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A NOTE ON SPAN UNDER REFINABLE MAPS

By

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1. Introduction.

All spaces considered in this note are metric, and all maps are continuous functions. A compactum is a compact metric space. A continuum is a connected compactum. In [1], Ford and Rogers defined a map $r: X\to Y$ from a compactum X onto a compactum Y to be *refinable* if for each $\varepsilon>0$, there is an ε-map $f: X\to Y$ from X onto Y whose distance from r is less than ε. Refinable maps are useful in continuum theory, and many properties in continuum theory are preserved by refinable maps. For example, decomposability [1], aposyndesis [2], property [k], irreducibility, hereditary indecomposability and being the pseudo-arc [6] (see for other properties [4] and [5]).

Lelek [8] defined the *surjective span* of a continuum X, $\sigma^*(X)$, (resp. the *surjective semi-span*, $\sigma_0^*(X)$) to be the least upper bound of all real numbers α with the following property; there exist a continuum C and maps $f_1, f_2: C\to X$ such that $f_1(C)=X=f_2(C)$ (resp. $f_1(C)=X$) and dist $(f_1(c), f_2(c))\geqq\alpha$ for every $c\in C$. The *span* $\sigma(X)$ and the *semi-span* $\sigma_0(X)$ of X are defined by the formulas;

$$\sigma(X)=\sup\{\sigma^*(A)\mid A \text{ is a subcontinuum of } X\},$$

$$\sigma_0(X)=\sup\{\sigma_0^*(A)\mid A \text{ is a subcontinuum of } X\}.$$

Recently, many authors have been investigating span theory and finding interesting properties. Concerning span and special classes of maps, the following problems are raised in the University of Houston Problem Book;

Problem 86. *Do confluent maps of continua preserve span zero?*

Problem 92. *If M is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate monotone continuous image of M have positive span?*

Ingram, [3, Theorem 2], showed that monotone maps of continua preserve span zero.

In this note we will show that refinable maps of continua preserve surjective

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(semi-) span zero, and refinable preimages of continua with surjective (semi-) span zero have surjective (semi-) span zero. We note that for a refinable map $r: X\to Y$, if Y has property $[k]$, then r is confluent ([6, Theorem (2.3)]), and moreover if Y is locally connected, r is montone ([1, Corollary 1.2]).

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2. Results.

In this note an ANR means an absolute neighborhood retract for the class of metric spaces. For a metric space X and points x, x' of X, $d(x, x')$ is the distance from x to x' under a metric of X.

THEOREM. *Let $r: X\to Y$ be a refinable map of continua. Then $\tau(X)=0$ if and only if $\tau(Y)=0$, where $\tau=\sigma^*$, σ_0^*, σ or σ_0.*

For the proof we need the following lemmas:

LEMMA 1 ([10, Lemma 1]). *Let $f: X\to P$ be a map from a compactum X to a compact ANR P. Then for every $\varepsilon>0$, there is a positive number $\delta>0$ such that if $g: X\to Y$ is a δ-map from X onto a compactum Y, then there is a map $h: Y\to P$ such that f and hg are ε-near.*

By a slight modification of the proof of [9, 3.1], we have the following.

LEMMA 2. *Let X be a non-empty continuum contained in a compactum Z. If β is a real number and for $n=1, 2, 3, \cdots$, there exists a continum Z_n in Z such that $\beta\leqq\tau(Z_n)$ and $\operatorname{Lim} Z_n=X$, then $\beta\leqq\tau(X)$, where $\tau=\sigma^*$, σ_0^*, σ or σ_0.*

PROOF OF THEOREM. Since surjective span zero is a topological invariant in the class of continua, we may assume that both X and Y are subsets of the Hilbert cube Q.

Suppose that $\sigma^*(X)=0$. Let C be a continuum, and let $f_1, f_2: C\to Y$ be maps such that $f_1(C)=Y=f_2(C)$. Let take a compact ANR neighborhood U of X in Q and a continuous extension $g: U\to Q$ of r. For each integer $n\geqq 1$, there is a positive number $\varepsilon_n>0$ such that

(2) if $d(x, x')<\varepsilon_n$, $x, x'\in U$, then $d(g(x), g(x'))<\dfrac{1}{n}$.

Since r is a refinable map, there exists a sequence $\{r_i\}$ of maps $r_i: X\to Y$ such that for each $i\geqq 1$,

(2) $r_i(X)=Y$,

(3) $d(r(x), r_i(x))<\frac{1}{i}$ for all $x\in X$, and

(4) $\operatorname{diam} r_i^{-1}(y)<\frac{1}{i}$ for all $y\in Y$.

Then by (2), (4) and Lemma 1, we have integers $n\leqq i(1)<i(2)<\cdots$ and maps $h_j: Y\to U$, $j=1, 2, \cdots$, such that

(5) $d(h_j r_{i(j)}(x), x)<\frac{1}{j}$ for all $x\in X$, $j=1, 2, \cdots$.

By (2) and (5), we easily have that $\operatorname{Lim}_j h_j(Y)=\operatorname{Lim}_j h_j r_{i(j)}(X)=X$. Since $\sigma^*(X)=0$, by Lemma 2, there exists an integer $j_0\geqq 1$ such that

(6) $\sigma^*(h_j(Y))<\varepsilon_n$ for all $j\geqq j_0$.

Now take an integer $j\geqq j_0$ with $1/j<\varepsilon_n$, and put the maps

$$r'=r_{i(j)}: X\longrightarrow Y \quad\text{and}\quad h=h_j: Y\longrightarrow h_j(Y)=h(Y).$$

Considering two maps $hf_1, hf_2: C\to h(Y)$, by (6), there exists a point $c_n\in C$ such that

(7) $d(hf_1(c_n), hf_2(c_n))<\varepsilon_n$.

Then by (1),

(8) $d(ghf_1(c_n), ghf_2(c_n))<\frac{1}{n}$.

By (2), take points $x_1, x_2\in X$ such that $r'(x_i)=f_i(c_n)$ for $i=1, 2$. Then by (3), (5) and (1), we have that for $i=1, 2$,

$$\begin{aligned}(9)\quad d(f_i(c_n), ghf_i(c_n))&=d(r'(x_i), ghr'(x_i))\\ &<d(r'(x_i), r(x_i))+d(r(x_i), ghr'(x_i))\\ &<\frac{1}{i(j)}+d(g(x_i), ghr'(x_i))\\ &<\frac{1}{n}+\frac{1}{n}=\frac{2}{n}\end{aligned}$$

Hence by (8) and (9),

(10) $d(f_1(c_n), f_2(c_n))<\frac{5}{n}$.

Let $c_0\in C$ be an accumulation point of the sequence $\{c_n\}$. Then by (10), $d(f_1(c_0), f_2(c_0))=0$. It follows that $\sigma^*(Y)=0$.

Conversely, we suppose that $\sigma^*(Y)=0$. Let C be a continuum, and let $f_1, f_2: C\to X$ be maps such that $f_1(C)=X=f_2(C)$. For each $n\geqq 1$, there is an $1/n$-map $r_n: X\to Y$ from X onto Y, since r is a refinable map. Since $r_n f_1(C)=Y=$

$r_n f_2(C)$ and $\sigma^*(Y)=0$, there exists a point $c_n \in C$ such that $r_n f_1(c_n)=r_n f_2(c_n)$. Then $d(f_1(c_n), f_2(c_n))<1/n$. Hence, as in the first part of the proof, we have the point $c_0 \in C$ such that $f_1(c_0)=f_2(c_0)$. Therefore $\sigma^*(X)=0$.

The above proof may be used to prove similar theorems for σ_0^*, σ and σ_0.

COROLLARY 1. *Let* $r: X \to Y$ *be a refinable map of continua. Then* $\tau(X)>0$ *if and only if* $\tau(Y)>0$, *where* $\tau=\sigma^*$, σ_0^*, σ *or* σ_0.

Therefore refinable maps of compacta preserve positive span.

In the latter part of the proof of the Theorem we needed only the fact that there exists an $1/n$-map from X onto Y for each $n \geqq 1$. Hence the following is obtained. The case $\tau=\sigma$ is included in [11, Lemma 21].

COROLLARY 2. *Let* X *and* Y *be continua. If* X *is* Y*-like and* $\tau(Y)=0$, *then* $\tau(X)=0$, *where* $\tau=\sigma^*$, σ_0^*, σ *or* σ_0.

By [4, Corollary 3.4], every hereditarily decomposable circle-like continuum admits a refinable map onto a circle. Therefore we have

COROLLARY 3. *Every hereditarily decomposable circle-like continuum has positive surjective (semi-) span.*

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