

THE L_2 -WELLPOSED CAUCHY PROBLEM FOR SCHRÖDINGER TYPE EQUATIONS

By

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§ 1. Introduction.

We study the Cauchy problem for a Schrödinger type operator

$$P = P(x, t, D_x, D_t) = D_t + \frac{1}{2} \sum_{j=1}^n (D_j - a_j(x, t))^2 + c(x, t),$$

where $a(x, t) = (a_1(x, t), \dots, a_n(x, t))$, $c(x, t) \in C_b^0([0, T_0]; \mathcal{B}^\infty(R^n))$, $(T_0 > 0)$, $D_t = -i\partial/\partial t$, $D_j = -i\partial/\partial x_j$ and $a_j(x, t) = a_j^R(x, t) + ia_j^I(x, t)$ ($a_j^R(x, t)$ and $a_j^I(x, t)$ are real valued functions in $R_x^n \times R_t$). Hence $\mathcal{B}^\infty(R^n)$ denotes the set of C^∞ -functions whose derivatives of any order are all bounded and $g(x, t) \in C_b^k([0, T_0]; X)$ ($k=0, 1, 2, \dots$) means that the mapping: $[0, T_0] \ni t \rightarrow g(x, t) \in X$ is k -times continuously differentiable in the topology of X .

In this paper we give a sufficient condition for the Cauchy problem

$$(*) \quad \begin{cases} P(x, t, D_x, D_t)u(x, t) = f(x, t) & (x, t) \in R^n \times [0, T] \quad (T > 0), \\ u(x, 0) = u_0(x) & x \in R^n \end{cases}$$

to be L_2 -wellposed.

We say that the Cauchy problem (*) is L_2 -wellposed if there exists $T > 0$ such that for any initial data $u_0 \in H_2$ and for any $f(x, t) \in C_b^0([0, T]; H_2)$ there exists a unique solution $u(x, t) \in C_b^1([0, T]; L_2) \cap C_b^0([0, T]; H_2)$ satisfying

$$\|u(\cdot, t)\|_{L_2} \leq C(t) \left\{ \|u_0\|_{L_2} + \int_0^t \|f(\cdot, s)\|_{L_2} ds \right\} \quad t \in [0, T],$$

where $L_2 = L_2(R^n)$, and H_2 is the Sobolev space which defined by $H_2 = \{v \in \mathcal{S}' ; \langle D_x \rangle^2 v \in L_2(R^n)\}$. Here, \mathcal{S}' is the dual space of the rapidly decreasing functions \mathcal{S} , and $\langle D_x \rangle^2$ is the pseudo-differential operator of symbol $\langle \xi \rangle^2$ ($\langle \xi \rangle = \sqrt{1 + |\xi|^2}$).

When the coefficients $a_j(x) = a_j^R(x) + ia_j^I(x)$ ($j=1, \dots, n$) are independent of t , Mizohata [3], [4] proved that the condition

$$(C_0) \quad \sup_{\rho \geq 0, x \in R^n, \omega \in S^{n-1}} \left| \int_0^\rho \sum_{j=1}^n a_j^I(x - s\omega) \omega_j ds \right| < \infty$$

is necessary for the Cauchy problem (*) to be L_2 -wellposed. S^{n-1} denotes the unit sphere in R^n . He also proved (C_0) and the following condition

$$(C_\alpha) \quad \begin{cases} |\alpha| \geq 1, \\ \max_{1 \leq j \leq n} \left\{ \sup_{x \in R^n, \omega \in S^{n-1}} \int_0^\infty |D_x^\alpha a_j(x + s\omega)| ds \right\} < \infty, \end{cases}$$

are sufficient for the Cauchy problem (*) to be L_2 -wellposed. Takeuchi [5] proved that if there is $\varepsilon_0 > 0$ such that $a_j(x) = a_j^R(x) + ia_j^I(x)$ ($j=1, \dots, n$) satisfy

$$(T) \quad \begin{cases} |D_x^\alpha a_j^I(x)| \leq \frac{C_\alpha}{\langle x \rangle^{1+\varepsilon_0+|\alpha|}} & (all \ \alpha), \\ |D_x^\alpha a_j^R(x)| \leq \frac{C_\alpha}{\langle x \rangle} & (|\alpha| \geq 1), \end{cases}$$

for $x \in R^n$, then the Cauchy problem (*) is L_2 -wellposed.

In the present paper we give a sufficient condition for the Cauchy problem (*) to be L_2 -wellposed.

THEOREM. Suppose that there are $\varepsilon_1, \varepsilon_2 > 0$ such that

$$(S) \quad \begin{cases} |D_x^\alpha a_j^I(x, t)| \leq \begin{cases} \frac{C_0}{\langle x \rangle^{1+\varepsilon_1}} & (|\alpha|=0), \\ \frac{C_\alpha}{\langle x \rangle^{2+\varepsilon_1}} & (|\alpha| \geq 1), \end{cases} \\ |D_x^\alpha a_j^R(x, t)| \leq \frac{C_\alpha}{\langle x \rangle^{1+\varepsilon_2}} & (|\alpha| \geq 2), \end{cases}$$

for $(x, t) \in R^n \times [0, T_0]$ and $j=1, \dots, n$. Then the Cauchy problem (*) is L_2 -wellposed.

We can see from the above Theorem that the condition for $a_j^R(x)$ in (T) is removed for $|\alpha|=1$.

To prove our Theorem, we reduce the operator P to an operator of which imaginary part of first order term vanishes identically, by use of pseudo-differential operator of type $S_{0,0}^0$. To do so, we need some properties of a solution of a characteristic equation for P . In §2 we investigate the characteristic equations, in §3 conjugate P by a pseudo-differential operator $e^A(x, D_x)$ of type $S_{0,0}^0$ and in §4 prove that the Cauchy problem (*) for P with $a_j^I \equiv 0$ is L_2 -wellposed.

§2. Properties of characteristic curve.

We consider a characteristic equation for P

$$(2.1) \quad A_t(x, \xi; t) + \xi \cdot A_x(x, \xi; t) + a_x^R(x, t, \xi) \cdot A_\xi(x, \xi; t) + a^I(x, t, \xi) = 0,$$

where

$$a^R(x, t, \xi) = \sum_{j=1}^n a_j^R(x, t) \xi_j, \quad a^I(x, t, \xi) = \sum_{j=1}^n a_j^I(x, t) \xi_j,$$

$$a_x^R(x, t, \xi) = \left(\sum_{j=1}^n a_{j, x_1}^R(x, t) \xi_j, \dots, \sum_{j=1}^n a_{j, x_n}^R(x, t) \xi_j \right),$$

$$x \cdot \xi = \sum_{j=1}^n x_j \xi_j, \quad (x, \xi \in R^n).$$

The characteristic curve $(x(t), \xi(t)) = (x(y, \eta, t), \xi(y, \eta, t))$ for (2.1) is defined by

$$(2.2) \quad \begin{cases} \dot{x} = \xi, & x(0) = y, \\ \dot{\xi} = a_x^R(x, t, \xi), & \xi(0) = \eta, \end{cases}$$

where $\dot{x} = \partial x(t) / \partial t$, $\dot{\xi} = \partial \xi(t) / \partial t$.

Put

$$\tilde{A}(y, \eta; t) = - \int_0^t a^I(x(s), s, \xi(s)) ds.$$

Then we can prove the following

LEMMA 2.1. Assume that the conditions (S) in Theorem are valid. Then the solution of (2.2) satisfies the following properties.

$$(i) \quad x(y, \eta, t), \quad \xi(y, \eta, t) \in C^\infty(R^{2n} \times [0, T_0]).$$

(ii) There is a constant C such that

$$C^{-1} |t\eta| \leq |x(y, \eta, t) - y| \leq C |t\eta|,$$

$$C^{-1} |\eta| \leq |\xi(y, \eta, t)| \leq C |\eta|, \quad (y, \eta, t) \in R^{2n} \times [0, T_0].$$

(iii) There are $T > 0$ ($T \leq T_0$) and $C > 0$ such that

$$\int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(y, \eta, s) \rangle^{1+\varepsilon_2}} ds \leq C, \quad (y, \eta, t) \in R^{2n} \times [0, T].$$

(iv) There is $T > 0$ ($T \leq T_0$) such that for $|\alpha + \beta| \geq 1$

$$x_{\{\beta\}}^{(\alpha)}(y, \eta, t), \quad \xi_{\{\beta\}}^{(\alpha)}(y, \eta, t) \in \mathcal{B}_t^0([0, T]; S_{0,0}^0 \cap \mathcal{B}_t^1([0, T]; S_{0,0}^1),$$

where $x_{\{\beta\}}^{(\alpha)}(y, \eta, t) = \partial_\eta^\alpha D_\eta^\beta x(y, \eta, t)$, $\xi_{\{\beta\}}^{(\alpha)}(y, \eta, t) = \partial_\eta^\alpha D_\eta^\beta \xi(y, \eta, t)$, $\mathcal{B}_t^k([0, T]; X)$ is the set of symbols of which k -th derivatives are bounded in X for $t \in [0, T]$ and $S_{\rho, \delta}^m$ is the symbol class of pseudo-differential operator which defined by $S_{\rho, \delta}^m = \{p(x, \xi) \in C^\infty(R^{2n}); |p_{\{\beta\}}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|} \text{ in } R^{2n}\}$, for some ρ and δ between 0 and 1.

(v) There exist $T > 0$ ($T \leq T_0$) and $C > 0$ such that

$$C^{-1} \leq \left| \frac{\partial(x(y, \eta, t), \xi(y, \eta, t))}{\partial(y, \eta)} \right| \leq C, \quad (y, \eta, t) \in R^{2n} \times [0, T].$$

(vi) There is $T > 0$ ($T \leq T_0$) such that

$$\tilde{A}(y, \eta, t) \in \mathcal{B}_t^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_t^1([0, T]; S_{0,0}^1).$$

PROOF. (i) This fact is well known (see [1], Chapter 1, Theorem 4.2).

(ii) From (2.2) we have

$$\begin{aligned} 2|\xi(t)| \frac{d}{dt} |\xi(t)| &= \frac{d}{dt} |\xi(t)|^2 \\ &\leq 2|\xi(t)| |\dot{\xi}(t)| \\ &\leq 2|\xi(t)|^2 |a_x^R(x(t), t)|, \end{aligned}$$

which shows

$$\begin{aligned} |\xi(t)| &\leq |\xi(0)| \exp \left[\int_0^t |a_x^R(x(s), s)| ds \right] \\ &\leq C |\eta|. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \frac{d}{dt} |\xi(t)|^2 &\geq -2|\xi(t)| |\dot{\xi}(t)| \\ &\geq -2|\xi(t)|^2 |a_x^R(x(t), t)| \end{aligned}$$

and

$$\begin{aligned} |\xi(t)| &\geq |\xi(0)| \exp \left[- \int_0^t |a_x^R(x(s), s)| ds \right] \\ &\geq C' |\eta|. \end{aligned}$$

Therefore, we can get

$$C |\eta| \geq |\xi(t)| \geq C^{-1} |\eta|.$$

From (2.2) and the mean value theorem, there exists a θ ($0 < \theta < 1$) such that

$$\begin{aligned} x_j(t) &= y_j + \int_0^t \xi_j(y, \eta, s) ds \\ &= y_j + t \xi_j(y, \eta, \theta t), \end{aligned}$$

for $j=1, \dots, n$. Therefore we get

$$C^{-1} |t\eta| \leq |x(t) - y| \leq C |t\eta|.$$

(iii) From (ii) we have

$$\begin{aligned}
& \sup_{y \in R^n, \eta \in R^n} \int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(s) \rangle^{1+\varepsilon_2}} ds \\
& \leq \sup_{y \in R^n, |\eta| \geq 1} \int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(s) \rangle^{1+\varepsilon_2}} ds + \sup_{y \in R^n, |\eta| \leq 1} \int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(s) \rangle^{1+\varepsilon_2}} ds \\
& \leq n \max_{1 \leq i \leq n} \left\{ \sup_{y \in R^n, |\eta_i| \geq |\eta|/\sqrt{n}} \int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(s) \rangle^{1+\varepsilon_2}} ds \right\} + C.
\end{aligned}$$

When $|\eta_i|/|\eta| \geq 1/\sqrt{n}$, we have

$$\begin{aligned}
\left| \frac{\xi_i}{|\eta|} - \frac{\eta_i}{|\eta|} \right| &= \left| \frac{\int_0^t a_{x_i}^R(x(s), s, \xi(s)) ds}{|\eta|} \right| \\
&\leq CT \leq \frac{1}{2\sqrt{n}}
\end{aligned}$$

for $T \leq 1/2\sqrt{n}C$. Hence we obtain $|\xi_i| \geq |\eta|/2\sqrt{n}$. Put $\sigma = \int_0^s \xi_i(\tau) d\tau$.

Since $x(y, \eta, s) = y + \int_0^s \xi(\tau) d\tau$, we have

$$\begin{aligned}
\int_0^t \frac{|\xi(y, \eta, s)|}{\langle x(y, \eta, s) \rangle^{1+\varepsilon_2}} ds &\leq \int_0^t \frac{C|\eta|}{\left\langle y + \int_0^s \xi(\tau) d\tau \right\rangle^{1+\varepsilon_2}} ds \\
&\leq \int_{-\infty}^{\infty} \frac{C}{\langle y_i + \sigma \rangle^{1+\varepsilon_2}} \frac{|\eta|}{|\xi_i(s)|} d\sigma \\
&\leq \int_{-\infty}^{\infty} \frac{C'}{\langle y_i + \sigma \rangle^{1+\varepsilon_2}} d\sigma \leq C''.
\end{aligned}$$

This completes the proof of (iii).

(iv) Taking the derivative of (2.2) with respect to y ,

$$(2.3) \quad \begin{cases} \dot{x}_y = \xi_y, \\ \dot{\xi}_y = a_{xx}^R(x, t, \xi) \times x_y + a_x^R(x, t) \times \xi_y, \end{cases}$$

where

$$\xi_y = \begin{pmatrix} \partial \xi_1(t)/\partial y_1 & \cdots & \partial \xi_1(t)/\partial y_n \\ \vdots & \ddots & \vdots \\ \partial \xi_n(t)/\partial y_1 & \cdots & \partial \xi_n(t)/\partial y_n \end{pmatrix},$$

$$a_{xx}^R(x, t, \xi) \times x_y$$

$$= \begin{pmatrix} a_{x_1 x_1}^R(x, t, \xi) & \cdots & a_{x_1 x_n}^R(x, t, \xi) \\ \vdots & \ddots & \vdots \\ a_{x_n x_1}^R(x, t, \xi) & \cdots & a_{x_n x_n}^R(x, t, \xi) \end{pmatrix} \begin{pmatrix} \partial x_1(t)/\partial y_1 & \cdots & \partial x_1(t)/\partial y_n \\ \vdots & \ddots & \vdots \\ \partial x_n(t)/\partial y_1 & \cdots & \partial x_n(t)/\partial y_n \end{pmatrix}$$

and

$$a_x^R(x, t) \times \xi_y = \begin{pmatrix} a_{1x_1}^R(x, t) & \cdots & a_{nx_1}^R(x, t) \\ \vdots & \ddots & \vdots \\ a_{1x_n}^R(x, t) & \cdots & a_{nx_n}^R(x, t) \end{pmatrix} \begin{pmatrix} \partial \xi_1(t)/\partial y_1 & \cdots & \partial \xi_1(t)/\partial y_n \\ \vdots & \ddots & \vdots \\ \partial \xi_n(t)/\partial y_1 & \cdots & \partial \xi_n(t)/\partial y_n \end{pmatrix}.$$

Let

$$\rho(t) = \sqrt{|x_y(t)|^2 + |\xi_y(t)|^2},$$

where $|x_y(t)| = \{\sum_{i,j=1}^n |\partial x_i(t)/\partial y_j|^2\}^{1/2}$, $|\xi_y(t)| = \{\sum_{i,j=1}^n |\partial \xi_i(t)/\partial y_j|^2\}^{1/2}$.

Then we have

$$\begin{aligned} \frac{d}{dt} \rho(t)^2 &= \frac{d}{dt} |x_y(t)|^2 + \frac{d}{dt} |\xi_y(t)|^2 \\ &\leq 2|x_y| |\dot{x}_y(t)| + 2|\xi_y| |\dot{\xi}_y(t)| \\ &\leq 2|x_y| |\xi_y(t)| + 2|\xi_y| (|a_{xx}^R(x, t, \xi)| |x_y| + |a_x^R(x, t) \times \xi_y|) \\ &\leq 2\rho(t)^2 \{1 + |a_{xx}^R(x, t, \xi)| + |a_x^R(x, t)|\} \\ &\leq 2C \rho(t)^2 \{1 + \alpha(t)\}. \end{aligned}$$

where $\alpha(t) = |a_{xx}^R(x(t), t, \xi(t))|$.

Therefore

$$\frac{d}{dt} \rho(t) \leq C \rho(t) (1 + \alpha(t)).$$

Since $\rho(0) = (|x_y(0)|^2 + |\xi_y(0)|^2)^{1/2} = \sqrt{n}$, we get

$$\rho(t) \leq \rho(0) \exp \left[C \int_0^t \alpha(s) ds \right] = \sqrt{n} \exp \left[C \int_0^t \alpha(s) ds \right].$$

By (iii) and (S) we have

$$\int_0^t \alpha(s) ds = \int_0^t \frac{C |\xi(y, \eta, s)|}{\langle x(y, \eta, s) \rangle^{1+\varepsilon_2}} ds \leq C'.$$

Thus we have $\rho(t) \leq C$ which yields $|x_y|, |\xi_y| \leq C$. Similarly, $|x_\eta|, |\xi_\eta| \leq C$.

For $l \geq 1$ we suppose

$$|x_{\{\beta\}}(y, \eta, t)|, \quad |\xi_{\{\beta\}}(y, \eta, t)| \leq C$$

for $1 \leq |\alpha + \beta| \leq l$. It follows from (2.2)

$$\partial_y \partial_\eta^\alpha D_y^\beta \dot{x}(y, \eta, t) = \partial_y \partial_\eta^\alpha D_y^\beta \xi(y, \eta, t).$$

We put $f(y, \eta, t) = a_{xx}^R(x(y, \eta, t), t, \xi(y, \eta, t))$ and $g(y, \eta, t) = a_x^R(x(y, \eta, t), t)$.

Then we have

$$\begin{aligned}
\partial_y \partial_\eta^\alpha D_y^\beta \xi(y, \eta, t) &= \partial_\eta^\alpha D_y^\beta (f \times x_y + g \times \xi_y) \\
&= \sum_{\substack{\alpha^1 + \alpha^2 = \alpha \\ \beta^2 + \beta^3 = \beta}} C_{\beta^1 \beta^2}^{\alpha^1 \alpha^2} f_{\{\beta^1\}}^{\{\alpha^1\}}(y, \eta, t) \times \partial_\eta^{\alpha^2} D_y^{\beta^2} x_y(y, \eta, t) \\
&\quad + \sum_{\substack{\alpha^3 + \alpha^4 = \alpha \\ \beta^3 + \beta^4 = \beta}} C_{\beta^3 \beta^4}^{\alpha^3 \alpha^4} g_{\{\beta^3\}}^{\{\alpha^3\}}(y, \eta, t) \times \partial_\eta^{\alpha^4} D_y^{\beta^4} \xi_y(y, \eta, t),
\end{aligned}$$

where

$$\begin{aligned}
|f_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| &\leq C_{\alpha\beta} \frac{1 + |\xi|}{\langle x(t) \rangle^{1+\varepsilon_2}}, \\
|g_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| &\leq C_{\alpha\beta} \frac{1}{\langle x(t) \rangle^{1+\varepsilon_2}}.
\end{aligned}$$

Let

$$\tilde{\rho}(t) = \{ |\partial_y x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)|^2 + |\partial_y \xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)|^2 \}^{1/2}.$$

Then we have

$$\begin{aligned}
\frac{d}{dt} \tilde{\rho}(t)^2 &= \frac{d}{dt} |\partial_y x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)|^2 + \frac{d}{dt} |\partial_y \xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)|^2 \\
&\leq 2 |\partial_y x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| |\partial_y \dot{x}_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| + 2 |\partial_y \xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| |\partial_y \dot{\xi}_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| \\
&\leq 2 \tilde{\rho}(t)^2 + 2 \tilde{\rho}(t) \{ C \alpha(t) + C' \alpha(t) \tilde{\rho}(t) \}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\tilde{\rho}(t) &\leq \int_0^t (C \alpha(s)) ds + \int_0^t (1 + C' \alpha(s)) \tilde{\rho}(s) ds \\
&\leq C + \int_0^t (1 + C' \alpha(s)) \tilde{\rho}(s) ds.
\end{aligned}$$

Therefore, by Gronwall's inequality we have

$$\tilde{\rho}(t) \leq C \exp \left[\int_0^t (1 + C' \alpha(s)) ds \right] \leq \tilde{C},$$

which implies

$$|\partial_y x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)|, |\partial_y \xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| \leq C.$$

Similarly, we obtain

$$|\partial_\eta x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)|, |\partial_\eta \xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)| \leq C.$$

Thus $x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)$ and $\xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)$ are in $\mathcal{B}^0([0, T]; S_{0,0}^0)$ and (2.2) implies that $x_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)$ and $\xi_{\{\beta\}}^{\{\alpha\}}(y, \eta, t)$ are in $\mathcal{B}^1([0, T]; S_{0,0}^0)$. This completes the proof of (iv).

(v) Let

$$X(t) = \begin{pmatrix} \partial x(t)/\partial y & \partial x(t)/\partial \eta \\ \partial \xi(t)/\partial y & \partial \xi(t)/\partial \eta \end{pmatrix},$$

where

$$X(0) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then taking the derivative of (2.2) with respect to y and η we have

$$\dot{X}(t) = \begin{pmatrix} 0 & I \\ a_{xx}^R(x(t), t, \xi) & a_x^R(x(t), t) \end{pmatrix} X(t) \equiv A(t)X(t),$$

where $\dot{X}(t) = \partial X(t) / \partial t$.

On the other hand,

$$\int_0^t |A(s)| ds \leq C \int_0^t (1 + \alpha(s)) ds \leq C_1,$$

where $|A(t)| = \{\sum_{i,j=1}^{2n} (a_{ij}(t))^2\}^{1/2}$ and $a_{ij}(t)$ ($i, j=1, \dots, 2n$) are components of $A(t)$.

Consequently,

$$\begin{aligned} \frac{d}{dt} |X(t)|^2 &= 2X(t)\dot{X}(t) \\ &\geq -2|A(t)||X(t)|^2, \end{aligned}$$

and

$$\begin{aligned} |X(t)| &\geq |X(0)| \exp \left[- \int_0^t |A(s)| ds \right] \\ &\geq C |X(0)| = C'. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d}{dt} |X(t)|^2 &= 2X(t)\dot{X}(t) \\ &\leq 2|A(t)||X(t)|^2, \end{aligned}$$

and

$$\begin{aligned} |X(t)| &\leq |X(0)| \exp \left[\int_0^t |A(s)| ds \right] \\ &\leq C |X(0)| = C''. \end{aligned}$$

This completes the proof of (v).

(iv) We have

$$\tilde{A}_{\{\beta\}}^{\{\alpha\}}(y, \eta, t) = \sum_{j=1}^n \sum_{\substack{\beta^1 + \beta^2 = \alpha \\ \beta^1 + \beta^2 = \beta}} C_{\beta^1 \beta^2}^{\alpha} \int_0^t \partial_{\eta}^{\alpha^1} D_{\eta}^{\beta^1} a_j^I(x(y, \eta, s), s) \xi_{j\{\beta^2\}}^{\{\alpha^2\}}(y, \eta, s) ds,$$

where

$$|\partial_{\eta}^{\alpha^1} D_{\eta}^{\beta^1} a^I(x(y, \eta, s), s)| \leq \frac{C_{\alpha\beta}}{\langle x(y, \eta, s) \rangle^{1+\epsilon_1}}.$$

Then

$$|\tilde{A}_{\{\beta\}}(y, \eta, t)| \leq C_{\alpha\beta} \int_0^t (\alpha(s)+1) ds \leq C.$$

It implies (vi).

Let $y(x, \xi, t)$, $\eta(x, \xi, t)$ be the inverse functions of $x(y, \eta, t)$, $\xi(y, \eta, t)$ and put

$$(2.4) \quad A(x, \xi; t) = \tilde{A}(y(x, \xi, t), \eta(x, \xi, t); t).$$

LEMMA 2.2. Assume that the condition (S) in Theorem is valid. Then there is $T > 0$ ($T \leq T_0$) such that

- (i) $y(x, \xi, t), \eta(x, \xi, t) \in C^\infty(R^{2n} \times [0, T])$
- (ii) $y_{\{\beta\}}(x, \xi, t), \eta_{\{\beta\}}(x, \xi, t) \in \mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1), (|\alpha + \beta| \geq 1)$
- (iii) $|\partial_{\xi}^\alpha D_x^\beta x(y(x, \xi, t), \eta(x, \xi, t), s)| \leq C_\alpha |t - s|, (|\alpha| \geq 1, \text{ all } \beta)$

$$(v) \quad \begin{cases} A(x, \xi, t) \in \mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1) \\ |A_{\{\beta\}}(x, \xi, t)| \leq \begin{cases} C_{\alpha\beta} t, & (|\alpha| \geq 1) \\ C_{0\beta}, & (|\alpha| = 0). \end{cases} \end{cases}$$

i. e. $A(x, \xi; t) \in S_{0,0}^0$.

PROOF. (i) It is well known the proof of the differentiability of solutions with respect to parameters (see [1], Chapter 1, Theorem 4.2).

(ii) Set

$$F = \begin{pmatrix} x(y, \eta) - x \\ \xi(y, \eta) - \xi \end{pmatrix}.$$

Then,

$$\frac{\partial F}{\partial(y, \eta)} = \frac{\partial(x, \xi)}{\partial(y, \eta)}, \quad \frac{\partial F}{\partial(x, \xi)} = -I.$$

By the implicit function theorem,

$$\begin{aligned} \frac{\partial(y, \eta)}{\partial(x, \xi)} &= - \left(\frac{\partial F}{\partial(y, \eta)} \right)^{-1} \times \left(\frac{\partial F}{\partial(x, \xi)} \right) \\ &= \left(\frac{\partial(x, \xi)}{\partial(y, \eta)} \right)^{-1}. \end{aligned}$$

Thus the conclusions of (ii) follow from (iv) and (v) of Lemma 2.1.

(iii) Since $x(y(x, \xi, t), \eta(x, \xi, t), t) = x$, we have for $|\alpha| \geq 1$,

$$\begin{aligned}
& |\partial_{\xi}^{\alpha} D_x^{\beta} x(y(x, \xi, t), \eta(x, \xi, t), s)| \\
&= \left| \partial_{\xi}^{\alpha} D_x^{\beta} \left\{ x + \int_t^s \dot{x}(y(x, \xi, t), \eta(x, \xi, t), \tau) d\tau \right\} \right| \\
&= \left| \partial_{\xi}^{\alpha} D_x^{\beta} \left\{ \int_t^s \xi(y(x, \xi, t), \eta(x, \xi, t), \tau) d\tau \right\} \right| \\
&\leq C_{\alpha\beta} |t-s|.
\end{aligned}$$

(iv) The estimate (iii) and (S) imply

$$\begin{aligned}
& |D_x^{\beta} \partial_{\xi}^{\alpha} A(x, \xi, t)| \\
&= |D_x^{\beta} \partial_{\xi}^{\alpha} \tilde{A}(y(x, \xi, t), \eta(x, \xi, t), t)| \\
&= \left| D_x^{\beta} \partial_{\xi}^{\alpha} \int_0^t a^I(x(y(x, \xi, t), \eta(x, \xi, t), s), s) \cdot \xi(y(x, \xi, t), \eta(x, \xi, t), s) ds \right| \\
&\begin{cases} \leq C_{\alpha\beta} \left\{ \int_0^t \frac{|(t-s)\xi(s)|}{\langle x(s) \rangle^{2+\varepsilon_1}} ds + \int_0^t \frac{ds}{\langle x(s) \rangle^{1+\varepsilon_1}} \right\} \leq C_{\alpha\beta} t & \text{if } |\alpha| \geq 1, \\ \leq C_{\alpha\beta} \left\{ \int_0^t \frac{|\xi(s)|}{\langle x(s) \rangle^{2+\varepsilon_1}} ds + \int_0^t \frac{ds}{\langle x(s) \rangle^{1+\varepsilon_1}} \right\} \leq C_{0\beta} & \text{if } |\alpha| = 0. \end{cases}
\end{aligned}$$

This implies (iv).

§3. Transform by e^A .

Let $\sigma(K)(x, \xi; t) = e^{A(x, \xi; t)}$ and $\sigma(\tilde{K})(x, \xi; t) = e^{-A(x, \xi; t)}$. Where $A(x, \xi; t)$ is given in (2.4). Then, we have

$$\begin{aligned}
\sigma(\tilde{K} \circ K)(x, \xi; t) &= Os - \iint e^{-iy \cdot \eta} e^{-A(x, \xi + \eta; t)} e^{A(x+y, \xi; t)} dy d\eta, \\
&= 1 + \int_0^1 \sum_{|\gamma|=1} Os - \iint e^{-iy \cdot \eta} \{-D_{\eta}^{\gamma} A(x, \xi + \eta; t)\} e^{-A(x, \xi + \eta; t)} \\
&\quad \times \partial_x^{\gamma} A(x + \theta y, \xi; t) e^{A(x + \theta y, \xi; t)} dy d\eta d\theta \\
&\equiv 1 + \sigma(R)(x, \xi; t),
\end{aligned}$$

where

$$\begin{aligned}
\sigma(R)(x, \xi; t) &= \int_0^1 \sum_{|\gamma|=1} Os - \iint e^{-iy \cdot \eta} \{-D_{\eta}^{\gamma} A(x, \xi + \eta; t)\} e^{-A(x, \xi + \eta; t)} \\
&\quad \times \partial_x^{\gamma} A(x + \theta y, \xi; t) e^{A(x + \theta y, \xi; t)} dy d\eta d\theta.
\end{aligned}$$

Here

$$\tilde{d}\eta = (2\pi)^{-n} d\eta = (2\pi)^{-n} d\eta_1 \cdots d\eta_n,$$

and

$$Os - \iint e^{-iy \cdot \eta} a(y, \eta) dy d\eta = \lim_{\varepsilon \rightarrow 0} \iint e^{-iy \cdot \eta} \chi(\varepsilon \eta, \varepsilon y) a(y, \eta) dy d\eta,$$

for $\chi \in \mathcal{S}$ in R^{2n} such that $\chi(0, 0) = 1$. Then we can get the following Proposition 3.1 from Lemma 2.2 (iv).

PROPOSITION 3.1. Assume that the same condition (S) as in Theorem is valid. Then, for any α, β ($|\alpha| \geq 1$) we have

$$(3.1) \quad |r_{\{\beta\}}^{\{\alpha\}}(x, \xi; t)| \leq C_{\alpha\beta} t, \quad (t \in [0, T]),$$

where $r(x, \xi; t) = \sigma(R)(x, \xi; t)$.

From (3.1) and the Calderón-Vaillancourt theorem, we obtain $\|R(x, D_x; t)\|_{L_2} < 1$ for $t \in [0, T]$, if we take $T > 0$ sufficiently small. Hence we can define

$$(3.2) \quad Q(x, D_x; t) = \sum_{j=0}^{\infty} (-R(x, D_x; t))^j$$

which converges in the sense of L_2 norm. Moreover by virtue of estimates of the symbols of multiple products of pseudo-differential operators we can show that (3.2) is convergent in the symbol class $S_{0,0}^0$.

PROPOSITION 3.2 (Kumano-go [2]). Let $q_j(x, \xi)$ ($j=1, \dots, \nu+1$) be in $S_{0,0}^0$. Define for $\nu \geq 0$

$$(3.3) \quad p_{\nu+1}(x, \xi) = Os - \iint \exp\left(-i \sum_{j=1}^{\nu} y^j \cdot \eta^j\right) \cdot \prod_{j=1}^{\nu+1} q_j(x + \bar{y}^{j-1}, \xi + \eta^j) dy^1 \dots dy^{\nu} d\eta^1 \dots d\eta^{\nu},$$

where $\bar{y}^0 = 0$, $\bar{y}^j = y^1 + \dots + y^j$, ($j=1, \dots, \nu$), $\eta^{\nu+1} = 0$ and $y^j, \eta^j \in R^n$. Then there is $C > 0$ independent of ν such that

$$(3.4) \quad |p_{\nu+1}(x, \xi)| \leq C^{\nu+1} \prod_{j=1}^{\nu+1} |q_j|_{n_0}^{(0)}, \quad n_0 = 2[n/2 + 1],$$

for $x, \xi \in R^n$.

Define $p_{\nu+1}(x, D_x; t) = (-R(x, D_x; t))^{\nu+1}$. Then its symbol $p_{\nu+1}(x, \xi; t)$ is given by (3.3) with $q_j = -r(x, \xi; t)$ ($j=1, \dots, \nu+1$). Moreover $p_{\nu+1}^{\{\beta\}}(x, \xi; t)$ is given by (3.3) with $q_j = r_{\{\beta\}}^{\{\alpha\}}(x, \xi; t)$ ($\sum \alpha^j = \alpha$, $\sum \beta^j = \beta$). Therefore we have by virtue of (3.4)

$$|p_{\nu+1}^{\{\beta\}}(x, \xi; t)| \leq \begin{cases} C^{\nu+1} (|r|_{\alpha+\beta+n_0}^{(0)})^{\nu+1} & \text{if } |\alpha+\beta| > \nu, \\ C^{\nu+1} (|r|_{\alpha+\beta+n_0}^{(0)})^{|\alpha+\beta|} (|r|_{n_0}^{(0)})^{\nu+1-|\alpha+\beta|} & \text{if } |\alpha+\beta| \leq \nu \end{cases}$$

for $x, \xi \in R^n$.

From (3.1) we have $|r|_{n_0}^{(0)} \leq C_0 t$ ($t \in [0, T]$). Hence

$$\left| \sum_{\nu=0}^{\infty} p_{\nu+1}(\beta)(x, \xi; t) \right| \leq \sum_{\nu=0}^{|\alpha+\beta|} |p_{\nu+1}(\beta)| + \sum_{\nu=|\alpha+\beta|+1}^{\infty} |p_{\nu+1}(\beta)|$$

$$\leq C_{\alpha\beta} \left(1 + \sum_{\nu=1}^{\infty} (C_0 t)^{\nu} \right) \leq C'_{\alpha\beta}, \quad t \in [0, T]$$

if T is sufficiently small, which implies

$$Q(x, \xi; t) = \sum_{\nu=0}^{\infty} p_{\nu+1}(x, \xi; t) + 1 \in \mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1).$$

Now we can construct the inverse K^{-1} of $K(x, D_x; t)$ as follows

$$K(x, D_x; t)^{-1} = (1 + R(x, D_x; t))^{-1} \tilde{K}(x, D_x; t),$$

where $\tilde{K} = e^{-A(x, \xi; t)}$. The symbols of $K^{-1}(x, D_x; t)$ is in $\mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1)$.

Put $u(x, t) = Kv(x, t)$. We have

$$\begin{aligned} Pu(x, t) &= P \circ Kv(x, t) \\ &= \int e^{ix \cdot \xi} e^{A(x, \xi; t)} \left\{ D_t A + \sum_{j=1}^n \xi_j (D_j A - i a_j^R) + D_t + \frac{1}{2} \sum_{j=1}^n (\xi_j - a_j^R)^2 \right. \\ &\quad \left. + c(x) + \frac{1}{2} \sum_{j=1}^n (D_j A - a_j)^2 + \frac{1}{2} \sum_{j=1}^n D_j (D_j A - a_j) - \frac{1}{2} \sum_{j=1}^n (a_j^R)^2 \right\} \\ &\quad \times \hat{v}(\xi, t) d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} &K \circ \left\{ D_t + \frac{1}{2} \sum_{j=1}^n (D_j - a_j^R)^2 \right\} v(x, t) \\ &= K \circ \left\{ D_t + \frac{1}{2} \sum_{j=1}^n (D_j^2 - 2a_j^R D_j + (a_j^R)^2 - D_j a_j^R) \right\} v(x, t) \\ &= \int e^{ix \cdot \xi + A(x, \xi; t)} \left(D_t + \frac{1}{2} \sum_{j=1}^n \xi_j^2 \right) \hat{v}(\xi, t) d\xi \\ &\quad - \sum_{j=1}^n K \circ (a_j^R(x, t) D_j) v(x, t) + \frac{1}{2} \sum_{j=1}^n K \circ ((a_j^R)^2 - D_j a_j^R) v(x, t), \end{aligned}$$

and

$$\begin{aligned} &\sigma(K \circ a_j^R D_j)(x, \xi; t) \\ &= \sum_{j=1}^n O_s - \iint e^{-iy \cdot \eta} e^{A(x, \xi + \eta; t)} a_j^R(x + y, t) \xi_j dy d\eta \\ &= \sum_{j=1}^n e^{A(x, \xi; t)} a_j^R(x, t) \xi_j + \sum_{j=1}^n \sum_{|\gamma|=1} (e^{A(x, \xi; t)})^{(\gamma)} (a_j^R(x, t) \xi_j)_{(\gamma)} + b(x, \xi; t), \end{aligned}$$

where

$$\begin{aligned}
& b(x, \xi; t) \\
&= 2 \sum_{j=1}^n \sum_{|\gamma|=2} \int_0^1 \frac{1-\theta}{\gamma!} \left\{ O s - \iint e^{-i y \cdot \eta} (e^{A(x, \xi + \eta; t)})^{(\gamma)} a_j^R(x + \theta y, t) \xi_j d y d \eta \right\} d \theta, \\
& \sigma \left(\sum_{j=1}^n K \circ (a_j^R)^2 \right) (x, \xi; t) = \sum_{j=1}^n O s - \iint e^{-i y \cdot \eta} e^{A(x, \xi + \eta; t)} (a_j^R(x + y, t))^2 d y d \eta \\
&= \sum_{j=1}^n e^{A(x, \xi; t)} (a_j^R(x, t))^2 + c_1(x, \xi; t)
\end{aligned}$$

and

$$c_1(x, \xi; t) = \sum_{j=1}^n \sum_{|\gamma|=1} \int_0^1 O s - \iint e^{-i y \cdot \eta} (e^{A(x, \xi + \eta; t)})^{(\gamma)} (a_j^R(x + \theta y, t))^2 d y d \eta d \theta.$$

Therefore we have

$$K^{-1} \circ P \circ K v(x, t) = \left\{ D_t + \frac{1}{2} \sum_{j=1}^n (D_j - a_j^R)^2 \right\} v(x, t) + \tilde{c}(x, D_x; t) v(x, t),$$

where

$$\begin{aligned}
(3.5) \quad & \tilde{c}(x, D_x; t) v(x, t) \\
&= K^{-1} \circ \int e^{i x \cdot \xi + A(x, \xi; t)} (-i) \{ A_t(x, \xi; t) + \xi \cdot A_x(x, \xi; t) + a_x^R(x, t, \xi) \\
&\quad \cdot A_\xi(x, \xi; t) + a^I(x, t, \xi) \} \hat{v}(\xi, t) d \xi \\
&\quad + K^{-1} \circ b(x, D_x; t) v(x, t) \\
&\quad + \frac{1}{2} \sum_{j=1}^n D_j a_j^R(x, t) v(x, t) - \frac{1}{2} K^{-1} \circ c_1(x, D_x; t) v(x, t) \\
&\quad + \sum_{j=1}^n K^{-1} \circ \int e^{i x \cdot \xi + A(x, \xi; t)} \left\{ c(x) + \frac{1}{2} (D_j A - a_j)^2 + \frac{1}{2} D_j (D_j A - a_j) - \frac{1}{2} (a_j^R)^2 \right\} \\
&\quad \times \hat{v}(\xi, t) d \xi.
\end{aligned}$$

LEMMA 3.3. Assume that the condition (S) in Theorem is valid. Then $\tilde{c}(x, \xi; t)$ is in $\mathcal{B}_l^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_l^1([0, T]; S_{0,0}^1)$.

PROOF. Note that the first term of the right side of (3.5) vanishes because of (2.1) and it is evident that the terms in (3.5) except $b(x, \xi; t)$ are in $\mathcal{B}_l^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_l^1([0, T]; S_{0,0}^1)$. Hence it suffices to prove that $b(x, \xi; t)$ is in $\mathcal{B}_l^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_l^1([0, T]; S_{0,0}^1)$. By Lemma 2.1 (iii), Lemma 2.2 (iii), (iv) and $e^{-y \cdot \eta} = \langle y \rangle^{-2l} \langle D_\eta \rangle^{2l} e^{-y \cdot \eta} = \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} e^{-y \cdot \eta}$, where $\langle D_\eta \rangle^2 = 1 - \Delta_\eta$, $\langle D_y \rangle^2 = 1 - \Delta_y$, we have

$$\begin{aligned}
& |b_{\{\beta\}}^{\{\alpha\}}(x, \xi; t)| \\
&= \left| 2 \sum_{|l|=2} \frac{1-\theta}{l!} \sum_{\substack{\alpha' \leq \alpha \\ \beta' \leq \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \right. \\
&\quad \times \int_0^1 \left\{ Os - \iint \langle y \rangle^{-2l} \langle D_\eta \rangle^{2l} \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} e^{-iy \cdot \eta} \partial_{\xi}^{\alpha-\alpha'} D_x^{\beta-\beta'} (e^{A(x, \xi+\eta; t)}) \right. \\
&\quad \left. \times \partial_{\xi'}^{\alpha'} D_x^{\beta'} a_{(r)}^R(x+\theta y, t) \cdot \xi dy d\eta \right\} d\theta \Big| \\
&\leq C_{\alpha\beta} \iint \langle y \rangle^{-2l} \langle \eta \rangle^{-2l} \left\{ \int_0^t \frac{(t-s)|\xi(s)+\eta(s)|}{\langle x(s) \rangle^{2+\varepsilon_1}} ds + \int_0^t \frac{ds}{\langle x(s) \rangle^{1+\varepsilon_1}} \right\} \\
&\quad \times \frac{|\xi|+1}{\langle x+\theta y \rangle^{1+\varepsilon_2}} dy d\eta d\theta \\
&\leq C'_{\alpha\beta} \iint \langle y \rangle^{-2l+2} \langle \eta \rangle^{-2l+1} dy d\eta \int_0^t \frac{|\xi(s)|}{\langle x(s) \rangle^{2+\varepsilon_1}} ds \times \frac{(t-s)(|\xi|+1)}{\langle x \rangle^{1+\varepsilon_2}} \\
&\leq \tilde{C}_{\alpha\beta}.
\end{aligned}$$

Here we take $l=[n/2+2]$ and we use the inequality

$$\begin{aligned}
\frac{(t-s)(|\xi|+1)}{\langle x(s) \rangle \langle x \rangle^{1+\varepsilon_2}} &\leq \frac{\langle (t-s)\xi \rangle}{\left\langle x + \int_t^s \xi(y, \eta, \tau) d\tau \right\rangle \langle x \rangle} \\
&\leq \frac{C \langle (t-s)\xi \rangle}{\langle (t-s)\xi(y, \eta, \theta(t-s)+s) \rangle} \\
&\leq C, \\
\frac{1}{\langle x+\theta y \rangle^{1+\varepsilon_2} \langle y \rangle^2} &\leq \frac{1}{\langle x+\theta y \rangle^{1+\varepsilon_2} \langle y \rangle^{1+\varepsilon_2}} \\
&\leq \frac{1}{\langle x+\theta y \rangle^{1+\varepsilon_2} \langle \theta y \rangle^{1+\varepsilon_2}} \\
&\leq \frac{1}{\langle x \rangle^{1+\varepsilon_2}},
\end{aligned}$$

and

$$|D_x^{\beta} \partial_{\xi}^{\alpha} A(x, \xi, t)| \leq C_{\alpha\beta} \left\{ \int_0^t \frac{|(t-s)\xi(s)|}{\langle x(s) \rangle^{2+\varepsilon_1}} ds + \int_0^t \frac{ds}{\langle x(s) \rangle^{1+\varepsilon_1}} \right\},$$

for any α ($|\alpha| \geq 1$) (see the proof of Lemma 2.2 (iv)). This completes the proof.

By Lemma 3.3 we have $\sigma(\tilde{c})(x, \xi; t) \in S_{0,0}^0$. Finally put $u(x, t) = Kv(x, t)$, then we can transform the Cauchy problem (*) to the problem

$$(**) \quad \begin{cases} K^{-1} \circ P \circ Kv(x, t) = K^{-1} f, \\ v(x, 0) = v_0(x) \quad (=u_0(x)), \end{cases}$$

where

$$K^{-1} \circ P \circ K(x, D_x; t) = D_t + \sum_{j=1}^n (D_j - a_j^R(x, t))^2 + \tilde{c}(x, D_x; t),$$

and

$$\tilde{c}(x, \xi; t) \in \mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1).$$

§ 4. The existence theorem for Schrödinger type operators of real valued coefficients.

Denote

$$P(t) = \sum_{j=1}^n (D_j - a_j(x, t))^2 + c(x, D_x; t).$$

Assume that $a_j(x, t)$ are real valued and $c(x, \xi; t)$ is in $\mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1)$, then $P(t)$ satisfies

$$(4.1) \quad \sigma(P^*(t)) - \sigma(P(t)) \in \mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1),$$

where $P^*(t)$ is the adjoint operator of $P(t)$ which is defined as $(P(t)u, v)_{L_2} = (u, P^*(t)v)_{L_2}$ for any $u, v \in \mathcal{S}$. We consider the Cauchy problem

$$(4.2) \quad \begin{cases} \left(\frac{d}{dt} - iP(t) \right) u(x, t) = f(x, t) & x \in R^n, t \in [0, T], \\ u(x, 0) = u_0(x) & x \in R^n. \end{cases}$$

Then we obtain the following.

THEOREM 4.1. *Assume that $a_j(x, t)$ are real valued functions in $C_i^0([0, T]; \mathcal{B}^\infty(R^n))$ and $c(x, \xi; t)$ in $\mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^1)$. Then for any $u_0 \in H_2(R^n)$ and any $f(x, t) \in C_i^0([0, T]; H_2)$ there exists a unique solution $u(x, t)$ of (4.2) which belongs to $C_i^0([0, T]; H_2) \cap C_i^1([0, T]; L_2)$ and satisfies*

$$(4.3) \quad \|u(t)\|_{H_k} \leq C \left(\|u_0\|_{H_k} + \int_0^t \|f(\tau)\|_{H_k} d\tau \right)$$

for $t \in [0, T]$ and $k=0, 1, 2$.

Following the idea of Kumano-go [1], [2], we shall prove the above theorem. We need several lemmas. First, we define $\{\zeta_\nu(\xi)\}_{\nu=1}^\infty$ as

$$(4.4) \quad \zeta_\nu(\xi) = \left(\nu \sin \frac{\xi_1}{\nu}, \dots, \nu \sin \frac{\xi_n}{\nu} \right)$$

and $P_\nu(t) = p_\nu(x, D_x; t)$ as

$$(4.5) \quad p_\nu(x, \xi; t) = p_1(x, \zeta_\nu(\xi); t) + c(x, \xi; t),$$

where $p_1(x, D_x; t) = \sum_{j=1}^n (D_j - a_j(x, t))^2$.

Then, we consider the following Cauchy problem

$$(4.6) \quad \begin{cases} L_\nu u_\nu = \partial_t u_\nu - i P_\nu(t) u_\nu = f(t) & (t \in [0, T]), \\ u_\nu|_{t=0} = u_0. \end{cases}$$

We define the series of weight function $\{\lambda_\nu(\xi)\}_{\nu=1}^\infty$ as

$$(4.7) \quad \lambda_\nu(\xi) = \langle \zeta_\nu(\xi) \rangle = \left\{ 1 + \sum_{j=1}^n \left(\nu \sin \frac{\xi_j}{\nu} \right)^2 \right\}^{1/2},$$

then we have

$$(4.8) \quad \begin{cases} \text{i)} & 1 \leq \lambda_\nu(\xi) \leq \min(\langle \xi \rangle, \sqrt{1+n\nu^2}), \\ \text{ii)} & |\partial_\xi^\alpha \lambda_\nu(\xi)| \leq A_\alpha \lambda_\nu(\xi)^{1-|\alpha|}, \\ \text{iii)} & \lambda_\nu(\xi) \longrightarrow \langle \xi \rangle \quad (\nu \rightarrow \infty) \text{ on } R_\xi^n, \\ & \text{(uniform convergence in a compact set),} \end{cases}$$

In fact $\zeta_\nu(\xi)$ satisfies

$$(4.9) \quad \begin{cases} \text{i)} & |\zeta_\nu(\xi)| \leq \min(|\xi|, \sqrt{n\nu}), \\ \text{ii)} & |\partial_\xi^\alpha \zeta_\nu(\xi)| \leq A'_\alpha \lambda_\nu(\xi)^{1-|\alpha|}, \\ \text{iii)} & \zeta_\nu(\xi) \longrightarrow \xi \quad (\nu \rightarrow \infty), \\ & \text{(uniform convergence in a compact set).} \end{cases}$$

Denote by $S_{\lambda_\nu}^m$ the set of symbols $p(x, \xi) \in C^\infty(R^{2n})$ satisfying

$$|p\{\beta\}(x, \xi)| \leq C_{\alpha\beta} \lambda_\nu(\xi)^{m-|\alpha|}$$

for any multi-index α, β . Then we get the following lemma.

LEMMA 4.1. For $p(x, \xi) \in S^m$ put $p_\nu(x, \xi) = p(x, \zeta_\nu(\xi))$. Then $p(x, \xi) \in S_{\lambda_\nu}^m$, and for any α, β there is constant $A_{\alpha, \beta}$ which is independent of ν and p , we have

$$(4.10) \quad \begin{cases} |p_\nu\{\beta\}(x, \xi)| \leq (A_{\alpha, \beta} |p\{\alpha+\beta\}|) \lambda_\nu(\xi)^{m-|\alpha|}, \\ p_\nu(x, \xi) \longrightarrow p(x, \xi) \quad (\text{uniformly}) \quad (\nu \rightarrow \infty) \text{ at } R_x^n \times K_\xi, \end{cases}$$

where K_ξ is an arbitrary compact set of R_ξ^n .

Denote $H_{\lambda_\nu, s} = \{u \in S'; \lambda_\nu(\xi)^s \hat{u} \in L_2\}$.

LEMMA 4.2. $P = p(x, D_x) \in S_{\lambda_\nu}^m$ is a continuous mapping from the Sobolev space $H_{\lambda, s+m}$ to $H_{\lambda, s}$ and for a constant $C_{s, m}$ and $l = l(s, m)$ we have

$$(4.11) \quad \|Pu\|_{\lambda_\nu, s} \leq (C_{s, m} \|p\|_{l^{(n)}}) \|u\|_{\lambda_\nu, s+m}, \quad u \in H_{\lambda_\nu, s+m}.$$

Especially for $m=0$ and $s=0$ we have

$$(4.12) \quad \|Pu\|_{L_2} \leq (C \|p\|_{l^{(0)}}) \|u\|_{L_2}, \quad u \in L_2(R^n).$$

LEMMA 4.3. For $c(x, \xi; t) \in \mathcal{B}_l^0([0, T]; S_{0,0}^m)$, $q_{j\nu}(\xi) \in S_{\nu}^{l_j}$ ($j=1, 2$) and $l_1+l_2 \geq 0$ we have

$$\sigma(q_{1\nu} \circ c \circ q_{2\nu})(x, \xi; t) \in \mathcal{B}_l^0([0, T]; S_{0,0}^{m+l_1+l_2}).$$

PROOF. We take $2l > \tilde{l}_1 + n$ ($\tilde{l}_1 = \max(l_1, 0)$), then by Lemma 4.1 and $\lambda_\nu(\xi + \eta)^\pm \leq C \langle \eta \rangle \lambda_\nu(\xi)^\pm$ we have

$$\begin{aligned} & | \langle q_{1\nu} \circ c \circ q_{2\nu} \rangle_{\{\beta\}}^{\alpha}(x, \xi; t) | \\ &= \left| D_x^{\beta} \partial_{\xi}^{\alpha} \left\{ O_s - \iint e^{-iy \cdot \eta} q_{1\nu}(\xi + \eta) c(x+y, \xi; t) q_{2\nu}(\xi) dy d\eta \right\} \right| \\ &\leq \Sigma C_{\beta}^{\alpha^1, \alpha^2, \alpha^3} \left| O_s - \iint \langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} \{ \langle y \rangle^{-2l} \langle D_{\eta} \rangle^{2l} e^{-iy \cdot \eta} \right. \\ &\quad \times q_{1\nu}^{(\alpha^1)}(\xi + \eta) c_{\{\beta\}}^{(\alpha^2)}(x+y, \xi; t) q_{2\nu}^{(\alpha^3)}(\xi) dy d\eta \left. \right| \\ &\leq C \times O_s - \iint e^{-iy \cdot \eta} \langle y \rangle^{-2l} \langle \eta \rangle^{-2l} \lambda_\nu(\xi + \eta)^{l_1} \langle \xi \rangle^m \lambda_\nu(\xi)^{l_2} dy d\eta \\ &\leq C' \langle \xi \rangle^m \lambda_\nu(\xi)^{l_1+l_2} \iint \langle y \rangle^{-2l} \langle \eta \rangle^{-2l+l_1} dy d\eta \\ &\leq C'' \langle \xi \rangle^{m+l_1+l_2}, \end{aligned}$$

where $\alpha^1 + \alpha^2 + \alpha^3 = \alpha$.

LEMMA 4.4. For any $u_0 \in L_2$ and any $f(t) \in C_l^0([0, T]; L_2)$ there exists a solution $u_\nu(t) \in C_l^1([0, T]; L_2)$ of (4.6) which satisfies the energy inequalities

$$(4.13) \quad \|u_\nu(t)\| \leq e^{\gamma t} \|u_0\| + \int_0^t e^{\gamma(t-\tau)} \|f(\tau)\| d\tau \quad (t \in [0, T])$$

$$(4.14) \quad \|A_\nu^j u_\nu(t)\| \leq e^{\gamma_1 t} \|A_\nu^j u_0\| + \int_0^t e^{\gamma_1(t-\tau)} \|A_\nu^j f(\tau)\| d\tau \quad (t \in [0, T]; j=1, 2, 3, \dots)$$

$$(4.15) \quad \left\| \frac{d}{dt} A_\nu^j u_\nu(t) \right\| \leq C_T \{ \|A_\nu^{j+2} u_0\| + \max_{[0, T]} \|A_\nu^{j+2} f(\tau)\| \} \quad (t \in [0, T]; j=0, 1, 2, \dots)$$

$$(4.16) \quad \|A_\nu^j(u_\nu(t) - u_\nu(t'))\| \leq C'_T |t - t'| \{ \|A_\nu^{j+2} u_0\| + \max_{[0, T]} \|A_\nu^{j+2} f(\tau)\| \} \\ (t, t' \in [0, T]; j=0, 1, 2, \dots),$$

where $\gamma, \gamma_1, C_T, C'_T$ are constants which are independent of ν , and $A_\nu = \lambda_\nu(D_x)$,

$$\|\cdot\| = \|\cdot\|_{L_2}.$$

PROOF OF LEMMA 4.4. I) If we fix ν arbitrarily, we have $p_\nu(x, \xi; t) \in \mathcal{B}_i^k([0, T]; \mathcal{B}^\infty(R_{x, \xi}^{2n}))$ ($k=1, 2$). We note $\mathcal{B}^\infty(R_{x, \xi}^{2n}) = S_\lambda^0$ ($\lambda=1$), and use Lemma 4.2. Then $P_\nu(t)$ is an L_2 -bounded operator uniformly with respect to t . Therefore $u_\nu(t)$ to be a unique solution of the integral equation

$$(4.17) \quad u_\nu(t) = u_0 + i \int_0^t P_\nu(\tau) u_\nu(\tau) d\tau + \int_0^t f(\tau) d\tau,$$

which can be solved as follows;

$$\begin{aligned} u_\nu(t) = & u_0 + \sum_{k=1}^{\infty} i^k \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} P_\nu(\tau_1) \cdots P_\nu(\tau_k) u_0 d\tau_k \cdots d\tau_1 + \int_0^t f(\tau) d\tau \\ & + \sum_{k=2}^n i^k \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} P_\nu(\tau_1) \cdots P_\nu(\tau_{k-1}) f(\tau_k) d\tau_k \cdots d\tau_1 \quad (\tau_0=t). \end{aligned}$$

Then we have $u_\nu(t) \in C_i^1([0, T]; L_2)$.

II) From (4.6) we have

$$\begin{aligned} \frac{d}{dt} \|u_\nu\|^2 &= 2 \operatorname{Re} \left(\frac{d}{dt} u_\nu, u_\nu \right) \\ &= \operatorname{Re} (i(P_\nu - P_\nu^*) u_\nu, u_\nu) + 2 \operatorname{Re} (f, u_\nu). \end{aligned}$$

By the way, from the definitions of (4.1) and (4.5) we have

$$(4.18) \quad J_\nu \equiv \sigma(P_{1\nu}) - \sigma(P_{1\nu}^*) \in C_i^0([0, T]; S_{\lambda_\nu}^0).$$

Thus by Lemma 4.2 and the Calderón-Vaillancourt theorem, for a constant $\gamma > 0$ which independent of ν, t we have

$$\frac{d}{dt} \|u_\nu(t)\|^2 \leq 2\gamma \|u_\nu(t)\|^2 + 2\|f(t)\| \|u_\nu(t)\| \quad (t \in [0, T]; \nu=1, 2, \dots)$$

and

$$\frac{d}{dt} \|u_\nu(t)\| \leq \gamma \|u_\nu(t)\| + \|f(t)\|.$$

Therefore we get (4.13). Moreover from (4.6) we have

$$\frac{d}{dt} A_\nu^j u_\nu = i(P_{1\nu} + [A_\nu^j, P_{1\nu}] A_\nu^{-j} + A_\nu^j c A_\nu^{-j}) A_\nu^j u_\nu + A_\nu^j f$$

and $P_{1\nu, j} = P_{1\nu} + [A_\nu^j, P_{1\nu}] A_\nu^{-j}$. Here, $[A_\nu^j, P_{1\nu}] A_\nu^{-j} \in C_i^0([0, T]; S_{\lambda_\nu}^0)$ and by Lemma 4.3 $A_\nu^j c A_\nu^{-j} \in \mathcal{B}_i^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_i^1([0, T]; S_{0,0}^0)$ uniformly with respect to ν and $P_{1\nu, j}$ satisfies (4.18). Hence we get (4.14) similarly to (4.13). On the other hand, noting

$$\frac{d}{dt} A_\nu^j u_\nu = i(A_\nu^j P_{1\nu} A_\nu^{-j-2} + A_\nu^j c A_\nu^{-j-2}) A_\nu^{j+2} u_\nu + A_\nu^j f$$

where $\sigma(A_\nu^j P_{1\nu} A_\nu^{-j-2}) \in C_b^0([0, T]; S_{\lambda_\nu}^0)$ and $A_\nu^j c A_\nu^{-j-2} \in \mathcal{B}_b^0([0, T]; S_{0,0}^0) \cap \mathcal{B}_b^1([0, T]; S_{0,0}^1)$ (uniformly on ν), we have

$$\left\| \frac{d}{dt} A_\nu^j u_\nu(t) \right\| \leq C_1 \|A_\nu^{j+2} u_\nu\| + \|A_\nu^j f\|.$$

Then from this and (4.14), we get (4.15). Put $u_\nu(t) - u_\nu(t') = \int_{t'}^t \frac{d}{d\tau} u_\nu(\tau) d\tau$. Then we get (4.16) from (4.15). This completes the proof of Lemma 4.4.

PROOF OF THEOREM 4.1. 1) First we assume that $u_0 \in H_4$, $f(t) \in C_b^0([0, T]; H_4)$. Then it follows from Lemma 4.4 that there is a solution $u_\nu(t)$ of (4.6). Then, from (4.9) we have

$$(4.19) \quad \|A_\nu^j u_0\| \leq \|A^j u_0\|, \quad \|A_\nu^j f\| \leq \|A^j f\|$$

for $j \leq 4$, where $A = \langle D_x \rangle$.

Therefore by (4.13), (4.14), (4.16) we have

$$(4.20) \quad \|A_\nu^j u_\nu(t)\| \leq C_1 \{ \|A^j u_0\| + \max_{[0, T]} \|A^j f\| \},$$

$$(4.21) \quad \|A_\nu^j (u_\nu(t) - u_\nu(t'))\| \leq C_2' |t - t'| \{ \|A^{j+2} u_0\| + \max_{[0, T]} \|A^{j+2} f\| \}$$

for $j \leq 4$.

Therefore if we fix $t_0 \in [0, T]$ arbitrarily, then from (4.20) with $j=0$, $\{u_\nu(t_0)\}_{\nu=1}^\infty$ is a bounded sequence in L_2 . Hence there exists $u(t_0) \in L_2$ and a subsequence such that $\{u_{\nu_m}\}_{m=1}^\infty$ of L_2 we have

$$u_{\nu_m} \longrightarrow u(t_0) \text{ (weakly) in } L_2 \quad (m \rightarrow \infty).$$

Let $\{t_k\}_{k=1}^\infty$ be a dense set in $[0, T]$. Then by the diagonal method there exists $u(t_k) \in L_2$ and we have

$$(4.22) \quad u_{\nu_m}(t_k) \longrightarrow u(t_k) \text{ (weakly) in } L_2 \quad (m \rightarrow \infty).$$

Then from (4.21) with $j=0$, for any $t, t' \in \{t_k\}_{k=1}^\infty$ we have

$$(4.23) \quad \|u(t) - u(t')\| \leq C_2 |t - t'| \{ \|A^2 u_0\| + \max_{[0, T]} \|A^2 f\| \}.$$

For any $t_0 \in [0, T]$ we choose subsequence $\{t_{k_s}\}_{s=1}^\infty$ of $\{t_k\}$ such that $t_{k_s} \rightarrow t_0$. Then by (4.23), $\{u(t_{k_s})\}_{s=1}^\infty$ becomes a Cauchy sequence in $L_2(R^n)$. Thus there exists $u(t_0) \in L_2$ and we have $u(t_{k_s}) \rightarrow u(t_0)$ in L_2 ($s \rightarrow \infty$). Moreover for all $u(t)$

we obtain (4.23) and

$$u_{\nu_m}(t) \longrightarrow u(t) \text{ (weakly) in } L_2 \quad (m \rightarrow \infty).$$

Thus for $\varphi \in \mathcal{S}$ we have

$$(4.24) \quad (A_{\nu_m}^k u_{\nu_m}(t), \varphi) = (u_{\nu_m}(t), A_{\nu_m}^k \varphi) \longrightarrow (u(t), A^k \varphi) \quad (m \rightarrow \infty).$$

Therefore noting $|(A_{\nu_m}^k u_{\nu_m}(t), \varphi)| \leq \|A_{\nu_m}^k u_{\nu_m}(t)\| \|\varphi\|$, we get from (4.20) and (4.24),

$$(4.25) \quad |(u(t), A^k \varphi)| \leq C_1 \{\|A^k u_0\| + \max_{[0, T]} \|A^k f\|\} \|\varphi\| \quad (\varphi \in \mathcal{S})$$

for $k \leq 4$. We define $u_{R, k}(t, x) \in L^2$ for any $t \in [0, T]$ and $R > 0$ such that $\hat{u}_{R, k}(t, \xi) = \langle \xi \rangle^k \hat{u}(t, \xi)$ ($|\xi| \leq R$), and $\hat{u}_{R, k}(t, \xi) = 0$ ($|\xi| > R$) and take the sequence $\{\varphi_j\}_{j=1}^\infty$ of \mathcal{S} as $\text{supp } \hat{\varphi} \subset \{|\xi| \leq R\}$, $\hat{\varphi}_j \rightarrow \hat{u}_{R, k}(t, \xi)$ in L_2 ($j \rightarrow \infty$). Then we have

$$(u(t), A^k \varphi_j) = \int \hat{u}(t, \xi) \langle \xi \rangle^k \overline{\hat{\varphi}_j(\xi)} d\xi \longrightarrow \int_{|\xi| \leq R} \langle \xi \rangle^{2k} |\hat{u}(t, \xi)|^2 d\xi \quad (j \rightarrow \infty).$$

Therefore from (4.25) we get

$$\|u_{R, k}(t)\| \leq C_1 \{\|A^k u_0\| + \max_{[0, T]} \|A^k f\|\} \quad (k \leq 4).$$

Hence taking $R \rightarrow \infty$, we have $A^k u \in L_2$ and

$$(4.26) \quad \|A^k u\| \leq C_1 \{\|A^k u_0\| + \max_{[0, T]} \|A^k f\|\} \quad (k \leq 4).$$

From (4.21) we have similarly to (4.26)

$$(4.27) \quad \|A^k(u(t) - u(t'))\| \leq C_2' |t - t'| \{\|A^{k+2} u_0\| + \max_{[0, T]} \|A^{k+2} f\|\} \quad (k \geq 2).$$

Thus we obtain $u(t), Au(t), A^2 u(t) \in C_l^0([0, T]; L_2)$.

We take $\varphi(t, x) = \varphi_1(t) \varphi_2(x) \in C_0^\infty((0, T) \times R^n)$ arbitrarily. Noting

$$L_{\nu_m}^* \varphi \longrightarrow L^* \varphi \quad \text{in } C_l^0([0, T]; L_2),$$

we have

$$\begin{aligned} \iint_{\Omega_T} Lu \cdot \bar{\varphi} dx dt &= \iint_{\Omega_T} u \cdot \overline{L^* \varphi} dx dt \\ &= \lim_{m \rightarrow \infty} \iint_{\Omega_T} u_{\nu_m} \cdot \overline{L_{\nu_m}^* \varphi} dx dt = \lim_{m \rightarrow \infty} \iint_{\Omega_T} L_{\nu_m} u_{\nu_m} \cdot \bar{\varphi} dx dt \\ &= \iint_{\Omega_T} f \cdot \bar{\varphi} dx dt. \end{aligned}$$

Thus $Lu = f$ ($t \in [0, T]$). Therefore u is a solution of (4.2). Also noting $u(t), Au(t), A^2 u(t) \in C_l^0([0, T]; L_2)$ and $\partial_t u = iP(t)u + f$, we get $u(t) \in C_l^1([0, T]; L_2) \cap C_l^0([0, T]; H_2)$.

II) If $u(t) \in C_l^1([0, T]; L_2) \cap C_l^0([0, T]; H_2)$ is a solution of (4.2), we obtain

the energy inequality (4.3) similarly to the proof of Lemma 4.4.

Assume $u_0 \in H_2$ and $f(t) \in C^0([0, T]; H_2)$. Put $u_{0,\varepsilon} = \chi_\varepsilon(D_x)u_0$ and $f_\varepsilon(t) = \chi_\varepsilon(D_x)f(t)$, where $\chi(\xi) \in \mathcal{S}'$ satisfies $\chi(0) = 1$ and $\chi_\varepsilon(\xi) = \chi(\varepsilon\xi)$. Then $u_{0,\varepsilon} \in H_4$, $f_\varepsilon(t) \in C^0([0, T]; H_4)$. Therefore from 1), we get the solution $u_\varepsilon(t)$ of (4.2) for $u_{0,\varepsilon}$ and f_ε . Then by (4.3) we have

$$\|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_2 \leq e^{\gamma_1 T} \{ \|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_2 + T \max_{[0, T]} \|f_\varepsilon(\tau) - f_{\varepsilon'}(\tau)\| \}.$$

On the other hand when $\varepsilon, \varepsilon' \rightarrow 0$ we have

$$\begin{cases} \|u_{0,\varepsilon} - u_{0,\varepsilon'}\|_2 = \|(\chi_\varepsilon(D_x) - \chi_{\varepsilon'}(D_x))u_0\|_2 \longrightarrow 0, \\ \max_{[0, T]} \|f_\varepsilon(\tau) - f_{\varepsilon'}(\tau)\|_2 = \max_{[0, T]} \|(\chi_\varepsilon(D_x) - \chi_{\varepsilon'}(D_x))f(\tau)\|_2 \longrightarrow 0. \end{cases}$$

Therefore, noting $\partial_t u_\varepsilon = iPu_\varepsilon + f_\varepsilon$, we can see $\{u_\varepsilon(t)\}_{0 < \varepsilon < 1}$ becomes a Cauchy sequence in $C^1([0, T]; L_2) \cap C^0([0, T]; H_2)$ and the limit of this series $u(t)$ is a solution of (4.2). This completes the proof of Theorem 4.1.

PROOF OF THEOREM. By Theorem 4.1 we can see that there exists a unique solution $v(x, t) \in C^1([0, T]; H_2) \cap C^0([0, T]; L_2)$ of the Cauchy problem (**) with initial data $u_0(x)$ and we get the energy inequality

$$\|v(\cdot, t)\|_{H_k} \leq C(T) \left(\|u_0\|_{H_k} + \int_0^t \|K^{-1}f(\cdot, \tau)\|_{H_k} d\tau \right)$$

for $t \in [0, T]$ and $k=0, 1, 2$. Hence, we obtain the unique solution $u(x, t) = Kv(x, t)$ of the equation (*) with initial data $u_0(x)$ and by using the Calderón-Vaillancourt theorem we get the energy inequality

$$\|u(\cdot, t)\|_{H_k} \leq C'(T) \left(\|u_0\|_{H_k} + \int_0^t \|f(\cdot, \tau)\|_{H_k} d\tau \right)$$

for $t \in [0, T]$ and $k=0, 1, 2$. Thus we complete the proof of our Theorem.

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References

- [1] H. Kumano-go, Partial differential operators, Kyôritsu shuppan, Tokyo, 1978, (in Japanese).
- [2] H. Kumano-go, Pseudo-Differential Operators, MIT Press, Cambridge, 1982.
- [3] S. Mizohata, On the Cauchy Problem, Notes and Reports in Math., 3, Academic Press, 1985.

- [4] S. Mizohata, Sur quelques équations du type Schrödinger, Séminaire J. Vaillant, 1980/1981, Univ. de Paris-VI.
- [5] J. Takeuchi, Le problème de Cauchy pour quelques équations aux dérivées partielles du type de Schrödinger, C.R. Acad. Sci. Paris, t. 310, Série I, p. 823-826, 1990.

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