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UNIQUENESS OF CERTAIN 3-DIMENSIONAL HOMOLOGICALLY VOLUME MINIMIZING SUBMANIFOLDS IN COMPACT SIMPLE LIE GROUPS

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1. Introduction.

The purpose of this paper is to prove uniqueness of certain 3-dimensional homologically volume minimizing submanifolds in compact simple Lie groups.

Let G be a connected compact simple Lie group whose rank is greater than 1 and G_1 be an analytic subgroup of G associated with the highest root of G. The explicit definition of G_1 will be found in Section 2. It is well known that the homology class $[G_1]$ represented by G_1 generates the real homology group $H_3(G;\boldsymbol{R})$ of G. Furnishing G with a bi-invariant Riemannian metric $\langle\ ,\ \rangle$, we consider a volume minimizing submanifold contained in the real homology class $[G_1]$. Using the notion of calibration introduced by Harvey-Lawson [1], the second named author has proved the following theorem in his paper [5].

THEOREM 1. *If M is a compact oriented 3-dimensional submanifold of G contained in the real homology class $[G_1]$, then*

$$\mathrm{vol}\,(G_1)\leqq \mathrm{vol}\,(M)\,.$$

In this paper we investigate submanifolds M contained in $[G_1]$ which satisfy the equality:

$$\mathrm{vol}\,(G_1)=\mathrm{vol}\,(M)$$

and obtain the following theorem.

THEOREM 2. *Let M be a compact oriented 3-dimensioual submanifold of G contained in $[G_1]$. The equality*

$$\mathrm{vol}\,(G_1)=\mathrm{vol}\,(M)$$

holds if and only if M is congruent with G_1 in G. In particular, G_1 is a unique volume minimizing submanifold contained in $[G_1]$ up to congruence in G.

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REMARK. Theorem 2 is an affirmative answer to the problem posed in [5, p. 126 Remark].

2. Preliminaries.

Let $\mathfrak{g}$ be the Lie algebra of G. Take a maximal Abelian subalgebra $\mathfrak{t}$ in $\mathfrak{g}$, then the complexification $\mathfrak{t}^C$ of $\mathfrak{t}$ is a Cartan subalgebra of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. For each element α in $\mathfrak{t}$, put

$$\mathfrak{g}_\alpha=\{X\in\mathfrak{g}^C ; [H, X]=\sqrt{-1}\langle\alpha, H\rangle X \text{ for each } H\in\mathfrak{t}\}.$$

An element α in $\mathfrak{t}-\{0\}$ is called a *root* if $\mathfrak{g}_\alpha\neq\{0\}$. Let Δ denote the set of all roots. We obtain a direct sum decomposition of $\mathfrak{g}^C$:

$$\mathfrak{g}^C=\mathfrak{t}^C+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha .$$

Fix a lexicographic ordering on $\mathfrak{t}$ and denote by Δ_+ the set of all positive roots in Δ.

The following lemma follows from the above direct sum decomposition of $\mathfrak{g}^C$. For details of the proof, see Section 3 of Chapter VI in Helgason [2].

LEMMA 3. *There exist unit vectors E_α, F_α in $\mathfrak{g}$ for each $\alpha\in\Delta_+$ in such a way that:*

i) $$\mathfrak{g}=\mathfrak{t}+\sum_{\alpha\in\Delta_+}\boldsymbol{R}E_\alpha+\sum_{\alpha\in\Delta_+}\boldsymbol{R}F_\alpha$$

is an orthogonal direct sum decomposition of $\mathfrak{g}$;

ii) $$[H, E_\alpha]=\langle\alpha, H\rangle F_\alpha , \quad [H, F_\alpha]=-\langle\alpha, H\rangle E_\alpha , \quad [E_\alpha, F_\alpha]=\alpha$$

for $\alpha\in\Delta_+$ *and* $H\in\mathfrak{t}$.

Let δ be the highest root in Δ_+ and set

$$\mathfrak{g}_1=\boldsymbol{R}\delta+\boldsymbol{R}E_\delta+\boldsymbol{R}F_\delta .$$

Then $\mathfrak{g}_1$ is a compact 3-dimensional simple Lie subalgebra of $\mathfrak{g}$. Let G_1 be the analytic subgroup of G corresponding to $\mathfrak{g}_1$. Wolf has proved that G_1 is simply connected when G is centerless in the proof of Theorem 5.4 in [6]. Therefore G_1 is simply connected, even if G has a nontrivial center.

Put

$$\phi(X, Y, Z)=\frac{1}{|\delta|}\langle[X, Y], Z\rangle$$

for X, Y, and Z in $\mathfrak{g}$. By regarding an element of $\mathfrak{g}$ as a left-invariant vector field on G, ϕ is a bi-invariant 3-form on G. In particular, ϕ is a closed form on G.

We introduce an orientation on $\mathfrak{g}_1$ such that $\{\delta, E_\delta, F_\delta\}$ is a positive basis of $\mathfrak{g}_1$.

LEMMA 4. ([5]) *For each* 3*-dimensional oriented subspace* ξ *in* $\mathfrak{g}$, *the inequality*

$$\phi|_\xi \leqq \mathrm{vol}_\xi$$

holds. The equality holds if and only if there is an element g *in* G *such that*

$$\xi = \mathrm{Ad}\,(g)\mathfrak{g}_1$$

and that $\mathrm{Ad}\,(g): \mathfrak{g}_1 \to \xi$ *is orientation preserving.*

3. Proof of Theorem 2.

At first we review the proof of Theorem 1.

Let M be a compact oriented 3-dimensional submanifold of G contained in the real homology class $[G_1]$. Since ϕ is a bi-invariant form on G, the inequality of ϕ stated in Lemma 4 holds at every point in G. The proof of Theorem 1 is as follows:

$$\mathrm{vol}\,(G_1) = \int_{G_1} \mathrm{vol}_{G_1} = \int_{G_1} \phi = \int_M \phi \leqq \int_M \mathrm{vol}_M = \mathrm{vol}\,(M)\,.$$

The equality holds if and only if $\phi|_M = \mathrm{vol}_M$. A 3-dimensional oriented submanifold M of G which satisfies $\phi|_M = \mathrm{vol}_M$ is called a *ϕ-submanifold* of G. So the following lemma completes the proof of Theorem 2.

LEMMA 5. *If* M *is a* ϕ*-submanifold of* G, *then* M *is congruent with a piece of* G_1 *in* G. *Furthermore, if* M *is complete, then* M *is congruent with* G_1.

PROOF. We show that M is totally geodesic in G. Let x be any point of M. Since G is a homogeneous Riemannian manifold, we may suppose that x is the identity element e of G. We may show that the second fundamental form h of M vanishes at e. It follows from Lemma 4 that there is an element g in G such that $T_e(M) = \mathrm{Ad}\,(g)\mathfrak{g}_1$. For simplicity we set $T_e(M) = T_e(G_1)$ and identify $T_e(G)$ with $\mathfrak{g}$. Then the following equations hold:

$$T_e(M) = \mathfrak{g}_1 \tag{1}$$

and

$$T_e^\perp(M) = \{H \in \mathfrak{t}\,;\, \langle \delta, H \rangle = 0\} + \sum_{\alpha \in \Delta_+ - \{\delta\}} (\boldsymbol{R}E_\alpha + \boldsymbol{R}F_\alpha)\,.$$

The curvature tensor R of G is given by

$$R(X, Y)Z = -\frac{1}{4}[[X, Y], Z] \tag{2}$$

for X, Y and Z in $\mathfrak{g}$. See, for example, Milnor [3]. So the assumption on M and Lemma 4 imply that $R(X, Y)Z$ is contained in the tangent space of M for any tangent vectors X, Y, and Z of M, that is, M is a curvature invariant submanifold of G. As M is curvature invariant and G is locally symmetric, the equation

$$(3)\qquad h(W, R(X, Y)Z)=R(h(W, X), Y)Z+R(X, h(W, Y))Z +R(X, Y)h(W, Z)$$

holds for tangent vectors X, Y, Z, and W of M. The above equation is due to Ohnita [4]. Putting $W=X$ and $Z=Y$ in (3), we obtain

$$(4)\qquad h(X, R(X, Y)Y)=R(h(X, X), Y)Y+R(X, h(X, Y))Y +R(X, Y)h(X, Y).$$

From now on we shall consider h at e. Let X and Y be orthonormal vectors in $\mathfrak{g}_1$. Then

$$R(X, Y)Y=\frac{1}{4}|\delta|^2 X.$$

By (2) and (4),

$$(5)\qquad |\delta|^2 h(X, X)=-[[h(X, X), Y], Y]-[[X, h(X, Y)], Y] -[[X, Y], h(X, Y)].$$

Let $\mathfrak{z}$ be the centralizer of $\mathfrak{g}_1$ in $\mathfrak{g}$ and $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{z}$ in $T_e^{\perp}(M)$. Then

$$\mathfrak{z}=\{H\in\mathfrak{k}\,;\langle\delta, H\rangle=0\}+\sum_{\substack{\alpha\in\Delta_+\\ \langle\alpha,\delta\rangle=0}}(\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha)$$

and

$$\mathfrak{m}=\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}(\boldsymbol{R}E_\alpha+\boldsymbol{R}F_\alpha).$$

According to Wolf [6], $(\mathfrak{g}, \mathfrak{z}+\mathfrak{g}_1)$ is a compact quaternionic symmetric pair. So the right hand side of the equation (5) is contained in $\mathfrak{m}$. Therefore the image of h is contained in $\mathfrak{m}$.

Since $(\mathfrak{g}, \mathfrak{z}+\mathfrak{g}_1)$ is a quaternionic symmetric pair, $\mathfrak{m}$ has a quaternionic vector space structure and $\mathfrak{g}_1$ acts on $\mathfrak{m}$ as the multiplications by purely quaternionic numbers. In particular, we obtain

$$(6)\qquad [X, [Y, h(X, Y)]]=-[Y, [X, h(X, Y)]],$$

because X and Y are orthogonal vectors in $\mathfrak{g}_1$ and $h(X, Y)\in\mathfrak{m}$.

The equation

$$(7)\qquad |\delta|^2 h(X, X)=-[Y, [Y, h(X, X)]]+3[Y, [X, h(X, Y)]]$$

follows from (5), (6), and the Jacobi identity.

On the other hand

(8) $$[U,[U,V]]=-\frac{1}{4}|\delta|^2 V$$

for any unit vector U in $\mathfrak{g}_1$ and any vector V in $\mathfrak{m}$. Indeed, since each unit vector in $\mathfrak{g}_1$ is G_1-conjugate to $\delta/|\delta|$ and $\mathfrak{m}$ is $\mathrm{Ad}(G_1)$-invariant, we may suppose $U=\delta/|\delta|$. Put

$$V=\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}(s_\alpha E_\alpha+t_\alpha F_\alpha).$$

By ii) of Lemma 3 and the fact that

$$\frac{2\langle\alpha,\delta\rangle}{\langle\delta,\delta\rangle}=1$$

for each α in $\Delta_+-\{\delta\}$ with $\langle\alpha,\delta\rangle\neq 0$ (cf. Wolf [6]), we have

$$\begin{aligned}[U,[U,V]]&=-\frac{1}{|\delta|^2}\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}\langle\alpha,\delta\rangle^2(s_\alpha E_\alpha+t_\alpha F_\alpha)\\ &=-\frac{|\delta|^2}{4}\sum_{\substack{\alpha\in\Delta_+-\{\delta\}\\ \langle\alpha,\delta\rangle\neq 0}}(s_\alpha E_\alpha+t_\alpha F_\alpha)\\ &=-\frac{|\delta|^2}{4}V.\end{aligned}$$

It follows from (7) and (8) that

(9) $$\frac{1}{4}|\delta|^2 h(X,X)=[Y,[X,h(Y,X)]]$$

for any orthonormal vectors X and Y in $\mathfrak{g}_1$. By (9) we also have

(10) $$\frac{1}{4}|\delta|^2 h(Y,Y)=[X,[Y,h(Y,X)]].$$

Thus by (6), (9) and (10) we get

$$h(X,X)=-h(Y,Y).$$

Since $\mathfrak{g}_1$ is 3-dimensional, we have

$$h(X,X)=0$$

for any unit vector X in $\mathfrak{g}_1$ and $h=0$.

Hence M is totally geodesic in G. Since G_1 is also totally geodesic in G, M is a piece of G_1. Q.E.D.

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