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PROPERTIES OF NORMAL EMBEDDINGS CONCERNING STRONG SHAPE THEORY, II

By

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Abstract. We show that any topological pair with normally embedded subspace has the strong shape of a pair, such that the inclusion map of the subspace into the total space is a cofibration. Furthermore we prove that a strong shape morphism of pairs is a strong shape equivalence if and only if it operates as strong shape equivalence of the total spaces and of the subspaces considered seperately.

Subject classification: Shape, strong shape, normal embeddings, topological pairs, ANR-spaces. 54C56, 55P55.

1. Preliminaries.

This paper is a continuation of [2] dealing with strong shape theory of pairs as defined by Lisica and Mardešić in [4]. One major property of strong shape theory is the existence of a functor $T:\mathbf{ssh}^2\to\mathbf{HTop}^2$ right adjoint to the strong shape functor $\eta:\mathbf{HTop}^2\to\mathbf{ssh}^2$. The authors do not state this explicitly, but in case of spaces it is a consequence of [5, theorem 6, p. 371]. We only give a brief description of T:

To a given pair (X, A) we choose a resolution in ANR-pairs $\{f_\lambda\}:(X, A)\to\{g_\lambda^\mu:(P_\mu, Q_\mu)\to(P_\lambda, Q_\lambda)\mid\mu\geqq\lambda\in\Lambda\}$ and form the simplicial complex $\mathcal{K}$, whose vertices are the indices $\lambda\in\Lambda$, and whose simplices are the finite, linearly ordered subsets of Λ. For a *finite* subcomplex $\mathcal{L}\subseteqq\mathcal{K}$ we denote by $\mathcal{L}_\lambda\subseteqq\mathcal{L}$ the full subcomplex spanned by all vertices μ with $\mu\leqq\lambda$ and consider the subspace

$$\tilde{P}_{\mathcal{L}}\subseteqq\prod_{\lambda\in\Lambda}\{\omega_\lambda:|\mathcal{L}_\lambda|\to P_\lambda\}$$

consisting of all families of maps on the geometric realizations $|\mathcal{L}_\lambda|$ subject to the condition $g_\lambda^\mu\omega_\mu=\omega_\lambda$ on $|\mathcal{L}_\mu|$ for $\mu\geqq\lambda$. By replacing P_λ with Q_λ we get a closed subspace $\tilde{Q}_{\mathcal{L}}\subseteqq\tilde{P}_{\mathcal{L}}$. The pairs $(\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ form an inverse system over a cofinite directed index set, whose bonding maps are the restriction maps

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$r^{\mathcal{M}}_{\mathcal{L}}: (\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) \to (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$. We set $T(X, A) := \lim (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$, and it follows directly from the definitions that there is a natural equivalence

$$\mathbf{ssh}^2(-, - ; X, A) \approx \mathbf{HTop}^2(-, - ; T(X, A)).$$

This allows the definition of induced mappings turning T into a right adjoint functor,

We need to take a closer look at the restriction maps $r^{\mathcal{M}}_{\mathcal{L}}$; $(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}}) \to (\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ in the particular case, where $\mathcal{M}$ is obtained from $\mathcal{L}$ by attaching one additional simplex σ with boundary $\dot{\sigma} \subseteq \mathcal{L}$. If λ is the lowest element of σ we have a pullback diagram:

$$\begin{array}{ccc} \tilde{P}_{\mathcal{M}} & \longrightarrow & P_{\lambda}^{|\sigma|} \\ \Big\downarrow{\scriptstyle r^{\mathcal{M}}_{\mathcal{L}}} & & \Big\downarrow \\ \tilde{P}_{\mathcal{L}} & \longrightarrow & P_{\lambda}^{|\dot{\sigma}|}, \end{array}$$

similarly for $\tilde{Q}_{\mathcal{M}}$. We see that $r^{\mathcal{M}}_{\mathcal{L}}: \tilde{P}_{\mathcal{M}} \to \tilde{P}_{\mathcal{L}}$ and $r^{\mathcal{M}}_{\mathcal{L}}: \tilde{Q}_{\mathcal{M}} \to \tilde{Q}_{\mathcal{L}}$ are fibrations, and by induction on the number of elements of $\mathcal{L}$ observing [1, Lemma 3.6] we see that $\tilde{P}_{\mathcal{L}}$ and $\tilde{Q}_{\mathcal{L}}$ are ANR-spaces.

2. Strong shape equivalences of pairs.

LEMMA 1. *We suppose that* $\{f_\lambda\}; X \to \{g^{\mu}_{\lambda}: X_\mu \to X_\lambda\}$ *is a strong expansion and that* $\pi: E \to B$ *is a fibration between ANR-spaces.*

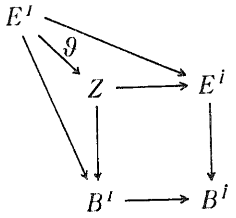
a) *If we are given maps* $\alpha: X \to E$ *and* $\beta: X_\lambda \to B$ *for some index* λ, *and a homotopy* $H: \pi\alpha \simeq \beta f_\lambda$, *then we can find some index* $\mu \geqq \lambda$, *a map* $\gamma: X_\mu \to E$ *with* $\pi\gamma = \beta g^{\mu}_{\lambda}$ *and a homotopy* $G: \alpha \simeq \gamma f_\mu$ *with* $\pi G = H$.

b) *If two maps* $\alpha_0, \alpha_1: X_\lambda \to E$ *and two homotopies* $\Gamma: \pi\alpha_0 \simeq \pi\alpha_1$ *and* $C: \alpha_0 f_\lambda \simeq \alpha_1 f_\lambda$ *with* $\pi C = \Gamma(f_\lambda \times \mathrm{id})$ *are given, then for a suitable index* $\mu \geqq \lambda$ *there is a homotopy* $\Sigma: \alpha_0 g^{\mu}_{\lambda} \simeq \alpha_1 g^{\mu}_{\lambda}$ *with* $\pi\Sigma = \Gamma(g^{\mu}_{\lambda} \times \mathrm{id})$ *and* $\Sigma(f_\mu \times \mathrm{id}) \simeq C$ *relative* $X \times \dot{I}$ *and relative* π.

POOOF. a) At first we take a map $\gamma': X_\nu \to E$ and a homotopy $\Gamma: \gamma' f_\nu \simeq \alpha$ for some suitable index $\nu \geqq \lambda$. This provides us with two maps $\pi\gamma'$, $\beta g^{\nu}_{\lambda}: X_\nu \to B$, whose compositions with f_ν are connected by the homotopy $\pi\Gamma \circ H: \pi\gamma' f_\nu \simeq \beta f_\lambda$. We conclude the existence of a homotopy $\Sigma: \pi\gamma' g^{\mu}_{\lambda} \simeq \beta g^{\mu}_{\lambda}$ for some $\mu \geqq \nu$ with $\Sigma(f_\mu \times \mathrm{id}) \simeq \pi\Gamma \circ H$ relative $X \times \dot{I}$. The fibration property of π ensures the existence of a homotopy $\Omega: X_\mu \times I \to E$ with $\Omega_0 = \gamma' g^{\mu}_{\lambda}$ and $\pi\Omega = \Sigma$. If we set $\gamma := \Omega_1$ then we have $\pi\gamma = \beta g^{\mu}_{\lambda}$, and there is a homotopy $G' := \Gamma^{-1} \circ \Omega(f_\mu \times \mathrm{id}):$

$\alpha\cong\gamma f_\mu$ with $\pi G'\cong H$ relative $X\times\dot{I}$. Using the fibration property of π again we can replace G' with another homotopy $G:\alpha\cong\gamma f_\mu$ with $\pi G=H$.

To prove (b) we take a look at the following diagram, where the square is formed as pullback diagram:



It is elementary to check that $\vartheta: E^I\to Z$ is a fibration, and by [1, Lemma 3.6] Z is an ANR-space. Therefore we can replace the fibration π in (a) by ϑ, and this proves (b). q.e.d.

REMARK. If the fibration π in Lemma 1.a has the form either of the zero map from an ANR-space P to a one point space or of the restriction map $P^I\to P^{\dot{I}}$, then we rediscover the defining conditions of a strong expansion (c) and (d) in part 1 of this paper. Therefore Lemma 1.a characterizes strong expansions.

We consider a pair (X, A) consisting of an arbitrary topological space X and a normally embedded subspace A.

As usual $\Delta^n=\{(t_0, \cdots t_n)\in I^{n+1}\,|\,\sum_{k=0}^n t_k=1\}$ is the topological standard-n-simplex with its faces $\Delta_k^n=\{(t_0, \cdots t_n)\in\Delta^n\,|\,t_k=0\}$; by $k\in\Delta^n$ we mean the vertex determined by $t_k=1$.

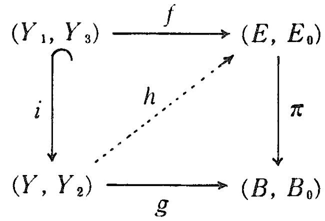
We set:

$$Y\ := X\times\Delta_0^2\cup A\times\Delta^2$$
$$Y_1 := X\times\{1\}\cup A\times\Delta_2^2\subseteq Y$$
$$Y_2 := A\times\Delta_1^2\subseteq Y$$
$$Y_3 := A\times\{0\}=Y_1\cap Y_2\,.$$

The space Y carries the weak topology determined by the subspaces $X\times\Delta_0^2$ and $A\times\Delta^2$.

An ANR-pair is a topological pair consisting of two ANR-spaces such that the subspace is closed. A fibration between ANR-pairs $\pi:(E, E_0)\to(B, B_0)$ is a map between these two ANR-pairs such that $\pi: E\to B$ and its restriction $\pi': E_0\to B_0$ are Hurewicz-fibrations.

LEMMA 2. a) *For any fibration of ANR-pairs* $\pi:(E, E_0)\to(B, B_0)$ *the following lifting problem has a solntion:*



b) *If in* (a) *two maps* $h_0, h_1:(Y, Y_2)\to(E, E_0)$ *and two homotopies* $H: h_0 i\cong h_1 i$ *and* $G:\pi h_0\cong\pi h_1$ *(as maps of pairs!) with* $\pi H=C(i\times\mathrm{id})$ *are prescribed, then there exists* $\Gamma: h_0\cong h_1$ *with* $\Gamma(i\times\mathrm{id})=H$ *and* $\pi\Gamma=G$.

PROOF. a) There is a deformation $D: B\times I\to B$ relative B_0 with $D_0=\mathrm{id}$ and a neighborhood V of B_0 in B with $D_1(V)\subseteq B_0$. Since B is a normal space the set $U_1:=\{x\in X\,|\,g(x, 2)\in V\}$ is a normal neighborhood of A in X. We observe that by corollary 2.5 in part 1 of this paper the system of normal neighborhoods of A in X forms a strong expansion of A and apply Lemma 1.a to the maps $\alpha:=f_{|A\times\{0\}}: A\to E_0$, $\beta:=D_1 g_{|U_1\times\{2\}}: U_1\to B_0$ and the homotopy $H:=g_{|A\times\Delta_1^2}$; we get a normal neighborhood U_2 of A in X contained in U_1, a map $\gamma: U_2\times\{2\}\to E_0$ with $\pi\gamma=D_1 g_{|U_2\times\{2\}}$ and a homotopy $G: A\times\Delta_1^2\to E_0$ with $G=f$ on $A\times\{0\}$, $G=\gamma$ on $A\times\{2\}$ and $\pi G=g$ on $A\times\Delta_1^2$.

We set $g':=g_{|U_2\times\{2\}}: U_2\to B$ and consider the homotopy $D(g'\times\mathrm{id}): U_2\times I\to B$; it is stationary on $A\times I$ and ends at $\pi\gamma$. Since the fibration $\pi: E\to B$ has a metrizable base space it is regular and we can find a homotopy $\Phi: U_2\times I\to E$ with $\Phi_1=\gamma$ and $\pi\Phi=D(g'\times\mathrm{id})$, which is stationary on $A\times I$. In particular we have $\Phi_0=\gamma$ on A and $\pi\Phi_0=g'$.

Since the inclusion map $A\times(\Delta_1^2\cup\Delta_2^2)\hookrightarrow A\times\Delta^2$ is a cofibration and a homotopy equivalence we can find a map $\Psi: A\times\Delta^2\to E$ with $\Psi=f$ on $A\times\Delta_2^2$, $\Psi=G$ on $A\times\Delta_1^2$ and $\pi\Psi=g$ on $A\times\Delta^2$. Applying Lemma 1.b to the maps $\alpha_0:=f_{|U_2\times\{1\}}: U_2\to E$, and $\alpha_1:=\Phi_0: U_2\to E$ and the homotopies $\Gamma:=g_{|U_2\times\Delta_0^2}$ and $C:=\Psi_{|A\times\Delta_0^2}$ we get a normal neighborhood U_3 of A in X contained in U_2 and a homotopy $\Sigma: U_3\times\Delta_0^2\to E$ with $\Sigma=f$ on $U_3\times\{1\}$, $\Sigma=\Phi_0$ on $U_3\times\{2\}$, $\pi\Sigma=g$ on $U_3\times\Delta_0^2$ and $\Sigma_{|A\times\Delta_0^2}\cong\Psi_{|A\times\Delta_0^2}$ relative $A\times\partial\Delta_0^2$ and relative π. The last condition allows to replace Ψ with a map $\Psi': A\times\Delta^2\to E$ with $\Psi'=f$ on $A\times\Delta_2^2$, $\Psi'=G$ on $A\times\Delta_1^2$, $\Psi'=\Sigma$ on $A\times\Delta_0^2$ and $\pi\Psi'=g$ on $A\times\Delta^2$.

We now take an Urysohn function $\varphi: X\to I$ with $\varphi=1$ on A and $\varphi=0$ on $X\setminus U_3$ and set $\psi:=\min(1, 2(1-\varphi))$; furthermore we consider the map $\omega: X\times\Delta_0^2$

$\to E$ and the homotopy $\Omega: X\times\Delta_0^2\times I\to B$ given by

$$\omega(x, 0, t_1, t_2) := \begin{cases} f(x, 0, 1, 0) & \text{for } \varphi(x)\leqq 1/2 \\ \Sigma(x, 0, t_1+t_2\phi(x), t_2(1-\phi(x))) & \text{for } \varphi(x)\geqq 1/2\,, \end{cases}$$

$$\Omega(x, 0, t_1, t_2, s) := g(x, 0, t_1+st_2\phi(x), t_2(1-s\phi(x)))\,.$$

We observe: $\Omega_1=\pi\omega$, and Ω is stationary on $(X\times\{1\}\cup A\times\Delta_0^2)\times I$. Since the fibration $\pi: E\to B$ is regular there is a homotopy $\Theta: X\times\Delta_0^2\times I\to E$ with $\Theta_1=\omega$ and $\pi\Theta=\Omega$, which is stationary on $(X\times\{1\}\cup A\times\Delta_0^2)\times I$. In particular we have $\pi\Theta_0=g_{|X\times\Delta_0^2}$, $\Theta_0=f$ on $X\times\{1\}$ and $\Theta_0=\Psi'$ on $A\times\Delta_0^2$. Therefore we can define a map $h: Y\to E$ meeting all requirements by $h:=\Theta_0$ on $X\times\Delta_0^2$ and $h:=\Psi'$ on $A\times\Delta^2$.

(b) is deduced from (a) by replacing the fibration $\pi: (E, E_0)\to(B, B_0)$ with $\vartheta: (E^I, E_0^I)\to(Z, Z_0)$, constructed as in the proof of Lemma 1.b. q.e.d.

PROPOSITION 1. *We consider a map of pairs* $f: (Y, \tilde{Y})\to(Z, \tilde{Z})$, *such that* $f: Y\to Z$ *and the restricted map* $\tilde{f}: \tilde{Y}\to\tilde{Z}$ *are homotopy equivalences. If* (X, A) *is a pair such that the inclusion map* $i: A\hookrightarrow X$ *is a cofibration, then* $f_*: \mathbf{HTop}^2(X, A; Y, Y)\to\mathbf{HTop}^2(X, A; Z, \tilde{Z})$ *is bijective.*

PROOF. In this proof a homotopy of the form $H(g\times \mathrm{id}_I)$ will simply be denoted Hg, and homotopies between homotopies will always be understood relative to the boundary. Furthermore we will repeatedly make use of the following two statements, taken from [6]:

i) For any homotopy equivalence $f: Y\to Z$ there exist a map $g: Z\to Y$ and homotopies $\Phi: fg\simeq\mathrm{id}$, $\Psi: gf\simeq\mathrm{id}$ with $f\Psi\simeq\Phi f$. ("f is a strong homotopy equivalence.")

ii) Suppose we are given maps $\alpha_0, \alpha_1: Y\to Z$, $\beta_0, \beta_1: Z\to T$ and homotopies $A: \alpha_0\simeq\alpha_1$, $B: \beta_0\simeq\beta_1$. Then: $\beta_0A\circ B\alpha_1\simeq B\alpha_0\circ\beta_1A$. ("Godement interchange law" or "commutativity lemma".)

Now we consider a map $f: (Y, \tilde{Y})\to(Z, \tilde{Z})$ with the property stated in the proposition. By (i) we can find maps and homotopies as follows:

$$\begin{array}{ll} g: Z\longrightarrow Y & \tilde{g}: \tilde{Z}\longrightarrow\tilde{Y} \\ \Phi: fg\simeq\mathrm{id} & \tilde{\Phi}: \tilde{f}\tilde{g}\simeq\mathrm{id} \\ \Psi: gf\simeq\mathrm{id} & \tilde{\Psi}: \tilde{g}\tilde{f}\simeq\mathrm{id} \\ \quad f\Psi\simeq\Phi f & \quad \tilde{f}\tilde{\Psi}\simeq\tilde{\Phi}\tilde{f}\,. \end{array}$$

Let $j:\tilde{Y}\hookrightarrow Y$ and $k:\tilde{Z}\hookrightarrow Z$ be the inclusion maps. We observe $fj=k\tilde{f}$ and define a homotopy:

$$G:=\Psi^{-1}j\tilde{g}\circ gk\tilde{\Phi}:j\tilde{g}\simeq gk\,.$$

Applying (ii) to the homotopies $\tilde{\Phi}:\tilde{f}\tilde{g}\simeq\mathrm{id}$ and $\Phi k:fgk\simeq k$ we get $fgk\tilde{\Phi}\circ\Phi k\simeq\Phi k\tilde{f}\tilde{g}\circ k\tilde{\Phi}=\Phi fj\tilde{g}\circ k\tilde{\Phi}\simeq f\Psi j\tilde{g}\circ k\tilde{\Phi}$ and hence:

$$fG\simeq g\tilde{\Phi}\circ\Phi^{-1}k\,.$$

Then we apply (ii) to the homotopies $\tilde{\Psi}:\tilde{g}\tilde{f}\simeq\mathrm{id}$ and $\Psi j:gfj\simeq j$ and conclude $\Psi j\tilde{g}\tilde{f}\circ j\tilde{\Psi}\simeq gfj\tilde{\Psi}\circ\Psi j=gk\tilde{f}\tilde{\Psi}\circ\Psi j\simeq gk\tilde{\Phi}\tilde{f}\circ\Psi j$ and therefore:

$$G\tilde{f}\simeq j\tilde{\Psi}\circ\Psi^{-1}j\,.$$

Now we consider our pair (X, A) with cofibration inclusion $i:A\hookrightarrow X$, and we suppose that a map of pairs $\alpha:(X, A)\to(Z, \tilde{Z})$ is prescribed. Let $\tilde{\alpha}:A\to\tilde{Z}$ be the restricted map and observe: $k\tilde{\alpha}=\alpha i$. The cofibration property allows the construction of a homotopy $H:X\times I\to Y$ with $H_1:=g\alpha$ and $H_{|A\times I}=G\tilde{\alpha}$. This means in particular $H_0=j\tilde{g}\tilde{\alpha}$ on A, so that $\beta:=H_0$ may be considered as map of pairs $\beta:(X, A)\to(Y, \tilde{Y})$, and we claim $f\beta\simeq\alpha$. Forgetting the subspaces for the moment these two maps are connected by the homotopy $\Gamma':=fH\circ\Phi\alpha$: $f\beta\simeq\alpha$; but on $A\times I$ we have $\Gamma' i=fG\tilde{\alpha}\circ\Phi k\tilde{\alpha}\simeq k\tilde{\Phi}\tilde{\alpha}\circ\Phi^{-1}k\tilde{\alpha}\circ\Phi k\tilde{\alpha}\simeq k\tilde{\Phi}\tilde{\alpha}$, and the cofibration property ensures the existence of a homotopy $\Gamma:f\beta\simeq\alpha$, $\Gamma\simeq\Gamma'$, with $\Gamma=k\tilde{\Phi}\tilde{\alpha}$ on $A\times I$, in particular $\Gamma(A\times I)\subseteq\tilde{Z}$. This proves f_* to be surjective.

Let us suppose two maps $\alpha, \beta:(X, A)\to(Y, \tilde{Y})$ and a homotopy $H:f\alpha\simeq f\beta$ with $H(A\times I)\subseteq\tilde{Z}$ are given; we denote by $\tilde{\alpha}, \tilde{\beta}:A\to\tilde{Y}$ and $\tilde{H}:A\times I\to\tilde{Z}$, $\tilde{H}$: $\tilde{f}\tilde{\alpha}\simeq\tilde{f}\tilde{\beta}$, the restrictions and observe: $\alpha i=j\tilde{\alpha}$, $\beta i=j\tilde{\beta}$ and $Hi=k\tilde{H}$. On the total space we can define a homotopy $\Gamma':\alpha\simeq\beta$ by $\Gamma':=\Psi^{-1}\alpha\circ gH\circ\Psi\beta$; then we have on $A\times I$: $\Gamma' i=\Psi^{-1}j\tilde{\alpha}\circ gk\tilde{H}\circ\Psi j\tilde{\beta}$. We apply (ii) to the homotopies $\tilde{H}:\tilde{f}\tilde{\alpha}\simeq\tilde{f}\tilde{\beta}$ and $G:j\tilde{g}\simeq gk$ and get $j\tilde{g}\tilde{H}\circ G\tilde{f}\tilde{\beta}\simeq G\tilde{f}\tilde{\alpha}\circ gk\tilde{H}$, hence $j\tilde{g}\tilde{H}\circ j\tilde{\Psi}\tilde{\beta}\circ\Psi^{-1}j\tilde{\beta}\simeq j\tilde{\Psi}\tilde{\alpha}\circ\Psi^{-1}j\tilde{\alpha}\circ gk\tilde{H}$ and finally $\Gamma' i\simeq j(\tilde{\Psi}^{-1}\tilde{\alpha}\circ\tilde{g}\tilde{H}\circ\tilde{\Psi}\tilde{\beta})$. The cofibration property allows us to replace Γ' by a homotopy $\Gamma:\alpha\simeq\beta$ with $\Gamma i=j\,(\tilde{\Psi}^{-1}\tilde{\alpha}\circ\tilde{g}\tilde{H}\circ\tilde{\Psi}\tilde{\beta})$ and in particular $\Gamma(A\times I)\subseteq\tilde{Y}$. This shows that f_* is injective. q.e.d.

Let (X, A) be a topological pair with normally embedded subspace and consider the space $X\times\{1\}\cup A\times I$ with the mapping cylinder topology. In general the projection map $p:(X\times\{1\}\cup A\times I, A\times\{0\})\to(X, A)$ is not a homotopy equivalence of pairs, although it operates as homotopy equivalence on the total spaces and as homeomorphism of the subspaces. From Lemma 2.9 in part 1 of our paper we know that the situation improves in the ordinary shape category, and the following theorem extends this result to the strong shape category:

THEOREM 1. *If A is normally embedded in X then $p:(X\times\{1\}\cup A\times I,\ A\times\{0\})\to(X, A)$ is a strong shape equivalence of pairs.*

COROLLARY 1. *Every topological pair with normally embedded subspace has the strong shape of a pair (Y, B), such that the inclution map $B\hookrightarrow Y$ is a closed cofibration.*

PROOF OF THEOREM 1. We consider our spaces Y, Y_1, Y_2, Y_3 constructed above. The natural projection map $q:(Y, Y_2)\to(X, A)$ is a homotopy equivalence of pairs, because $j:(X, A)\to(Y, Y_2)$, $j(x):=(x, 2)$ is an inverse up to homotopy. The pair (Y_1, Y_3) coincides with the mapping cylinder $(X\times\{1\}\cup A\times I, A\times\{0\})$, and the restriction of $q:(Y, Y_2)\to(X, A)$ to (Y_1, Y_3) equals $p:(X\times\{1\}\cup A\times I, A\times\{0\})\to(X, A)$. Hence it suffices to show that the inclusion map $i:(Y_1, Y_3)\hookrightarrow(Y, Y_2)$ is a strong shape equivalence of pairs, wherefore we make use of the right adjoint functor $T:\mathbf{ssh}^2\to\mathbf{HTop}^2$ described in the introduction. We have to show that for every topological pair (Z, C) the induced map $i^*:\mathbf{HTop}^2(Y, Y_2; T(Z, C))\to\mathbf{HTop}^2(Y_1, Y_3; T(Z, C))$ is bijective. We recall that $T(Z, C)$ is the limit of an inverse system of pairs $T(Z, C)=\lim(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ and consider a map $g:(Y_1, Y_3)\to\lim(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$. We are going to construct a family of maps $h_{\mathcal{M}}:(Y, Y_2)\to(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ with $r^{\mathcal{M}}_{\mathcal{L}}h_{\mathcal{M}}=h_{\mathcal{L}}$ for $\mathcal{M}\supseteqq\mathcal{L}$ and $h_{\mathcal{M}}=r_{\mathcal{M}}g$ on (Y_1, Y_3), $r_{\mathcal{M}}:\lim(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})\to(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ being the projection map. The induction is by number of simplices of $\mathcal{M}$, where we distinguish two cases: If $\mathcal{M}$ does not contain a largest proper subcomplex, then $\mathcal{M}$ equals the union of all its proper subcomplexes $\mathcal{L}$ and therefore $(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})=\lim(\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ and $h_{\mathcal{M}}=\lim h_{\mathcal{L}}$. If on the other hand there is a largest subcomplex $\mathcal{L}$, then we have to apply Lemma 2.a to the fibration of ANR-pairs $r^{\mathcal{M}}_{\mathcal{L}}:(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})\to(\tilde{P}_{\mathcal{L}}, \tilde{Q}_{\mathcal{L}})$ to get $h_{\mathcal{M}}$. This completes the induction and determines a map $h:(Y, Y_2)\to\lim(\tilde{P}_{\mathcal{M}}, \tilde{Q}_{\mathcal{M}})$ with $r_{\mathcal{M}}h=h_{\mathcal{M}}$; and we necessarily have $hi=g$. This shows that $i^*:\mathbf{HTop}^2(Y, Y_2; T(Z, C))\to\mathbf{HTop}^2(Y_1, Y_3; T(Z, C))$ is surjective. In a similar way, using Lemma 2.b instead of Lemma 2.a, we see that i^* is injective.

q.e.d.

It is obvious that every morphism in the homotopy category of pairs $[f]\in\mathbf{HTop}^2(X, A; Y, B)$ determines a morphism between the total spaces $[f_1]\in\mathbf{HTop}(X, Y)$ and another one between the relative spaces $[f_2]\in\mathbf{HTop}(A, B)$. We are going to explain that this carries over to the strong shape category, provided the subspaces are normally embedded. Let $\{f_\lambda\}:(X, A)\to\{g^\mu_\lambda:(P_\mu, Q_\mu)$

$\to(P_\lambda, Q_\lambda) \mid \mu \geqq \lambda \in \Lambda\}$ be a resolution of (X, A) in ANR-pairs; *if A is normally embedded in X* then $\{f_\lambda\} : X \to \{g^\mu_\lambda : P_\mu \to P_\lambda\}$ and $\{f'_\lambda\} : A \to \{g'^\mu_\lambda : Q_\mu \to Q_\lambda\}$ are ANR-resolutions. Hence for every strong shape morphism $\alpha \in \mathbf{ssh}^2(X, A ; Y, B)$ between pairs with normally embedded subspaces there are unique strong shape morphisms $\alpha_1 \in \mathbf{ssh}^2(X, \emptyset ; Y, \emptyset)$ and $\alpha_2 \in \mathbf{ssh}^2(A, A ; B, B)$ fitting commutatively into the following diagram:

$$\begin{array}{ccccc} (X, \emptyset) & \xrightarrow{i_1} & (X, A) & \xleftarrow{i_2} & (A, A) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_2} \\ (Y, \emptyset) & \xrightarrow{j_1} & (Y, B) & \xleftarrow{j_2} & (B, B) \end{array}$$

α_1 is called the total part of α, α_2 is the relative part; we observe that these two morphisms can be identified with strong shape morphisms $\alpha_1 \in \mathbf{ssh}(X, Y)$ and $\alpha_2 \in \mathbf{ssh}(A, B)$ in an obvious way. The uniqueness statement made above implies in particular that the assignments $\alpha \mapsto \alpha_1$ and $\alpha \mapsto \alpha_2$ are functorial.

THEOREM 2. *A strong shape morphism $\alpha \in \mathbf{ssh}^2(X, A ; Y, B)$ between pairs with normally embedded subspace is a strong shape equivalence of pairs if and only if its total part $\alpha_1 \in \mathbf{ssh}(X, Y)$ and its relative part $\alpha_2 \in \mathbf{ssh}(A, B)$ are strong shape equivalences.*

PROOF. Clearly the condition is necessary; we show that it is also sufficient. Our right adjoint functor $T : \mathbf{ssh}^2 \to \mathbf{HTop}^2$ transforms the diagram above into the following form:

$$\begin{array}{ccccc} T(X, \emptyset) & \xrightarrow{T(i_1)} & T(X, A) & \xleftarrow{T(j_2)} & T(A, A) \\ \downarrow{\scriptstyle T(\alpha_1)} & & \downarrow{\scriptstyle T(\alpha)} & & \downarrow{\scriptstyle T(\alpha_2)} \\ T(Y, \emptyset) & \xrightarrow{T(j_1)} & T(Y, B) & \xleftarrow{T(j_2)} & T(B, B) \end{array}$$

We observe that $T(i_1)$ and $T(j_1)$ induce homotopy equivalences of the total spaces and that $T(i_2)$ and $T(j_2)$ induce homotopy equivalences of the subspaces. If α_1 and α_2 are strong shape equivalences, then $T(\alpha_1)$ and $T(\alpha_2)$ are homotopy equivalences and therefore $T(\alpha)$ operates as homotopy equivalence on the total spaces and of the relative spaces of the pairs $T(X, A)$ and $T(Y, B)$. Proposition 1 implies that for every pair (Z, C), such that the inclusion map $C \hookrightarrow Z$ is is a cofibration, the induced function $T(\alpha)_* : \mathbf{HTop}^2(Z, C : T(X, A)) \to \mathbf{HTop}^2(Z, C ; T(Y, B))$ is bijective, and hence that $\alpha_* : \mathbf{ssh}^2(Z, C ; X, A) \to$

$\mathbf{ssh}^2(Z, C; Y, B)$ is bijective. By corollary 1 (X, A) and (Y, B) have the strong shape of pairs, such that the inclusion maps of the relative space into the total space are cofibrations, and therefore α must be a strong shape equivalence.

q.e.d.

COROLLARY 2. *A continuous map* $f:(X, A)\to(Y, B)$ *between pairs with normally embedded subspace is a strong shape equivalence if and only if* $f: X\to Y$ *and the restricted mapping* $f': A\to B$ *have the properties* (a) *and* (b) *from the introduction to part* 1 *of this paper. A pointed map* $f:(X, *)\to(Y, *)$ *is a strong shape equivalence if and only if the unpointed map* $f: X\to Y$ *is a strong shape equivalence.*

COROLLARY 3. *Let* (X, A) *be a pair with normally embedded subspace* A, *such that* A *has the strong shape of a point. Then the quotient map* $p: X\to X/A$ *is a strong shape equivalence.*

REMARK. By [3] a space has the strong shape of a point if and only if it has the ordinary shape of a point.

COROLLARY 4. *For every pair* (X, A) *with normally embedded subspace and every homology or cohomology theory* H *factoring over the strong shape category the quotient map* $p:(X, A)\to(X/A, *)$ *induces isomorphisms of the homology respectively cohomology groups* $H(p): H(X, A)\to H(X/A, *)$.

PROOF OF COROLLARY 3. Since A has strong shape of a point it cannot be empty, so choose a point $a\in A$. By corollary 2 the inclusion map $i:(X, a)\hookrightarrow(X, A)$ is a strong shape equivalence of pairs and therefore the induced function $i^*: \mathbf{HTop}^2(X, A; T(Y, B))\to\mathbf{HTop}^2(X, a; T(Y, B))$ is bijective for every pair (Y, B). From the description of $T(Y, B)$ given in the preliminaries it is readily seen that for a pointed space $(Y, *)$ the classifying space $T(Y, *)$ can be chosen to be a pointed space too. Then the function $p^*: \mathbf{HTop}^2(X/A, *; T(Y, *))\to \mathbf{HTop}^2(X, A; T(Y, *))$ is bijective, and the same holds for the composed function $(pi)^*: \mathbf{HTop}^2(X/A, *; T(Y, *))\to\mathbf{HTop}^2(X, a; T(Y, *))$. Making use of the right adjoint property this shows that $\eta(pi)^*: \mathbf{ssh}^2(X/A, *; Y, *)\to\mathbf{ssh}^2(X, a; Y, *)$ is a bijection. Since $(Y, *)$ was arbitrary $pi:(X, a)\to(X/A, *)$ must be strong shape equivalence and hence $p:(X, A)\to(X/A, *)$ is a strong shape equivalence.

q.e.d.

Corollary 4 follows from 3 applied to the pair $(X\cup CA, CA)$.

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