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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF WEAK SOLUTIONS TO SEMILINEAR HYPERBOLIC SYSTEMS WITH DAMPING TERMS

By

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1. Introduction

Let Ω be a bounded domain of $\boldsymbol{R}^k$ with Lipschitz boundary $\partial\Omega$. We consider the following system of hyperbolic equations for a map $u:\Omega\times(0,\infty)\to\boldsymbol{R}^\ell$:

$$
\begin{aligned}
&a_{ij}(x)D_t^2u^i(x,t)-D_\beta(b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t))+c_{ij}(x)\|u(x,t)\|_c^{m-2}u^i(x,t)\\
&\qquad+a_{ij}(x)D_tu^i(x,t)=0 \text{ in } \Omega\times(0,\infty),\ \ j=1,\dots,\ell,
\end{aligned}
\tag{1.1}
$$

where $D_t=\partial/\partial t, D_\alpha=\partial/\partial x^\alpha, \|u(x,t)\|_c=(c_{ij}(x)u^i(x,t)u^j(x,t))^{1/2}$ and $m>1$. Here and in the following, summation over repeated indices is understood, the greek indices run from 1 to k, and the latin ones from 1 to ℓ. We assume that the coefficients $a_{ij}(x)$, $b_{ij}^{\alpha\beta}(x)$ and $c_{ij}(x)$ are bounded functions defined on Ω and satisfy the conditions

$$
\begin{cases}
a_{ij}(x)\xi^i\xi^j\ge\lambda_0|\xi|^2 \text{ for all } \xi\in\boldsymbol{R}^\ell,\\
b_{ij}^{\alpha\beta}(x)\eta_\alpha^i\eta_\beta^j\ge\lambda_1|\eta|^2 \text{ for all } \eta\in\boldsymbol{R}^{k\ell},\\
c_{ij}(x)\xi^i\xi^j\ge\lambda_2|\xi|^2 \text{ for all } \xi\in\boldsymbol{R}^\ell,
\end{cases}
\tag{1.2}
$$

$$a_{ij}(x)=a_{ji}(x),\ b_{ij}^{\alpha\beta}(x)=b_{ji}^{\alpha\beta}(x),\ c_{ij}(x)=c_{ji}(x), \tag{1.3}$$

for some positive constants λ_0,λ_1 and λ_2. The initial and boundary conditions are

$$u(x,0)=u_0(x),\ D_tu(x,0)=v_0(x) \text{ in } \Omega, \tag{1.4}$$

$$u(x,t)=w(x) \text{ on } \partial\Omega\times(0,\infty), \tag{1.5}$$

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where $u_0(x), v_0(x)$ and $w(x)$ are given maps such that $u_0(x) = w(x)$ on $\partial\Omega$.

In §2 we shall construct global weak solutions to (1.1), (1.4) and (1.5) by the semi-discretization in time variable combining the variational method (Theorem 2.1). This construction was employed to hyperbolic equations without damping term by Tachikawa [15]. It is very powerful tool to construct global weak solutions, because we need not distinguish technically between single-valued equations and systems of equations. It applied to other various evolution equations in [10, 11, 12, 5, 1, 7, 8].

The method of semi-discretization in time variable, so-called Rothe's method, has been used to construct solutions of parabolic equations since about 60 years ago (see Rothe [13]). Moreover, by Rektorys [12] and Kačur[3], Rothe's method was applied to hyperbolic equations also.

Though the Faedo-Galerkin method is very common to construct weak solutions, it would be fruitful to consider various constructions, since weak solutions of hyperbolic systems are not uniquely determined in general.

In §3 we shall investigate the exponential decay property of solutions in case of $w \equiv 0$ and $m \geq 2$ (Theorem 3.1). For the case that $2 \leq m \leq 2(k-1)/(k-2)$, Zuazua [18, Example 2.6] shows that any weak solution has the exponential decay property. Moreover, it is known that the weak solutions which are given as limit functions of smooth approximate solutions satisfy the exponential decay property (see [9]). For example, the Faedo-Galerkin method gives us the weak solutions satisfying the exponential decay property. On the other hand, the weak solutions constructed in §2 are not given as limits of smooth approximate solutions. Therefore, it is not trivial that they have exponential decay property even if $m > 2(k-1)/(k-2)$. We shall utilize the discrete energy method to approximate solutions, and pass to the limit. In the time-discretized form we can employ various test functions and easily derive discrete energy method.

Other results on hyperbolic equation with damping term can be seen in [14, 6, 17, 19, 20, 4]. In [6, 17, 4] authors investigated global smooth or strong solutions and their asymptotic behavior. Zuazua [19, 20] dealt with equations with localized damping term. See also references cited therein.

2. Construction of weak solutions

In this article we denote $\boldsymbol{R}^l$-valued Sobolev and Lebesgue spaces $H^{1,2}(\Omega;\boldsymbol{R}^l)$, $L^p(\Omega;\boldsymbol{R}^l)$ etc. simply by $H^{1,2}(\Omega)$, $L^p(\Omega)$ etc. We define a weak solution of (1.1) satisfying the initial and boundary conditions (1.4) and (1.5) as follows.

DEFINITION. Let $\gamma_{\partial\Omega}$ and $\gamma_{t=0}$ denote the trace operators to $\partial\Omega$ and $\Omega\times\{0\}$ respectively. For $u_0, w\in H^{1,2}(\Omega)\cap L^m(\Omega)$ and $v_0\in L^2(\Omega)$ satisfying $\gamma_{\partial\Omega}u_0=\gamma_{\partial\Omega}w$, a map $u:\Omega\times[0,T)\to \boldsymbol{R}^\ell$ is called a *weak solution* of (1.1) on [0, T) with the initial and boundary conditions (1.4)–(1.5), if the following conditions are satisfied:

(i) $u\in L^\infty(0,T;L^m(\Omega))\cap L^\infty(0,T;H^{1,2}(\Omega))$ with $D_tu\in L^\infty(0,T;L^2(\Omega))$

(ii) $\gamma_{t=0}u(x,t)=u_0(x)$ and $\gamma_{\partial\Omega}u(x,t)=\gamma_{\partial\Omega}w(x)$ for $0<t<T$

(iii) For any $\psi(x,t)\in C_0^1([0,T);C_0(\Omega))\cap C([0,T);C^1(\Omega))$,

$$\begin{aligned}&\int_0^T\int_\Omega(-a_{ij}(x)D_tu^i(x,t)D_t\psi^j(x,t)+b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)D_\beta\psi^j(x,t)\\ &+c_{ij}(x)\|u(x,t)\|_c^{m-2}u^i(x,t)\psi^j(x,t)+a_{ij}(x)D_tu^i(x,t)\psi^j(x,t))dxdt \qquad (2.1)\\ &=\int_\Omega a_{ij}(x)v_0^i(x)\psi^j(x,0)dx.\end{aligned}$$

We say u is a *global weak solution* if $u|_{\Omega\times[0,T)}$ is a weak solution on [0, T) for any $T>0$.

REMARK. It follows from (i) that $u\in C([0,T];L^2(\Omega))$ (see [16, Chapter III, Lemma 1.1]).

To construct a weak solution of (1.1), we proceed as in [15]. We determine a family $\{u_n\}$ as follows:

(I) (n = 1). Let $v_0(x)=(v_0^1(x),\cdots,v_0^\ell(x))$ be a given map of class $L^2(\Omega)$ as in the above definition. Take $v(x,t)\in L^\infty(\boldsymbol{R};H^{1,2}(\Omega))\cap L^\infty(\boldsymbol{R};L^m(\Omega))$ such that

$$(2.2)\qquad \begin{cases} v(x,0)=0,\ D_tv(x,0)=v_0(x) \text{ in } \Omega,\ v(x,t)=0 \text{ on } \partial\Omega\times\boldsymbol{R},\\ D_tv(\cdot,t) \text{is a weakly continuous map of } t \text{ with values in } L^2(\Omega).\end{cases}$$

Let us define $u_1(x)=u_0(x)+v(x,h)$.

REMARK. To get a map $v(x,t)$ satisfying (2.2), for example, we solve the initial-boundary value problems

$$(2.3)\qquad \begin{cases} D_t^2v^i(x,t)-\Delta v^i(x,t)+|v^i|^{m-2}v^i(x,t)=0 & \text{on } \Omega\times\boldsymbol{R},\\ v^i(x,0)=0,\quad D_tv^i(x,0)=v_0^i(x) & \text{in } \Omega,\\ v^i(x,t)=0 & \text{on } \partial\Omega\times\boldsymbol{R}\end{cases}$$

[14, Theorem 2] guarantees the existence of weak solutions $\{v^i(x,t)\}$ of (2.3) in the class $L^\infty(\boldsymbol{R};H^{1,2}(\Omega))\cap L^\infty(\boldsymbol{R};L^m(\Omega))$ with the weak continuous time derivatives $\{D_tv^i(x,t)\}$. Moreover, they satisfy the energy estimates

$$\int_\Omega \left(\frac{1}{2} | D_t v^i |^2 + \frac{1}{2} \| Dv^i \|^2 + \frac{1}{m} | v^i |^m \right) dx \le \int_\Omega \frac{1}{2} | v_0^i |^2 dx \tag{2.4}$$

for all t, where $\| \cdot \|$ denotes the Euclidean norm, and $D = (D_1, \ldots, D_k)$.

(II) $(n \ge 2)$. Given u_{n-2}, $u_{n-1} \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $h > 0$, we consider the functional

$$\mathscr{F}_n(u) = \int_\Omega \left(\frac{1}{2} \frac{\| u - 2u_{n-1} + u_{n-2} \|_a^2}{h^2} + \frac{1}{2} \| Du \|_b^2 + \frac{1}{m} \| u \|_c^m + \frac{1}{2} \frac{\| u - u_{n-2} \|_a^2}{2h} \right) dx$$

for $u \in H^{1,2}(\Omega) \cap L^m(\Omega)$ with $u = w$ on $\partial\Omega$. Here $\| u \|_a^2 = a_{ij}(x) u^i u^j$, $\| \eta \|_b^2 = b_{ij}^{\alpha\beta}(x) \eta_\alpha^i \eta_\beta^j$ For $n \ge 2$, let $u_n(x)$ be a minimizer of $\mathscr{F}_n$ in the class $\{ u \in H^{1,2}(\Omega) \cap L^m(\Omega) : u = w \text{ on } \partial\Omega \}$.

The Euler-Lagrange equation of $\mathscr{F}_n(u)$ is

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \mathscr{F}_n(u + \varepsilon\varphi) \Big|_{\varepsilon = 0} \\ &= \int_\Omega \left\{ \frac{1}{h^2} a_{ij}(x)(u^i - 2u_{n-1}^i + u_{n-2}^i) \varphi^j + b_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta \varphi^j + c_{ij}(x) \| u \|_c^{m-2} u^i \varphi^j \right. \\ &\quad \left. + \frac{1}{2h} a_{ij}(x)(u^i - u_{n-2}^i) \varphi^j \right\} dx \qquad \text{for all } \varphi \in H_0^{1,2}(\Omega) \cap L^m(\Omega) \end{aligned} \tag{2.5}$$

The lower semicontinuity of L^p-norms guarantees the existence of a minimizer of $\mathscr{F}_n$. Moreover one can see that a minimizer satisfies (2.5) by means of differentiability of the integrand of $\mathscr{F}_n$ with respect to Du and u. About general theory of the direct method of calculus of variations see [2, Chapter I].

Thus $u_n (n \ge 2)$ satisfies (2.5) and we get the following lemma.

LEMMA 2.1. *Let $\{u_n\}$ be as above. Then we have the energy estimates*

$$\frac{1}{2} \left(\int_\Omega \frac{\| u_n - u_{n-1} \|_a^2}{h^2} dx + \mathcal{E}(u_n) + \mathcal{E}(u_{n-1}) \right) \le K_0 \tag{2.6}$$

for some positive constant K_0 depending on u_0 and v_0, where

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2} \| Du \|_b^2 + \frac{1}{m} \| u \|_c^m \right) dx$$

PROOF. Since u_n and u_{n-2} coincide on $\partial\Omega$, $u_n - u_{n-2} (n \ge 2)$ is an admissible test function for (2.5). Thus using Young's inequality, we get

$$0 = \frac{d}{d\varepsilon}\mathscr{F}_n(u_n + \varepsilon(u_n - u_{n-2}))\big|_{\varepsilon=0}$$

$$= \int_\Omega \left\{ \frac{1}{h^2} a_{ij}(u_n^i - 2u_{n-1}^i + u_{n-2}^i)(u_n^j - u_{n-2}^j) + b_{ij}^{\alpha\beta} D_\alpha u_n^i (D_\beta u_n^j - D_\beta u_{n-2}^j) \right.$$

$$\text{(2.7)} \qquad \left. + c_{ij} \| u_n \|_c^{m-2} u_n^i (u_n^j - u_{n-2}^j) + \frac{\| u_n - u_{n-2} \|_a^2}{2h} \right\} dx$$

$$\geq \int_\Omega \left\{ \left(\frac{\| u_n - u_{n-1} \|_a^2}{h^2} + \frac{1}{2} \| Du_n \|_b^2 + \frac{1}{m} \| u_n \|_c^m \right) + \frac{\| u_n - u_{n-2} \|_a^2}{2h} \right.$$

$$\left. - \left(\frac{\| u_{n-1} - u_{n-2} \|_a}{h^2} + \frac{1}{2} \| Du_{n-2} \|_b^2 + \frac{1}{m} \| u_{n-2} \|_c^m \right) \right\} dx$$

Now, let

$$\begin{cases} a_n = \int_\Omega \dfrac{\| u_n - u_{n-1} \|_a^2}{h^2} dx, \\ b_n = \int_\Omega \left(\dfrac{1}{2} \| Du_n \|_b^2 + \dfrac{1}{m} \| u_n \|_c^m \right) dx \end{cases}$$

Then (2.7) implies

$$a_n + b_n + b_{n-1} \leq a_{n-1} + b_{n-1} + b_{n-2} \leq \cdots \leq a_1 + b_1 + b_0$$

On the other hand, the definition of u_1 and (2.4) imply that

$$a_1 = \frac{1}{h^2} \int_\Omega \| v(x,h) \|_a^2 dx \leq \frac{c}{h^2} \int_\Omega \left(h \int_0^h \| D_t v(x,t) \|^2 dt \right) dx$$

$$\leq \frac{c}{h} \int_0^h \int_\Omega \| v_0(x) \|^2 dx dt \leq c' \int_\Omega h \| v_0(x) \|^2 dx,$$

where c and c' are constants depending only on (a_{ij}). From the above estimates, remarking (2.4) again, we get (2.6).

Now, using $\{u_n(x)\}$, we construct two maps u_h and $\bar{u}_h$ which approximate to a weak solution of (1.1). Let us define

$$\begin{cases} \bar{u}_h(x,t) = \begin{cases} u_0(x) & \text{for} \quad t = 0, \\ u_n(x) & \text{for} \quad (n-1)h < t \leq nh, \ n \geq 1, \end{cases} \\ u_h(x,t) = \begin{cases} u_0(x) + v(x,t) & \text{for} \quad -1 \leq t \leq h, \\ \dfrac{t-(n-1)h}{h} u_n(x) + \dfrac{nh-t}{h} u_{n-1}(x) & \text{for} \quad (n-1)h < t \leq nh, \ n \geq 2 \end{cases} \end{cases}$$

Then, we can proceed as in [15, §3] and see that $\bar{u}_h$ and u_h converge to a weak solution of the equation (1.1) which satisfies the conditions (1.4) and (1.5).

From (2.5), we can see that

$$\begin{aligned}
&\int_0^T \int_\Omega \Big\{ \frac{1}{h} a_{ij}(x)\big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h)\big)\varphi^j(x) \\
&\quad + b_{ij}^{\alpha\beta}(x) D_\alpha \bar{u}_h^i(x,t) D_\beta \varphi^j(x) + c_{ij}(x) \|\bar{u}_h(x,t)\|_c^{m-2} \bar{u}_h^i(x,t)\varphi^j(x) \\
&\quad + \frac{1}{2} a_{ij}(x)\big(D_t u_h^i(x,t) + D_t u_h^i(x,t-h)\big)\varphi^j(x) \Big\} \eta(t)\,dxdt \\
&- \int_0^h \int_\Omega \Big\{ \frac{1}{h} a_{ij}(x)\big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h)\big)\varphi^j(x) \\
&\quad + b_{ij}^{\alpha\beta}(x) D_\alpha \bar{u}_h^i(x,t) D_\beta \varphi^j(x) + c_{ij}(x) \|\bar{u}_h(x,t)\|_c^{m-2} \bar{u}_h^i(x,t)\varphi^j(x) \\
&\quad + \frac{1}{2} a_{ij}(x)\big(D_t u_h^i(x,t) + D_t u_h^i(x,t-h)\big)\varphi^j(x) \Big\} \eta(t)\,dxdt = 0
\end{aligned} \tag{2.8}$$

for any $T > 0$ and $\eta \in C_0^\infty[0,T)$.

On the other hand, from (2.6), we get the estimates

$$\operatorname*{ess\,sup}_{-1<t<T} \int_\Omega \| D_t u_h \|_a^2 dx \le 2K_0, \tag{2.9}$$

$$\int_{-1}^T \int_\Omega \| D_t u_h \|_a^2 dxdt \le 2K_0(T+1), \tag{2.10}$$

$$\int_{-1}^T \mathcal{E}(u_h)dt \le 2K_0(T+1), \tag{2.11}$$

$$\int_0^T \mathcal{E}(\bar{u}_h)dt \le 2K_0 T \tag{2.12}$$

Using the Banach-Alaoglu theorem, from (2.9), (2.10) and (2.11) we can deduce that

$$D_t u_h \rightharpoonup D_t u,\ D_\alpha u_h \rightharpoonup D_\alpha u \ \text{ weakly in } L^2(\Omega\times(-1,T)), \tag{2.13}$$

$$u_h \rightharpoonup u \ \text{ weakly in } L^{m'}(\Omega\times(-1,T)), \tag{2.14}$$

$$D_t u_h \rightharpoonup u' \ \text{ weakly star in } L^\infty(-1,T;L^2(\Omega)) \tag{2.15}$$

for some $u \in L^m(\Omega\times(-1,T)) \cap H^{1,2}(\Omega\times(-1,T))$ and $u' \in L^\infty(-1,T;L^2(\Omega))$ as $h \downarrow 0$ taking a subsequence if necessary. Here $m' = \max\{2,m\}$. In what follows $h \downarrow 0$ means always a limit along a suitable subsequence. Since (2.13) and (2.15) imply that $D_t u = u'$ almost everywhere on $\Omega\times(-1,T)$, we can see that $D_t u \in L^\infty(-1,T;$ $L^2(\Omega))$. Moreover, using Rellich's compactness theorem, from (2.13) and (2.14), we get

$$u_h \to u \quad \text{strongly in } L^2(\Omega\times(-1,T)) \quad \text{as} \quad h\downarrow 0 \tag{2.16}$$

Using the Banach-Alaoglu theorem again, by (2.12) we obtain that

$$\begin{cases} D_\alpha \bar{u}_h \rightharpoonup D_\alpha \tilde{u} & \text{weakly in } L^2(\Omega\times(0,T)), \\ \bar{u}_h \rightharpoonup \tilde{u} & \text{weakly in } L^{m'}(\Omega\times(0,T)) \end{cases}$$

as $h\downarrow 0$ for some $\tilde{u}\in L^{m'}(\Omega\times(0,T))$ with $D_\alpha \tilde{u}\in L^2(\Omega\times(0,T))$ taking a subsequence if necessary.

Moreover, by the definition of u_h and $\bar{u}_h$ and (2.9), we have

$$\int_0^T\int_\Omega \| \bar{u}_h - u_h \|_a^2 dxdt \le ch^2K_0T \to 0 \quad as \quad h\downarrow 0 \tag{2.17}$$

for some constant c depending only on the matrix (a_{ij}). Hence, using (2.16) and (2.17), we see that $\bar{u}_h \to u$ in $L^2(\Omega\times(0,T))$. This implies that $\tilde{u}=u$ almost everywhere and therefore $D_\alpha\tilde{u} = D_\alpha u$ almost everywhere on $\Omega\times(0,T)$. Thus we obtain

$$\begin{cases} \bar{u}_h \rightharpoonup u \text{ weakly in } L^{m'}(\Omega\times(0,T)), \\ \bar{u}_h \to u \text{ strongly in } L^2(\Omega\times(0,T)), \\ D_\alpha \tilde{u} \rightharpoonup D_\alpha u \text{ weakly in } L^2(\Omega\times(0,T)) \end{cases} \tag{2.18}$$

as $h\downarrow 0$.

For any $\eta(t)\in C_0^\infty[0,T)$, if h is small so that spt $\eta\subset[0,T-h)$, then

$$\begin{aligned} &\int_0^T\int_\Omega \frac{1}{h} a_{ij}(x)\big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h)\big)\varphi^j(x)\eta(t)dxdt \\ &= \int_0^T\int_\Omega a_{ij}(x) D_t u_h^i(x,t)\varphi^j(x)\frac{\eta(t)-\eta(t+h)}{h}dxdt \\ &\quad -\frac{1}{h}\int_{-h}^0\int_\Omega a_{ij}(x)D_t v^i(x,t)\varphi^j(x)\eta(t+h)dxdt \end{aligned} \tag{2.19}$$

The weak continuity of $D_t v$ implies that

$$\frac{1}{h}\int_{-h}^0\int_\Omega a_{ij}(x)D_t v^i(x,t)\varphi^j(x)\eta(t+h)dxdt \to \int_\Omega a_{ij}(x)v_0^i(x)\varphi^j(x)\eta(0)dx \tag{2.20}$$

as $h\downarrow 0$. From (2.19) and (2.20), we obtain

$$\begin{aligned} &\int_0^T\int_\Omega \frac{1}{h} a_{ij}(x)\big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h)\big)\varphi^j(x)\eta(t)dxdt \\ &\to -\int_0^T\int_\Omega a_{ij}(x)D_t u^i(x,t)\varphi^j(x)D_t\eta(t)dxdt - \int_\Omega a_{ij}(x)v_0^i(x)\varphi^j(x)\eta(0)dx \end{aligned} \tag{2.21}$$

as $h \downarrow 0$.

Because of (2.18), by means of Egoroff's theorem, we get

$$\left| \int_0^T \int_\Omega c_{ij} \| \bar{u}_h \|_a^{m-2} \bar{u}_h^i \varphi^j \eta dxdt - \int_0^T \int_\Omega c_{ij} \| u \|_c^{m-2} u^i \varphi^j \eta dxdt \right| \to 0 \text{ as } h \downarrow 0, \tag{2.22}$$

taking a subsequence if necessary.

Next, let us see the convergence of the damping term. Using (2.13), we have

$$\begin{aligned}
&\left| \int_0^T \int_\Omega a_{ij}(x)\big(D_t u_h^i(x,t-h) - D_t u^i(x,t)\big)\varphi^j(x)\eta(t)dxdt \right| \\
&\le \left| \int_\Omega a_{ij}(x) D_t u_h^i(x,t)\varphi^j(x)\eta(t)dxdt - \int_0^T \int_\Omega a_{ij}(x) D_t u^i(x,t)\varphi^j(x)\eta(t)dxdt \right| \\
&\quad + \left| \int_{-h}^{T-h} \int_\Omega a_{ij}(x) D_t u_h^i(x,t)\varphi^j(x)(\eta(t+h) - \eta(t))dxdt \right| \\
&\le \left| \int_0^T \int_\Omega a_{ij}(x)\big(D_t u_h^i(x,t) - D_t u^i(x,t)\big)\varphi^j(x)\eta(t)dxdt \right| \\
&\quad + \left| \int_{-h}^{0} \int_\Omega a_{ij}(x) D_t u_h^i(x,t)\varphi^j(x)\eta(t)dxdt \right| \\
&\quad + \left| \int_{T-h}^{T} \int_\Omega a_{ij}(x) D_t u_h^i(x,t)\varphi^j(x)\eta(t)dxdt \right| \\
&\quad + \left| \int_{-h}^{T-h} \int_\Omega a_{ij}(x) D_t u_h^i(x,t)\varphi^j(x)(\eta(t+h) - \eta(t))dxdt \right| \\
&\to 0
\end{aligned}$$

as $h \downarrow 0$. Therefore, we get

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{1}{2} a_{ij}(x)\big(D_t u_h^i(x,t) + D_t u_h^i(x,t-h)\big)\varphi^j(x)\eta(t)dxdt \\
&\to \int_0^T \int_\Omega a_{ij}(x) D_t u^i(x,t)\varphi^j(x)\eta(t)dxdt \text{ as } h \downarrow 0
\end{aligned} \tag{2.23}$$

Using (2.20) again, we can see that

$$\int_0^{\mathrm{h}} \int_\Omega \frac{1}{h} a_{ij}(x)\big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h)\big)\varphi^j(x)\eta(t)dxdt$$

$$(2.24) \quad = \int_0^h \int_\Omega \frac{1}{h} a_{ij}(x)\big(D_t v^i(x,t) - D_t v^i(x,t-h)\big)\varphi^j(x)\eta(t)dxdt$$
$$\to \int_\Omega a_{ij}(x)v_0^i(x)\varphi^j(x)\eta(0)dx - \int_\Omega a_{ij}(x)v_0^i(x)\varphi^j(x)\eta(0)dx = 0$$

as $h \downarrow 0$. Moreover, it is easy to see that

$$(2.25) \quad \begin{aligned} &\int_0^h \int_\Omega \Big\{ b_{ij}^{\alpha\beta}(x) D_\alpha \bar{u}_h^i(x,t) D_\beta \varphi^j(x) + c_{ij}(x) \| \bar{u}_h(x,t) \|_c^{m-2} \bar{u}_h^i(x,t)\varphi^j(x) \\ &+ \frac{1}{2} a_{ij}(x)\big(D_t u_h^i(x,t) + D_t u_h^i(x,t-h)\big)\varphi^j(x) \Big\} \eta(t)dxdt \to 0 \text{ as } h \downarrow 0 \end{aligned}$$

Now, letting $h \downarrow 0$ in (2.8) and using (2.13), (2.18), (2.21), (2.22), (2.23), (2.24) and (2.25) we obtain

$$(2.26) \quad \begin{aligned} &\int_0^T \int_\Omega (-a_{ij}(x)D_t u^i(x,t)\varphi^j(x)D_t\eta(t) + b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)D_\beta\varphi^j(x)\eta(t) \\ &+ c_{ij}(x) \| u(x,t) \|_c^{m-2} u^i(x,t)\varphi^j(x)\eta(t) + a_{ij}(x)D_t u^i \varphi^j(x)\eta(t)(t))dxdt \\ &= \int_\Omega a_{ij}(x)v_0^i(x)\varphi^j(x)\eta(0)dx, \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$, and for all $\eta \in C_0^\infty[0,T)$. Since functions of the form $\varphi(x)\eta(t)$ are total in the space $C^1([0,T);C_0(\Omega)) \cap C([0,T);C^1(\Omega))$, (2.26) means that u satisfies (2.1).

On the other hand, since $u_h(x,0) = u_0(x)$, $u_h|_{\partial\Omega\times[-1,\infty)} = w$ and $u_h \rightharpoonup u$ in $H^{1,2}(\Omega\times(-1,\mathrm{T}))$ as $h \downarrow 0$, we can see that u satisfies the initial condition $u(x,0) = u_0(x)$ and the boundary condition $u|_{\partial\Omega\times(0,\infty)} = w$ also. Using diagonal argument, we get a global weak solution.

THEOREM 2.1. *Let Ω be a bounded domain of $\boldsymbol{R}^k$ with Lipschitz boundary $\partial\Omega$. Suppose that* (1.2) *and* (1.3) *are satisfied. For any $v_0 \in L^2(\Omega)$ and $u_0, w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ with $\gamma_{\partial\Omega}u_0 = \gamma_{\partial\Omega}w$, there exists a global weak solution of* (1.1) *which satisfies the initial and boundary conditions* (1.4) *and* (1.5).

3. Asymptotic behavior

In this section we show the exponential decay property for the weak solution of (1.1) which is constructed in the previous section. In the following we treat only the case that $m \geq 2$ and the boundary conditions are

$$(3.1) \quad u(x,t) = 0 \text{ on } \partial\Omega\times(0,\infty)$$

We test (2.5) by $\varphi = u_n - u_{n-1}$ to get

$$0 = \frac{d}{d\varepsilon}\mathscr{F}_n(u_n + \varepsilon(u_n - u_{n-1}))\Big|_{\varepsilon=0}$$

$$= \int_\Omega \Big[\frac{1}{h^2}a_{ij}\{(u_n^i - u_{n-1}^i) - (u_{n-1}^i - u_{n-2}^i)\}(u_n^j - u_{n-1}^j)$$

$$+ b_{ij}^{\alpha\beta}D_\alpha u_n^i(D_\beta u_n^j - D_\beta u_{n-1}^j) + c_{ij}\| u_n \|_c^{m-2} u_n^i(u_n^j - u_{n-1}^j)$$

$$+ \frac{1}{2h}a_{ij}(u_n^i - u_{n-2}^i)(u_n^j - u_{n-1}^j)\Big]dx \tag{3.2}$$

$$= \int_\Omega \Big[\frac{1}{h^2}\{\| u_n - u_{n-1}\|_a^2 - a_{ij}(u_{n-1}^i - u_{n-2}^i)(u_n^j - u_{n-1}^j)\}$$

$$+ \| Du_n \|_b^2 - b_{ij}^{\alpha\beta}D_\alpha u_n^i D_\beta u_{n-1}^i + \| u_n \|_c^m - \| u_n \|_c^{m-2} c_{ij}u_n^i u_{n-1}^j$$

$$+ \frac{1}{2h}\| u_n - u_{n-1}\|_a^2 + \frac{1}{2h}a_{ij}(u_n^i - u_{n-1}^i)(u_n^j - u_{n-2}^j)\Big]dx$$

Thus dividing (3.2) by h and using Young's inequality, we get

$$0 \geq \int_\Omega \Big\{\frac{1}{h}\Big(\frac{\| u_n - u_{n-1}\|_a^2}{2h^2} - \frac{\| u_{n-1} - u_{n-2}\|_a^2}{2h^2}\Big)$$

$$+ \frac{1}{h}\Big(\frac{1}{2}\| Du_n \|_b^2 - \frac{1}{2}\| Du_{n-1}\|_b^2\Big) + \frac{1}{h}\Big(\frac{1}{m}\| u_n \|_c^m - \frac{1}{m}\| u_{n-1}\|_c^m\Big) \tag{3.3}$$

$$+ \frac{\| u_n - u_{n-1}\|_a^2}{2h^2} + \frac{1}{2h^2}a_{ij}(u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j)\Big\}dx$$

Since we are posing the homogeneous boundary condition, u_n is an admissible test function for (2.5) too. Therefore we can see that

$$0 = \frac{d}{d\varepsilon}\mathscr{F}_n(u_n + \varepsilon u_n)\Big|_{\varepsilon=0}$$

$$= \int_\Omega \Big\{\frac{1}{h^2}a_{ij}(u_n^i - 2u_{n-1}^i + u_{n-2}^i)u_n^j + \| Du_n \|_b^2 + \| u_n \|_c^m + \frac{1}{2h}a_{ij}(u_n^i - u_{n-2}^i)u_n^j\Big\}dx$$

$$= \int_\Omega \Big\{\frac{1}{h}\Big(a_{ij}u_n^i\frac{u_n^j - u_{n-1}^j}{h} - a_{ij}u_{n-1}^i\frac{u_n^j - u_{n-2}^j}{h}\Big) - \frac{1}{h^2}a_{ij}(u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j)$$

$$+ \| Du_n \|_b^2 + \| u_n \|_c^m + \frac{1}{h}a_{ij}u_n^i(u_n^j - u_{n-1}^j) - \frac{1}{2h}a_{ij}(u_n^i - 2u_{n-1}^i + u_{n-2}^i)u_n^j\Big\}dx$$

Thus we get

$$\int_\Omega \frac{1}{h^2}a_{ij}(u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j)dx$$

$$(3.4)\qquad = \int_\Omega \left\{ \frac{1}{h}\left(a_{ij}u_n^i \frac{u_n^j - u_{n-1}^j}{h} - a_{ij}u_{n-1}^i \frac{u_{n-1}^j - u_{n-2}^j}{h} \right) + a_{ij}u_n^i \frac{u_n^j - u_{n-1}^j}{h} \right.$$
$$\left. + \| Du_n \|_b^2 + \| u_n \|_c^m - \frac{1}{2h} a_{ij}(u_n^i - 2u_{n-1}^i + u_{n-2}^i)u_n^j \right\} dx$$

On the other hand, $0 = \frac{d}{d\varepsilon}\mathscr{F}_n(u_n + \varepsilon\ u_n)|_{\varepsilon=0}$ implies the estimates

$$\left| \int_\Omega \frac{1}{h^2} a_{ij}\left(u_n^i - 2u_{n-1}^i + u_{n-2}^i\right)u_n^j dx \right| \le \int_\Omega \left(\| Du_n \|_b^2 + \| u_n \|_c^m + \frac{\| u_n - u_{n-2} \|_a^2}{2h^2} + \frac{\| u_n \|_a^2}{2} \right) dx$$

Remarking that with the help of Poincaré's inequality the right-hand side of the above inequality is estimated by the energy estimate (2.6), we obtain

$$(3.5)\qquad \left| \int_\Omega \frac{1}{h} a_{ij}\left(u_n^i - 2u_{n-1}^i + u_{n-2}^i\right)u_n^j dx \right| \le hK_1,$$

where K_1 is a constant depending only on K_0 and Ω.
Now, inserting (3.4) into (3.3) and using (3.5), we obtain

$$0 \ge \int_\Omega \left\{ \frac{1}{h}\left(\frac{\| u_n - u_{n-1} \|_a^2}{2h^2} - \frac{\| u_{n-1} - u_{n-2} \|_a^2}{2h^2} \right) + \frac{\| u_n - u_{n-1} \|_a^2}{2h^2} \right.$$
$$+ \frac{1}{h}\left(\frac{1}{2}\| Du_n \|_b^2 - \frac{1}{2}\| Du_{n-1} \|_b^2 \right) + \frac{1}{2}\| Du_n \|_b^2$$
$$(3.6)\qquad + \frac{1}{h}\left(\frac{1}{m}\| u_n \|_c^m - \frac{1}{m}\| u_{n-1} \|_c^m \right) + \frac{1}{2}\| u_n \|_c^m$$
$$\left. + \frac{1}{h}\left(\frac{1}{2} a_{ij}u_n^i \frac{u_n^j + u_{n-1}^j}{h} - \frac{1}{2} a_{ij}u_{n-1}^i \frac{u_{n-1}^j + u_{n-2}^j}{h} \right) + \frac{1}{2} a_{ij}u_n^i \frac{u_n^j + u_{n-1}^j}{h} \right\} dx$$
$$- hK_1$$

Let u_h and $\bar{u}_h$ as in the previous section and put

$$(3.7)\qquad \Psi_h(t) := \int_\Omega \left(\frac{1}{2}\| D_t u_h \|_a^2 + \frac{1}{2} a_{ij}\bar{u}_h^i D_t u_h^j + \frac{1}{2}\| D\bar{u}_h \|_b^2 + \frac{1}{m}\| \bar{u}_h \|_c^m \right) dx$$

Then from (3.6) we can deduce that

$$\frac{\Psi_h(t) - \Psi_h(t-h)}{h} + \Psi_h(t) \le hK_1,$$

because $m \ge 2$. For any $t \in (0,\infty)$, putting $n = \lceil t/h \rceil$ ($\lceil\ \rceil$ denotes the ceiling *i.e.*, $\lceil x \rceil$ is the smallest integer greater than or equal to x), the above difference inequality implies that

$$\Psi_h(t)=\Psi_h(nh)\le\left(\frac{1}{1+h}\right)^n\Psi_h(+0)+\sum_{k=1}^{n}\left(\frac{1}{1+h}\right)^k h^2K_1 \tag{3.8}$$
$$\le\left(\frac{1}{1+h}\right)^n\Psi_h(+0)+hK_1$$

Remark that $\Psi_h(+0)$ is dominated by a constant $K_2(u_0,v_0)$ which is independent of h. Since we are assuming that Ω is bounded, we can use Poincaré's inequality. Therefore it follows from (3.8) that

$$\frac{1}{2}\int_\Omega a_{ij}\bar{u}_h^i D_t u_h^j(x,t)dx+C_0\int_\Omega \|\bar{u}_h\|_a^2(x,t)dx\le(1+h)^{-n}K_2+hK_1 \tag{3.9}$$

where C_0 depends only on $(a_{ij}),(b_{ij}^{\alpha\beta})$ and Ω. Multiplying the both side of (3.9) by $\eta\in C_0^\infty[0,\infty)$ with $\eta(t)\ge 0$, and integrating them from 0 to ∞ we get

$$\int_0^\infty\int_\Omega\left(\frac{1}{2}a_{ij}\bar{u}_h^i D_t u_h^j(x,t)+C_0\|\bar{u}_h\|_a^2(x,t)\right)\eta(t)dxdt \tag{3.10}$$
$$\le\int_0^\infty\left\{(1+h)^{-n}K_2+hK_1\right\}\eta(t)dt$$

Remark that $\bar{u}_h,u_h\to u$ and $D_t u_h\rightharpoonup D_t u$ in $L^2(\Omega\times(0,T))$ for any $T\in(0,\infty)$ taking subsequence if necessary (see [15]) and that

$$(1+h)^{-n}\le\{(1+h)^{1/h}\}^{-t}\to e^{-t}\quad\text{as } h\downarrow 0$$

Hence letting $h\downarrow 0$ in (3.10) and taking subsequence if necessary, we obtain

$$\int_0^\infty\int_\Omega\left(\frac{1}{2}a_{ij}u^i D_t u^j(x,t)+C_0\|u\|_a^2(x,t)\right)\eta(t)dxdt\le\int_0^\infty K_2e^{-t}\eta(t)dt, \tag{3.11}$$

for all $\eta\in C_0^\infty[0,\infty)$ with $\eta(t)\ge 0$. We recall that u belongs to $C([0,T];L^2(\Omega))$ and $D_t u$ to $L^\infty(0,T;L^2(\Omega))$. Therefore (3.11) implies that

$$D_t\int_\Omega\|u\|_a^2(x,t)dx+C_0\int_\Omega\|u\|_a^2(x,t)dx\le K_3e^{-t}\ \text{almost every } t\in(0,\infty), \tag{3.12}$$

where $K_3=K_3((a_{ij}),K_2)$. It is easy to see that the estimate (3.12) implies

$$\|u(\cdot,t)\|_{L^2(\Omega)}^2\le Ke^{-C_1t} \tag{3.13}$$

where K is a positive constant depending only on coefficients of the equation, the initial data and Ω, and C_1 is a positive constant depending only on coefficients of the equation and Ω.

From (3.7) and (3.8) we have

$$\int_\Omega \frac{1}{4} \| D_t u_h \|_a^2 (x,t)dx + \mathcal{E}(\overline{u}_h(\cdot,t)) \leq (1+h)^{-n} K_2 + hK_1 + C_2 \int_\Omega \| \overline{u}_h \|^2 (x,t)dx$$

Using the lower semicontinuity of the left-hand side, (2.13), (2.18) and (3.13), we get the exponential decay property of $\| D_t u \|^2_{L^2(\Omega)}$ and the energy $\mathcal{E}(u)$ by $h \to 0$.

Thus we obtain the following theorem.

THEOREM 3.1. *Let* $m \geq 2$ *and* $u(x,t)$ *be the weak solution of* (1.1) *with conditions* (1.4) *and* (3.1) *which is constructed in the previous section. Then* $u(x,t)$ *enjoys the following exponential decay property*

(3.14) $\| u(\cdot,t) \|^2_{L^2(\Omega)} + \| D_t u(\cdot,t) \|^2_{L^2(\Omega)} + \mathcal{E}(u(\cdot,t)) \leq K_{e^{-Ct}}$ for almost every $t \geq 0$

where K *is a positive constant depending only on coefficients of the equation, the initial data and* Ω, *and* C *is a positive constant depending only on coefficients of the equation and* Ω.

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