabla_j p_i=-h_{jr} J_i^r. \tag{1.3}$$

In the sequel, the ambient Kaehlerian manifold N is assumed to be of constant holomorphic sectional curvature c and real dimension $2n$, which is called a complex space form and denoted by $M_n(c)$. Then the components of the curvature tensor K of $M_n(c)$ take the following form:

$$K_{DCBA}=\frac{c}{4}(G_{DA}G_{CB}-G_{DB}G_{CA}+F_{DA}F_{CB}-F_{DB}F_{CA}-2F_{DC}F_{BA}).$$

Thus, the equations of Gauss and Codazzi for M are respectively obtained:

$$R_{kjih}=\frac{c}{4}(g_{kh}g_{ji}-g_{jh}g_{ki}+J_{kh}J_{ji}-J_{jh}J_{ki}-2J_{kj}J_{ih})+h_{kh}h_{ji}-h_{jh}h_{ki}, \tag{1.4}$$

$$\nabla_k h_{ji}-\nabla_j h_{ki}=\frac{c}{4}(p_k J_{ji}-p_j J_{ki}-2p_i J_{kj}), \tag{1.5}$$

where R_{kjih} are the components of the Riemannian curvature tensor R of M.

To be able to write our formulas in a convention form, the components X_{ji}^m of a tensor field X^m and a function X_m on M for any integer $m(\geqq 2)$ are introduced as follows:

$$X_{ji}^m=X_{ji_1}X_{i_2}^{i_1}\cdots X_i^{i_{m-1}}, \quad X_m=\sum_i X_{ii}^m.$$

In our notation, the Gauss equation (1.4) implies

$$S_{ji}=\frac{c}{4}\{(2n+1)g_{ji}-3p_jp_i\}+hh_{ji}-h^2_{ji}, \tag{1.6}$$

where S_{ji} denotes components of the Ricci tensor S of M, and h the trace of the shape operator A.

REMARK 1. We notice here that the structure vector P cannot be parallel provided that $c\neq0$. In fact, if P is parallel along M, then the second equation of (1.3) becomes $h_{jr}J^r_i=0$. Thus, it is not hard to see that $h_{ji}=hp_jp_j$ because of properties of the almost contact metric structure. Hence it follows that $\nabla_kh_{ji}=(\nabla_kh)p_jp_i$, which together with (1.5) give

$$\frac{c}{4}(p_kJ_{ji}-p_jJ_{ki}-2p_iJ_{kj})=\{(\nabla_kh)p_j-(\nabla_jh)p_k\}p_i.$$

By transvecting p^iJ^{kj}, we have $c(n-1)=0$. Thus the assumption $c\neq0$ will produce a contradiction.

2. Real hypersurfaces with harmonic curvature.

Let M be a real hypersurface with harmonic curvature of a complex space form $M_n(c)$, $c\neq0$, that is, the Ricci tensor S satisfies $\nabla_kS_{ji}=\nabla_jS_{ki}$. Then, we easily, using the second Bianchi identity, see that the scalar curvature r of M is constant everywhere. Moreover, the Ricci formula for S_{ji} gives rise to

$$\nabla_m\nabla_kS_{ji}=\nabla_j\nabla_iS_{mk}-R_{mjkr}S^r_i-R_{mjir}S^r_k,$$

which together with the first Bianchi identity and the Ricci formula imply that

$$R_{mkir}S^r_j+R_{kjir}S^r_m+R_{jmir}S^r_k=0, \tag{2.1}$$

where $S^h_j=S_{ji}g^{ih}$, g^{ji} being the contravariant components of g_{ji}. Therefore, it follows that

$$J^{kj}R_{kjih}S^h_m+2J^{rk}R_{kmih}S^h_r=0$$

and hence, in consequence of (1.4),

$$\left(-n+\frac{3}{2}\right)cS_{jr}J^r_i+\frac{c}{2}\{S_{ir}J^r_j-(r-A_1)J_{ji}-p_i(S_{rt}p^r)J^t_j-2p_j(S_{tr}p^r)J^t_i\}$$
$$+2h_{tr}h_{is}J^{rs}S^t_j-2h_{jt}h_{ir}J^{sr}S^t_s=0,$$

where we have put $A_1=S_{ji}p^jp^i$. By the way, the last two terms of this reduces to $-\frac{3}{2}cp_j(h_{rt}p^t)h_{is}J^{rs}$ by virtue of (1.6). Accordingly we have

$$S_{ir}J^r_j-(2n-3)S_{jr}J^r_i-(r-A_1)J_{ji}-S_{tr}p^r(p_iJ^t_j+2p_jJ^t_i)-3h_{rt}p^th_{is}J^{rs}p_j=0$$

because of the fact that $c\neq 0$ is assumed, which implies

$$3h_{rt}p^t h_{is}J^{rs}+(2n-1)S_{rt}p^t J_i^r=0.$$

Thus, the last equation can be written as

$$(2.2)\qquad (2n-3)\{S_{jr}J_i^r-(S_{tr}p^r)p_jJ_i^t\}-S_{ir}J_j^r+(S_{rt}p^t)p_iJ_j^r+(r-A_1)J_{ji}=0,$$

from which, taking the symmetric parts,

$$S_{jr}J_i^r+S_{ir}J_j^r=S_{tr}p^r(p_jJ_i^t+p_iJ_j^t).$$

Hence, the relationship (2.2) turns out to be

$$2(n-1)\{S_{jr}J_i^r-(S_{tr}p^r)p_jJ_i^t\}+(r-A_1)J_{ji}=0.$$

Transforming this by J_k^i and utilizing properties of the almost contact metric structure, it is reduced to

$$(2.3)\qquad 2(n-1)\{S_{ji}-p_iS_{jr}p^r-p_jS_{ir}p^r\}-(r-A_1)g_{ji}+\{r+(2n-3)A_1\}p_jp_i=0,$$

which implies immediately that

$$(2.4)\qquad 2(n-1)(S_2-2A_2+A_1^2)=(r-A_1)^2,$$

where $A_2=S_{ji}^2p^jp^i$.

PROPOSITION 2.1. *Let M be a real hypersurface with harmonic curvature of a complex space form $M_n(c)$, $c\neq 0$. If the structure vector P is an eigenvector of the Ricci tensor, namely, if*

$$(2.5)\qquad S_{jr}p^r=A_1p_j,$$

then M is Ricci parallel.

PROOF. By means of (2.5), the relationship (2.3) reduces to

$$(2.6)\qquad 2(n-1)S_{ji}-(r-A_1)g_{ji}+\{r-(2n-1)A_1\}p_jp_i=0,$$

which implies

$$(2.7)\qquad 2(n-1)S_{ji}^2-\{r+(2n-3)A_1\}S_{ji}+A_1(r-A_1)g_{ji}=0.$$

Differentiating (2.6) covariantly, we find

$$(2.8)\qquad 2(n-1)\nabla_kS_{ji}+(\nabla_kA_1)g_{ji}-(2n-1)(\nabla_kA_1)p_jp_i$$
$$+\{r-(2n-1)A_1\}\{(\nabla_kp_j)p_i+(\nabla_kp_i)p_j\}=0$$

because the scalar curvature r is constant. Since the Ricci tensor S is of Codazzi type, it is seen that

(2.9) $$(\nabla_k A_1)g_{ji}-(\nabla_j A_1)g_{ki}-(2n-1)\{(\nabla_k A_1)p_j-(\nabla_j A_1)p_k\}p_i + \{r-(2n-1)A_1\}\{(\nabla_k p_j-\nabla_j p_k)p_i+(\nabla_k p_i)p_j-(\nabla_j p_i)p_k\}=0.$$

If we transvect this with g^{ji}, then we obtain

$$\nabla_k A_1-(2n-1)(p^r\nabla_r A_1)p_k+\{r-(2n-1)A_1\}p^r\nabla_r p_k=0$$

and hence $p^r\nabla_r A_1=0$. Thus, it follows that $\nabla_k A_1+\{r-(2n-1)A_1\}p^r\nabla_r p_k=0$. Transvecting (2.9) with $p^j p^i$ and taking account of the last equation, we can verify that A_1 is constant everywhere. Therefore, by differentiating (2.7) covariantly, we find

$$2(n-1)\nabla_k S^2_{ji}-\{r+(2n-3)A_1\}\nabla_k S_{ji}=0,$$

which shows that S^2_{ji} is of Codazzi type. Thus, the Ricci tensor S is parallel because the scalar curvature of M is constant (see Umehara, Theorem 1.3 of [19]). This completes the proof of Proposition 2.1.

REMARK 2. If the structure vector P is principal, that is, $h_{jr}p^r=\alpha p_j$, we can see from (1.6) that P is the eigenvector of the Ricci tensor and hence the Ricci tensor is parallel.

Now, transforming (2.3) by S^i_k, we obtain

(2.10) $$2(n-1)\{S^2_{jk}-(S_{kt}p^t)(S_{jr}p^r)-p_j S^2_{kr}p^r\}-(r-A_1)S_{jk} + \{r+(2n-3)A_1\}p_j S_{kr}p^r=0,$$

which enables us to obtain

$$(2(n-1)S^2_{kr}p^r-\{r+(2n-3)A_1\}S_{kr}p^r)p_j-(2(n-1)S^2_{jr}p^r -\{r+(2n-3)A_1\}S_{jr}p^r)p_k=0.$$

Thus, it is seen that

(2.11) $$2(n-1)S^2_{kr}p^r-\{r+(2n-3)A_1\}S_{kr}p^r=(2(n-1)A_2-A_1\{r+(2n-3)A_1\})p_k.$$

Making use of the last equation, (2.10) turns out to be

(2.12) $$2(n-1)\{S^2_{jk}-(S_{jt}p^t)(S_{kr}p^r)\}-(r-A_1)S_{jk}+\mu p_j p_k=0,$$

where $\mu=A_1(r-A_1)-2(n-1)(A_2-A_1^2)$. Transforming (2.12) by S^k_i and utilizing (2.3), (2.11) and (2.12), we get

(2.13) $$4(n-1)^2 S^3_{ji}-4(n-1)\{r+(n-2)A_1\}S^2_{ji} + \{(r-A_1)(r+(4n-5)A_1)-4(n-1)^2(A_2-A_1^2)\}S_{ji}-\mu(r-A_1)g_{ji}=0,$$

or, equivalently

$$\left(S_j^r-\frac{r-A_1}{2(n-1)}\delta_j^r\right)\{2(n-1)S_{ir}^2-\lambda S_{ir}+\mu g_{ir}\}=0,$$

where we have put $\lambda=r+(2n-3)A_1$. Thus the minimal polynomial for S tells us that there exist at most three Ricci curvatures of M: $(r-A_1)/2(n-1)$, $(\lambda\pm\sqrt{D})/4(n-1)$, where

$$D=\{r-(2n-1)A_1\}^2+16(n-1)^2(A_2-A_1^2). \tag{2.14}$$

And their multiplicities are respectively denoted by $2n-1-l_1-l_2$, l_1 and l_2. Therefore the scalar curvature r of M satisfies

$$(l_1+l_2-2)\{r-(2n-1)A_1\}=\sqrt{D}(l_1-l_2). \tag{2.15}$$

We also have

$$4(n-1)^2S_2=\frac{1}{4}(\lambda^2+D)(l_1+l_2)+\frac{1}{2}\lambda\sqrt{D}(l_1-l_2)+(r-A_1)^2(2n-1-l_1-l_2),$$

which together with (2.4), (2.14) and (2.15) imply that

$$(A_2-A_1^2)(l_1+l_2-2)=0. \tag{2.16}$$

Now, suppose that the number of distinct Ricci curvatures does not exceed 2. Then we can easily see that $A_2=A_1^2$ because of (2.15). Thus, it follows that $S_{jr}p^r=A_1p_j$.

According to Proposition 2.1, we have

PROPOSITION 2.2. *Let M be a real hypersurface with harmonic curvature of a complex space form $M_n(c)$, $c\neq 0$. Then the number of distinct Ricci curvature is at most* 3. *In particular, it does not exceed* 2, *then M is Ricci parallel.*

3. Real hypersurfaces with parallel Ricci tensor.

In this section we devote to investigate the real hypersurfaces with parallel Ricci tensor of a complex space form $M_n(c)$, $c\neq 0$. Since the Ricci tensor S is assumed to be parallel, we have (2.13) and hence

$$4(n-1)^2S_3-4(n-1)rS_2-4(n-1)(n-2)S_2A_1+r(r-A_1)^2+4(n-1)rA_1(r-A_1)$$
$$+2(n-1)r(A_2-A_1^2)-2(n-1)(2n-1)A_1(A_2-A_1^2)-(2n-1)A_1(r-A_1)^2=0,$$

which together with (2.4) yield

$$\frac{1}{2(n-1)}(r-A_1)^3+2(n-1)A_1^3+3rA_1(r-A_1)-3(2n-3)S_2A_1-3rS_2$$
$$+4(n-1)S_3=0.$$

Thus, A_1 is a root of the cubic equation with constant coefficients because S_i is constant for each number i. Accordingly A_1 is constant. By the definition of A_1, it is not hard to see that

$$S_{ir}p^i\nabla_k p^r=0 \tag{3.1}$$

because the Ricci tensor is parallel. By differentiating (2.3) covariantly, we find

$$\begin{aligned}2(n-1)\{(\nabla_k p_i)S_{jr}p^r+(\nabla_k p_j)S_{ir}p^r+p_iS_{jr}\nabla_k p^r+p_jS_{ir}\nabla_k p^r\}\\ =\{r+(2n-3)A_1\}\{(\nabla_k p_j)p_i+(\nabla_k p_i)p_j\}.\end{aligned} \tag{3.2}$$

If we apply p^j to this and sum for j, and make use of (3.1), we obtain

$$2(n-1)S_{ir}\nabla_k p^r=(r-A_1)\nabla_k p_i.$$

Thus, (3.2) turns out to be

$$(\nabla_k p_i)S_{jr}p^r+(\nabla_k p_j)S_{ir}p^r=A_1(p_i\nabla_k p_j+p_j\nabla_k p_i).$$

Transvecting the last equation with $S_t^j p^t$ and utilizing (3.1), we get

$$(A_2-A_1^2)\nabla_k p_i=0. \tag{3.3}$$

By means of Remark 1, it follows that $A_2=A_1^2$ and hence $S_{jr}p^r=A_1p_j$. Therefore, the relationship (2.3) is reduced to

$$2(n-1)S_{ji}=(r-A_1)g_{ji}-\{r-(2n-1)A_1\}p_jp_i.$$

The Ricci tensor of M being parallel, it is seen that

$$\{r-(2n-1)A_1\}(p_i\nabla_k p_j+p_j\nabla_k p_i)=0$$

and hence $r-(2n-1)A_1=0$. Thus, M is Einstein. But, there are no Einstein real hypersurfaces of $M_n(c)$, $c\neq 0$ because of Theorem K and Theorem M (see also [10]). Hence Theorem A is completely proved.

PROOF OF THEOREM B. Due to Theorem A, Proposition 2.1 and Proposition 2.2.

By means of (2.16), Theorem A and Proposition 2.2, it is clear that $l_1=l_2=1$. Therefore we can state the following fact:

REMARK 3. Let M be a real hypersurface with harmonic curvature of $M_n(c)$, $c\neq 0$. Then M has three distinct Ricci curvatures: $(r-A_1)/2(n-1)$, $(\lambda+\sqrt{D})/4(n-1)$, $(\lambda-\sqrt{D})/4(n-1)$ with multiplicities $2n-3$, 1, 1 respectively.

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