TSUKUBA J. MATH.
Vol. 5 No. 1 (1981). 117–132

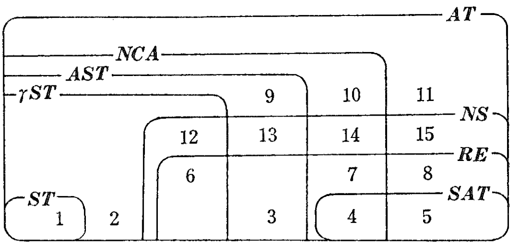
ON A CLASSIFICATION OF ARONSZAJN TREES II

By

Masazumi HANAZAWA

§ 1. Introduction.

In the former paper [3], we considered the classification of Aronszajn trees by the notions of Souslin trees, ω_1-trees with property γ, almost-Souslin trees, ω_1-trees with no club antichain, special Aronszajn trees and $\boldsymbol{R}$-embeddable trees. As we remarked in its last section, there is another interesting notion. It is the notion of non-Souslin trees which had been introduced by Baumgartner [1]. The classification of Aronszajn trees by this notion together with the previous ones is shown by the following:



where ST=the class of Souslin trees,

γST=the class of ω_1-trees with property γ,

AST=the class of almost-Souslin trees,

NCA=the class of ω_1-trees with no club anti-chain,

SAT=the class of special Aronszajn tree,

RE=the class of $\boldsymbol{R}$-embeddable ω_1-trees,

NS=the class of non-Souslin trees,

AT=the class of Aronszajn trees.

Under ZFC alone, none of the categories but Category 5 can be proved to be non-void. In the former paper we proved that if $V=L$, Categories $1\sim 11$ are all non-void (note that the trees constructed in Theorems 9, 10 and 11 [3], are the elements of Categories 9, 10 and 11 respectively). In this paper we shall prove that if $V=L$, remaining Categories $12\sim 15$ are also non-void. It is shown as a

Received May 29, 1980. Revised December 2, 1980.

by-product that $\diamondsuit$ suffices for the existence of non-Souslin trees which are not $\boldsymbol{R}$-embeddable.

§ 2. Preliminaries.

Most of the notions and the notations which are used here are described in the former paper. It is assumed that the reader knows them. Let $T=\langle T, <_T\rangle$ be a tree. $\langle X, <_T\rangle$ is called a subtree of T if $X\subset T$. $\langle X, <_T\rangle$ is called a transitive subtree of T if it is a subtree of T such that $(\forall x\in X\ \forall y\in T)[y<_T x\rightarrow y\in X]$ (in the paper [3], we called a transitive subtree a subtree). When $X\subset T$, we use

$\tilde{X}$ to denote $\langle X, <_T\rangle$,

$ht_X(x)$ to denote the height of x in $\tilde{X}$,

$\tilde{X}_\alpha$ to denote the set $\{x\in X: ht_X(x)=\alpha\}$,

$\tilde{X}\restriction\alpha$ to denote the set $\{x\in X: ht_X(x)<\alpha\}$.

But $\tilde{T}$, $ht_T(x)$ $\tilde{T}_\alpha$, $\tilde{T}\restriction\alpha$ will exceptionally be written as T, $ht(x)$, T_α, $T\restriction\alpha$ respectively. If $S\subset\omega_1$, $T\restriction S$ is the set $\{x\in T: ht(x)\in S\}$. Recall that Ω is the set of all limit ordinals $<\omega_1$. In this paper ω_1-trees are assumed to have only one minimal element (a root).

Before introducing more special notions, we shall raise well-known facts.

LEMMA 1. *If T is an $\boldsymbol{R}$-embeddable tree with $ht(T)\leqq\omega_1$, then the tree $\langle T\restriction(\omega_1\backslash\Omega), <_T\rangle$ is $\boldsymbol{Q}$-embeddable.*

PROOF. With each $x\in T\restriction(\omega_1\backslash\Omega)$, associate a $q\in\boldsymbol{Q}$ such that $e(x')<q<e(x)$, where $e: T\rightarrow\boldsymbol{R}$ is the embedding and x' means the immediate predecessor of x.

LEMMA 2. *If T is a $\boldsymbol{Q}$-embeddable uncountable tree, then T contains an uncountable anti-chain.*

PROOF. Let e embed T in $\boldsymbol{Q}$. Clearly $\{x\in T: e(x)=q\}$ is an anti-chain and is uncountable for some $q\in\boldsymbol{Q}$.

LEMMA 3. *Let T be an $\boldsymbol{R}$-embeddable tree. If X is an uncountable subset of T, then X contains an uncountable anti-chain of T.*

PROOF. $\tilde{X}$ is clearly $\boldsymbol{R}$-embeddable. If $\tilde{X}_\alpha$ is uncountable for some α, then $\tilde{X}_\alpha(\subset X)$ is an uncountable anti-chain of $\tilde{X}$ and hence an uncountable anti-chain of T. If $\tilde{X}_\alpha$ is countable for all α, then $\tilde{X}\restriction(\omega_1\backslash\Omega)$ is uncountable. Since

$\langle \tilde{X} \upharpoonright (\omega_1 \backslash \Omega), <_T \rangle$ is $\boldsymbol{Q}$-embeddable by Lemma 1, there is an uncountable anti-chain $\subset \tilde{X} \upharpoonright (\omega_1 \backslash \Omega) \subset X$ by Lemma 2.

LEMMA 4. *Let T be a tree with height ω_1. If T_α is finite for uncountably many α, then T has a confinal branch.*

PROOF. Put $T^* = \{x \in T : x$ has an extension in every higher level $T_\alpha\}$. It is easy to see by the assumption that the transitive subtree $\langle T^*, <_T \rangle$ of T has height ω_1. Pick a branch b of $\tilde{T}^*$. We shall show that the order type, say λ, of b is ω_1. Suppose $\lambda < \omega_1$. Pick $\alpha < \omega_1$ such that $\lambda < \alpha$ and T_α is finite. Put $Y_x = \{y \in T_\alpha ; x <_T y\}$ for each $x \in b$. Then $\cap \{Y_x : x \in b\}$ is non-empty, since (1) $Y_x \neq \emptyset$, (2) $x <_T y \to Y_x \supseteq Y_y$ and (3) Y_x is finite. Pick $y \in \cap \{Y_x : x \in b\}$. Then $b \subset \hat{y}$, this contradicts the assumption that b is a branch (a maximal linearly ordered subset of T), q. e. d.

Now recall that $\mathfrak{T}$ is the tree $\bigcup_{\alpha<\omega_1} \boldsymbol{R}^{\alpha+1}$ with the ordering defined by $x <_T y \leftrightarrow x \subset y$ and that if $x \in \mathfrak{T}$, $m(x)$ is the real number $x(ht(x))$. When $x \in \mathfrak{T}$ and a limit ordinal λ is in $\mathrm{dom}(x)$, we write $\lim_{\xi\to\lambda} x(\xi) = r$ instead of $(\forall q < r\, \exists \alpha < \lambda\, \forall \beta < \lambda)$ $[\alpha < \beta \to q < x(\beta) \leqq r]$. Now we define a transitive subtree $\mathfrak{T}_P$ of $\mathfrak{T}$ as follows:

$$\mathfrak{T}_P = \{x \in \mathfrak{T} : P(x)\},$$

where $P(x)$ is the conjunction of the following three:

(1) $x(\alpha) \geqq 0$ for all $\alpha \in \mathrm{dom}(x)$;

(2) $x(\alpha) < x(\alpha+1)$ for all α with $\alpha+1 \in \mathrm{dom}(x)$;

(3) for all limit ordinals $\lambda \in \mathrm{dom}(x)$,

$$(\forall r > 0)[\lim_{\xi\to\lambda} x(\xi) = r \leftrightarrow x(\lambda) = r].$$

For a transitive subtree T of $\mathfrak{T}_P$, we put

$$T^0 = \{x \in T : m(x) = 0\}.$$

We shall write

$$x \vec{<}_T y \text{ instead of } x <_T y \;\&\; (x, y] \cap T^0 = \emptyset.$$

LEMMA 5. *Let T be a transitive subtree of $\mathfrak{T}_P$. Then for every $x, y \in T$:*

(1) $m(x) \geqq 0$;

(2) $x \vec{<}_T y \to m(x) < m(y)$;

(3) *the function m increases monotonously on $[x, y)$ if $(x, y) \cap T^0 = \emptyset$;*

(4) $m(x) > 0 \to \exists y [y \vec{<}_T x]$;

(5) $\lambda \in \Omega \;\&\; x, y \in T_\lambda \;\&\; \hat{x} = \hat{y} \to x = y$;

(6) $m(x) = 0 \to ht(x) \in \Omega$;

(7) *if* $ht(y)\in\Omega$, *then for every* $r>0$,

$$\lim_{z\to y} m(z)=r \qquad \text{iff } m(y)=r\,,$$

where $\lim_{z\to y} m(z)=r$ *means*

$$(\forall\varepsilon>0)(\exists z<_T y)(\forall w\in[z,\ y))[m(w)>r-\varepsilon]\,;$$

(8) $y<_T x\ \&\ x\in T^0\ \&\ q\in\boldsymbol{Q}\rightarrow(\exists z)[y\rightleftarrows_T z<_T x\ \&\ m(z)>q]$.

PROOF. The first seven statements are easily checked. To show the last one, suppose that $y<_T x\in T^0$ and $q\in\boldsymbol{Q}$. Let w be the least of those elements z that $y<_T z\leqq_T x$ and $m(z)=0$. By (3), the function m increases monotonously on $[y,\ w)$, since $(y,\ w)\cap T^0=\emptyset$. Hence w increases monotonously on $[ht(y),\ ht(w))$. Hence $\lim_{\xi\to ht(w)} w(\xi)=\infty$ because of (7) and $w(ht(w))=0$. Pick ζ so that $ht(y)<\zeta<ht(w)$ and $w(\zeta)>q$. Put $z=w\upharpoonright(\zeta+1)$. Then

$$y\rightleftarrows_T z<_T w\leqq_T x \quad \& \quad m(z)=z(ht(z))=w(\zeta)>q\,, \qquad \text{q.e.d.}$$

If a transitive subtree of $\mathfrak{T}_P$ is an ω_1-tree, we call it a P-tree. Recall that an ω_1-tree T is called a non-Souslin tree if every uncountable subset of T contains an uncountable anti-chain. By $\boldsymbol{NS}$, we denote the class of all non-Souslin trees.

LEMMA 6. *Let T be an Aronszajn P-tree. If*

$$\{\alpha:\ T_\alpha\cap T^0 \text{ is finite}\}$$

is a stationary set, then

(i) *if X is an uncountable subset of T^0, $\tilde{X}_\alpha$ is uncountable for some $\alpha<\omega_1$,*

(ii) $T\in\boldsymbol{NS}$.

PROOF. (i) Let X be an uncountable subset of T^0 and suppose that $\tilde{X}_\alpha$ is countable for all $\alpha<\omega_1$. Put:

$$C=\{\lambda\in\Omega:\ \tilde{X}\upharpoonright\lambda\subseteq T\upharpoonright\lambda\}.$$

C is a club set since $\tilde{X}\upharpoonright\alpha$ is countable for all $\alpha<\omega_1$. Hence by the assumption of the lemma, the set

$$E=\{\lambda\in C:\ T_\lambda\cap T^0 \text{ is finite}\}$$

is stationary and hence uncountable. Put:

$$Y=\{y\in T:\ y\leqq_T x \text{ for some } x\in X\}.$$

CLAIM. If $\lambda\in E$, then $\tilde{Y}_\lambda$ is a subset of $T^0\cap T_\lambda$.

PROOF OF CLAIM. Since $\tilde{Y}$ is a transitive subtree of T, $\tilde{Y}_\lambda \subseteq T_\lambda$. Let $y \in \tilde{Y}_\lambda$. Let x be a minimal element of $\{x \in X : y \leqq_T x\}$. Then $ht_X(x)=\lambda$. (The reason: In general $ht_X(z) \leqq ht(z)$. Hence by the minimality of x, $ht_X(x) \leqq \lambda$. If $ht_X(x)<\lambda$ then $ht(x)<\lambda$ because $\lambda \in C$; this contradicts $y \leqq_T x$). Now suppose $y \notin T^0$. Then we can pick $w \lneqq_T y$ by Lemma 5-(4). Pick β so that $ht(w)<\beta<\lambda$ and pick $z \in X$ so that $ht_X(z)=\beta$ and $z <_T x$. Then $ht(w)<\beta=ht_X(z) \leqq ht(z)<ht(y)=\lambda$ and so $w <_T z <_T y \leqq_T x$. Thus $z \in (w, y] \cap T^0$, a contradiction. Claim is thus proved. Thus $\tilde{Y}_\lambda$ is finite for all $\lambda \in E$. By Lemma 4, $\tilde{Y}$ has a cofinal branch which is also a cofinal branch of T. This is absurd since $T \in \boldsymbol{AT}$.

(ii) Let X be an uncountable subset of T. For each $z \in T^0$, put:

$$X_{(z)}=\{x \in X : z \lneqq_T x\},$$

$$Z=\{z \in T^0 : X_{(z)} \neq \emptyset\}.$$

Case 1. Z is uncountable. By (i), we can find an uncountable subset Y (i.e. $\tilde{Z}_\alpha$ for some α) of Z such that Y is an anti-chain of T. With each $y \in Y$ associate an element, say $x(y)$, of $X_{(y)}$. Then the subset $\{x(y) : y \in Y\}$ of X is clearly an uncountable anti-chain of T.

Case 2. Z is countable. Since the uncountable set X is the union of $\{X_{(z)} : z \in Z\}$, we can find $z \in Z$ such that $X_{(z)}$ is uncountable. Note that $\tilde{X}_{(z)}$ is an $\boldsymbol{R}$-embeddable tree by Lemma 5-(3). By Lemma 3, $\tilde{X}_{(z)}$ contains an uncountable anti-chain which is also an antichain of T and is contained in X. Lemma 6 is thus proved.

COROLLARY 7. *Let T be an Aronszajn P-tree. If the set*

$$\{\alpha<\omega_1 : m(x)>0 \text{ for all } x \in T_\alpha\}$$

is stationary, then $T \in \boldsymbol{NS}$.

Though this corollary assumes a rather strong condition, it suffices for our purpose. In this sense Lemma 6 is redundant. Lemma 6 stands because of its own interest.

Recall that a $\diamondsuit_{\mathfrak{T}}$-sequence $\langle Z_\alpha : \alpha<\omega_1\rangle$ has the following properties: If T is an ω_1-tree and is a transitive subtree of $\mathfrak{T}$, then

(1) if X is a subset of T, then the set

$$\{\alpha<\omega_1 : X \cap T \upharpoonright \alpha = Z_\alpha\} \text{ is stationary};$$

(2) if e is a function which embeds T in $\boldsymbol{R}$, then

$$\{\alpha<\omega_1 : e \upharpoonright (T \upharpoonright \alpha)=Z_\alpha\} \text{ is a stationary set}.$$

Recall that a $\diamondsuit_{\mathfrak{T}}^{*}$-sequence $\langle\{W_i^\alpha : i\in\omega\} : \alpha<\omega_1\rangle$ has the following properties: If T is an ω_1-tree and is a transitive subtree of $\mathfrak{T}$, then

(1) if X is a subset of T, then

$$\{\alpha<\omega_1 : X\cap T\restriction\alpha=W_i^\alpha \text{ for some } i<\omega\} \text{ contains a club set.}$$

(2) if e is a function which embeds T in $\boldsymbol{R}$, then

$$\{\alpha<\omega_1 : e\restriction(T\restriction\alpha)=W_i^\alpha \text{ for some } i<\omega\} \text{ contains a club set.}$$

LEMMA 8. (1) ($\diamondsuit$) *There exists a* $\diamondsuit_{\mathfrak{T}}$*-sequence.*

(2) ($\diamondsuit^*$) *There exists a* $\diamondsuit_{\mathfrak{T}}^{*}$*-sequence.*

LEMMA 9. *Let T be a P-tree and $\langle Z_\alpha : \alpha<\omega_1\rangle$ a $\diamondsuit_{\mathfrak{T}}$-sequence. If for every $\lambda\in\Omega$ $(\forall x\in T_\lambda)[Z_\lambda\neq\hat{x}]$ holds, then $T\in\boldsymbol{AT}$.*

PROOF. Suppose that X were a cofinal branch of T. Then there is a $\lambda\in\Omega$ such that $Z_\lambda=X\cap T\restriction\lambda$. Let x be the unique element of $X\cap T_\lambda$. Then $Z_\lambda=X\cap T\restriction\lambda=\hat{x}$, a contradiction.

LEMMA 10. *Let T be a P-tree and $\langle Z_\alpha : \alpha<\omega_1\rangle$ a $\diamondsuit_{\mathfrak{T}}$-sequence. Let T satisfy the following condition:*

(1) *if $\lambda\in\Omega$ and Z_λ is a function which embeds $T\restriction\lambda$ in $[0, 1)$, then there is an $x\in T_\lambda$ such that*

$$(*)\qquad (\forall n\exists y<_T x)[Z_\lambda^{\sup}(y)-1/n<Z_\lambda(y)],$$

where $Z_\lambda^{\sup}(y)=\sup\{Z_\lambda(z) : y<_T z\in T\restriction\lambda\}$.

Then T is not $\boldsymbol{R}$-embeddable.

PROOF. Let e embed T in $\boldsymbol{R}$. We may assume $\operatorname{ran}(e)\subset[0, 1)$. Put:

$$C=\{\lambda\in\Omega : (\exists y\in T)[x<_T y \ \&\ q<e(y)]\rightarrow(\exists y\in T\restriction\lambda)[x<_T y \ \&\ q<e(y)]$$
$$\text{for every } q\in\boldsymbol{Q} \text{ and every } x\in T\restriction\lambda\}.$$

Clearly C is club and hence we can pick $\lambda\in C$ such that $e\restriction(T\restriction\lambda)=Z_\lambda$. Then Z_λ embeds $T\restriction\lambda$ in $[0, 1)$. So, by the assumption, we can take $x\in T_\lambda$ which satisfies (*). Let x' be one of the immediate successors of x. Pick n so that $1/n<e(x')-e(x)$. Pick $y<_T x$ so that $Z_\lambda^{\sup}(y)-1/n<Z_\lambda(y)$. Since $\lambda\in C$, $e(x')\leqq\sup\{e(z) : y<_T z\in T\}=\sup\{e(z) : y<_T z\in T\restriction\lambda\}=Z_\lambda^{\sup}(y)$. It follows that $1/n<e(x')-e(x)<Z_\lambda^{\sup}(y)-e(y)<1/n$, a contradiction.

LEMMA 11. *Let $\langle\{W_i^\alpha : i<\omega\} : \alpha<\omega_1\rangle$ be a $\diamondsuit_{\mathfrak{T}}^{*}$-sequence. Let T be a P-tree which satisfies the following condition:*

(1) *whenever* $\lambda\in\Omega$ *and* W_i^λ *is an anti-chain of* $T\restriction\lambda$ *and* $x\in T_\lambda$, *one of the following conditions holds*:

(a) $$(\exists y<_T x)(\forall z\in T\restriction\lambda)[y<_T z\rightarrow z\notin W_i^\lambda],$$

(b) $$(\exists y\precapprox_T x\exists q>0)[m(x)\leqq m(y)+q\ \&\ (\forall z\in W_i^\lambda)[y\precapprox_T z\rightarrow m(z)\geqq m(y)+2q]].$$

Then T has property γ.

The proof of this lemma is given separately in a later section, since it is rather long.

Finally we define for two ω_1-trees $(T, <_0)$ and $(T', <_1)$ an ω_1-tree $T\dotplus T'$ as follows: The field of $T\dotplus T'$ is $T\times\{0\}\cup T'\times\{1\}\setminus\{\langle 0_1, 1\rangle\}$, where 0_0, 0_1 are the roots of T, T' respectively; The ordering $<_T$ of $T\dotplus T'$ is defined by

$$\langle x, 0\rangle<_T\langle y, 0\rangle \quad \text{if } x, y\in T \text{ and } x<_0 y,$$

$$\langle x, 1\rangle<_T\langle y, 1\rangle \quad \text{if } x, y\in T'\setminus\{0_1\} \text{ and } x<_1 y,$$

$$\langle 0_0, 0\rangle<_T\langle y, 1\rangle \quad \text{if } y\in T'\setminus\{0_1\}.$$

§ 3. Theorems.

THEOREM 12 ($\diamondsuit^*$). $(\boldsymbol{NS}\setminus\boldsymbol{RE})\cap\gamma\boldsymbol{ST}\neq\emptyset$.

PROOF. Let $\langle Z_\alpha : \alpha<\omega_1\rangle$ be a $\diamondsuit_\Sigma$-sequence and $\langle\{W_i^\alpha : i<\omega\} : \alpha<\omega_1\rangle$ a $\diamondsuit_\Sigma^*$-sequence. We define a P-tree T by induction on levels so that T satisfies the following:

(1) if $\alpha<\beta<\omega_1$ and $x\in T_\alpha$ and $q\in\boldsymbol{Q}\cap(m(x), \infty)$, there is a $y\in T_\beta$ such that $x\precapprox_T y$ and $m(y)<q$.

Set $$T_0=\{0_T\};$$

$$T_{\alpha+1}=\{x\cup\{\langle q, \alpha+1\rangle\} : x\in T_\alpha\ \&\ m(x)<q\in\boldsymbol{Q}\}.$$

Let $\lambda\in\Omega$ and suppose $T\restriction\lambda$ has been defined so that (1) holds. Fix an increasing sequence $\langle\lambda_n : n<\omega\rangle$ such that $\lim_{n\in\omega}\lambda_n=\lambda$. For each $x\in T\restriction\lambda$ and each positive rational q, we define $t_\lambda(x, q)$ as follows: Let $x\in T\restriction\lambda$ and $0<q\in\boldsymbol{Q}$. We pick x_n, $x_n^*\in T\restriction\lambda$, q_n, $q_n^*>0$ inductively so that:

(a) $x_0=x$ and $q_0=q$;

(b) if W_n^λ is an anti-chain of $T\restriction\lambda$ and

$$(\exists z\in T\restriction\lambda)[x_n\precapprox_T z\in W_n^\lambda\ \&\ m(z)<m(x_n)+q_n],$$

then

$$x_n\precapprox_T x_n^*\in W_n^\lambda\ \&\ m(x_n^*)<m(x_n)+q_n \quad\text{and}\quad q_n^*=m(x_n)+q_n-m(x_n^*);$$

$$\text{otherwise, } x_n^*=x_n \text{ and } q_n^*=q_n/2;$$

(c) $x_n^* \lneqq_T x_{n+1}$ & $ht(x_{n+1}) > \lambda_n$ & $m(x_{n+1}) < m(x_n^*) + q_n^*$ (this is possible by (1));

(d) $q_{n+1} = m(x_n^*) + q_n^* - m(x_{n+1})$.

Put:
$$t_\lambda(x, q) = \bigcup_{n<\omega} x_n \cup \{\langle \sup_{n<\omega} m(x_n), \lambda\rangle\}.$$

Notice that $x \lneqq_T t_\lambda(x, q)$ and $0 < m(t_\lambda(x, q)) \leqq m(x) + q$. Now, we shall define T_λ.

Case 1. Z_λ is a cofinal branch of $T \upharpoonright \lambda$. For each $x \in T \upharpoonright \lambda$ and each positive rational q, pick $x^* \in T \upharpoonright \lambda$ and q^* so that:

$$x <_T x^* \notin Z_\lambda,\ ht(x^*) = ht(x) + 1,\ m(x^*) < m(x) + q \text{ and } q^* = m(x) + q - m(x^*).$$

And put:
$$u_\lambda(x, q) = t_\lambda(x^*, q^*).$$

We set:
$$T_\lambda = \{u_\lambda(x, q) : x \in T \upharpoonright \lambda,\ 0 < q \in \boldsymbol{Q}\}.$$

Note that if $u = u_\lambda(x, q)$, then $Z_\lambda \neq \hat{u}$, $x \lneqq_T u$ and $0 < m(u) \leqq m(x) + q$.

Case 2. Z_λ is a function which embeds $T \upharpoonright \lambda$ in $[0, 1)$. Pick y_n, $y_n^* \in T \upharpoonright \lambda$ inductively as follows:

(a) $y_0 = 0_T$;

(b) if W_n^λ is an anti-chain of $T \upharpoonright \lambda$ and

$$(\exists z \in W_n^\lambda)[y_n <_T z \ \&\ Z_\lambda^{\sup}(y_n) - 1/(n+1) < Z_\lambda(z)],$$

then

$$y_n <_T y_n^* \in W_n^\lambda \text{ and } Z_\lambda^{\sup}(y_n) - 1/(n+1) < Z_\lambda(y_n^*);$$

otherwise, $y_n <_T y_n^* \in T \upharpoonright \lambda$ and $Z_\lambda^{\sup}(y_n) - 1/(n+1) < Z_\lambda(y_n^*)$;

(c) $y_{n+1} >_T y_n^*$ & $ht(y_{n+1}) > \lambda_n$;

(see Lemma 10 for the definition of $Z_\lambda^{\sup}(y_n)$).

Put:
$$s_\lambda = \bigcup_{n<\omega} y_n \cup \{\langle r, \lambda\rangle\},$$

where the real r is taken so that $s_\lambda \in \mathfrak{T}_P$ (such an r is unique).

We set:
$$T_\lambda = \{s_\lambda\} \cup \{t_\lambda(x, q) : x \in T \upharpoonright \lambda,\ 0 < q \in \boldsymbol{Q}\}.$$

Case 3. Otherwise. We set:

$$T_\lambda = \{t_\lambda(x, q) : x \in T \upharpoonright \lambda,\ 0 < q \in \boldsymbol{Q}\}.$$

T_λ is thus defined. Now set:

$$T = \bigcup_{\alpha<\omega_1} T_\alpha.$$

T is clearly a P-tree. We can easily check that $T \in \boldsymbol{AT}$ by Lemma 9, $T \notin \boldsymbol{RE}$ by Lemma 10, $T \in \boldsymbol{NS}$ by Corollary 7, $T \in \gamma\boldsymbol{ST}$ by Lemma 11, using the following facts:

(a) even when Z_λ is a cofinal branch of $T \upharpoonright \lambda$, $Z_\lambda \neq \hat{x}$ for every $x \in T_\lambda$;

(b) if Z_λ is a function which embeds $T\restriction\lambda$ in $[0, 1)$, then for every $n<\omega$, $y_n<_T s_\lambda\in T_\lambda$ and $Z_\lambda^{\sup}(y_n)-1/n\leqq Z_\lambda^{\sup}(y_{n-1})-1/n<Z_\lambda(y_n)$;

(c) stationarily many ordinals $\in\Omega$ are put in Case 3 and for every such ordinal λ it holds that $(\forall x\in T_\lambda)[m(x)>0]$;

(d) if W_n^λ is an anti-chain of $T\restriction\lambda$ and $t=t_\lambda(x, q)$, then the one of the following holds:

1°. $x_n^*\in W_n^\lambda\cap\hat{t}$ and $(\forall z\in T\restriction\lambda)[x_n^*<_T z\to z\notin W_n^\lambda]$;

2°. $m(t)\leqq m(x_n^*)+q_n^*$ & $\forall z[x_n^*\lneqq_T z\in W_n^\lambda\to m(z)\geqq m(x_n)+2q_n^*]$;

(e) if W_n^λ is an anti-chain of $T\restriction\lambda$, Z_λ is a function which embeds $T\restriction\lambda$ in $[0, 1)$ and $t=s_\lambda$, then one of the following holds:

1°. $y_n^*\in W_n^\lambda\cap\hat{t}$ and hence $(\forall z\in T\restriction\lambda)[y_n^*<_T z\to z\notin W_n^\lambda]$;

2°. $Z_\lambda^{\sup}(y_n)-1/(n+1)<Z_\lambda(y_{n+1})$, $\forall z[y_n<_T z$ & $Z_\lambda^{\sup}(y_n)-1/(n+1)<Z_\lambda(z)\to z\notin W_n^\lambda]$ and hence $\forall z[y_{n+1}<_T z\to z\notin W_n^\lambda]$.

Theorem 12 is thus proved.

THEOREM 13 ($\diamondsuit^*$). $(\boldsymbol{NS}\backslash\boldsymbol{RE})\cap(\boldsymbol{AST}\backslash\gamma\boldsymbol{ST})\neq\emptyset$.

PROOF. Assume $\diamondsuit^*$. We can take $T\in(\boldsymbol{NS}\backslash\boldsymbol{RE})\cap\gamma\boldsymbol{ST}$ (Theorem 12) and $T'\in\boldsymbol{RE}\cap(\boldsymbol{AST}\backslash\gamma\boldsymbol{ST})$ (Devlin and Shelah [2, Theorem 4.4]). Then clearly $T\dotplus T'\in(\boldsymbol{NS}\backslash\boldsymbol{RE})\cap(\boldsymbol{AST}\backslash\gamma\boldsymbol{ST})$.

THEOREM 14 ($\diamondsuit$). $(\boldsymbol{NS}\backslash\boldsymbol{RE})\cap(\boldsymbol{NCA}\backslash\boldsymbol{AST})\neq\emptyset$.

PROOF. Let $\langle Z_\alpha:\alpha<\omega_1\rangle$ be a $\diamondsuit_{\mathfrak{X}}$-sequence. To define a P-tree, we construct each level T_α by induction on α ensuring that the following holds:

(1) if $\alpha<\beta<\omega_1$ & $x\in T_\alpha$ & $m(x)<q\in\boldsymbol{Q}$, there is a $y\in T_\beta$ such that $x\lneqq_T y$ & $m(y)<q$, and additionally if β is a successor ordinal, there is a $y'\in T_\beta$ such that $x\lneqq_T y'$ & $m(y')=q$.

Set: $T_0=\{0_T\}$;

$$T_{\alpha+1}=\{x\cup\{\langle q, \alpha+1\rangle\}: x\in T_\alpha,\ m(x)<q\in\boldsymbol{Q}\}.$$

Let $\lambda\in\Omega$ and suppose that $T\restriction\lambda$ has been defined. Fix an increasing sequence $\langle\lambda_n:n<\omega\rangle$ such that $\sup_{n<\omega}\lambda_n=\lambda$. For each $x\in T\restriction\lambda$ and each rational $q<m(x)$, we shall define $t_\lambda(x, q)$ as follows: First take an increasing sequence $\langle q_n:n<\omega\rangle$ such that $\lim_{n<\omega}q_n=q$ and $m(x)<q_0$. Pick x_n for every $n<\omega$ by induction so that:

$$x_0=x;$$

$$x_n\lneqq_T x_{n+1}\ \&\ ht(x_{n+1})>\lambda_n\ \&\ m(x_{n+1})=q_n,$$

(this is possible by (1)). We set:

$$t_\lambda(x, q)=\bigcup_{n<\omega} x_n \cup \{\langle q, \lambda\rangle\}.$$

Notice that $x \gneqq_T t_\lambda(x, q)$ and $m(t_\lambda(x, q))=q$.

Now we shall define T_λ.

Case 1. Z_λ is an cofinal branch of $T \restriction \lambda$. For each $x \in T \restriction \lambda$ and each rational $q>m(x)$, pick x^* so that $ht(x^*)=ht(x)+1$, $x<_T x^*$, $m(x^*)<q$ and $x^* \notin Z_\lambda$. Put:

$$s_\lambda(x, q)=t_\lambda(x^*, q).$$

We set:
$$T_\lambda=\{s_\lambda(x, q): x \in T \restriction \lambda,\ m(x)<q \in \boldsymbol{Q}\}.$$

Clearly $Z_\lambda \neq \{y \in T \restriction \lambda: y<_T s_\lambda(x, q)\}$.

Case 2. Z_λ is an anti-chain of $T \restriction \lambda$. For each $x \in T \restriction \lambda$ and each rational $q>m(x)$, pick x^* and $q^* \in \boldsymbol{Q}$ so that:

(a) if $(\exists w \in T \restriction \lambda)[x \leqq_T w \in Z_\lambda\ \&\ m(w)<q]$, then

$$x \leqq_T x^* \in Z_\lambda,\ m(x^*)<q \text{ and } m(x^*)<q^*<q;$$

(b) otherwise, $x^*=x$ and $m(x)<q^*<q$.

Put:
$$u_\lambda(x, q)=t_\lambda(x^*, q^*).$$

We set:
$$T_\lambda=\{u_\lambda(x, q): x \in T \restriction \lambda,\ m(x)<q \in \boldsymbol{Q}\}.$$

Case 3. Z_λ is a function which embeds $T \restriction \lambda$ in $[0, 1)$. Pick y_n for each $n<\omega$ by induction so that:

$$y_0=0_T;$$

$$y_{n+1}>_T y_n\ \&\ ht(y_{n+1})>\lambda_n\ \&\ Z_\lambda^{\sup}(y_n)-1/(n+1)<Z_\lambda(y_{n+1}).$$

Put:
$$v_\lambda=\bigcup_{n<\omega} y_n \cup \{\langle r, \lambda\rangle\},$$

where $r \in \boldsymbol{R}$ is taken so that $v_\lambda \in \mathfrak{T}_P$. We set:

$$T_\lambda=\{v_\lambda\} \cup \{t_\lambda(x, q): x \in T \restriction \lambda,\ m(x)<q \in \boldsymbol{Q}\}.$$

Case 4. Otherwise. We set:

$$T_\lambda=\{t_\lambda(x, q): x \in T \restriction \lambda,\ m(x)<q \in \boldsymbol{Q}\}.$$

T_λ is thus defined. Now we set:

$$T=\bigcup_{\alpha<\omega_1} T_\alpha.$$

Clearly T is P-tree. We can easily check that $T \in \boldsymbol{AT}$ by Lemma 9, $T \notin \boldsymbol{RE}$ by Lemma 10 and $T \in \boldsymbol{NS}$ by Corollary 7, using the following:

(a) $Z_\lambda \neq \hat{x}$ for every $x \in T_\lambda$, even if Z_λ is a cofinal branch of $T \restriction \lambda$;

(b) if Z_λ embeds $T\upharpoonright\lambda$ in $[0, 1)$, then $y_n < v_\lambda$ and

$$Z_\lambda^{\sup}(y_n) - 1/n \leqq Z_\lambda^{\sup}(y_{n-1}) - 1/n < Z_\lambda(y_n);$$

(c) stationarily many limit ordinals are put in Case 4, and for such an ordinal λ $m(x)>0$ for all $x\in T_\lambda$.

To see that $T\in \boldsymbol{NCA}$, suppose that there were a club anti-chain X of T. Put:

$C_1 = \{\lambda\in\Omega : (\forall x\in T\upharpoonright\lambda \forall q\in\boldsymbol{Q})[(\exists w\in T)R(x, w, X, q)\rightarrow(\exists w\in T\upharpoonright\lambda)R(x, w, X, q)]\}$,

where $R(x, w, X, q)$ stands for $x \gtrless_T w\in X$ & $m(w)<q$. Clearly C_1 is a club set. Hence so is $C = C_1\cap\{ht(x): x\in X\}$. So we can pick $\lambda\in\Omega$ so that $\lambda\in C\cap\{\alpha<\omega_1 : X\cap T\upharpoonright\alpha = Z_\alpha\}$. Then we can pick $t\in X\cap T_\lambda$ since $\lambda\in\{ht(t): t\in X\}$. Since $X\cap T\upharpoonright\lambda = Z_\lambda$, Z_λ is an anti-chain of $T\upharpoonright\lambda$ and so we can take $x\in T\upharpoonright\lambda$ and $q\in\boldsymbol{Q}$ so that $t = u_\lambda(x, q)$. Then $m(t) = q^* < q$. Thus $R(x, t, X, q)$ and hence $(\exists w\in T\upharpoonright\lambda)R(x, w, X, q)$ because $\lambda\in C_1$. Since $X\cap T\upharpoonright\lambda = Z_\lambda$, this implies that $x^*\in Z_\lambda$. Thus, x^*, $t\in X$ and $x^* <_T t$. This is absurd since X is an anti-chain. $T\in\boldsymbol{NCA}$ is thus shown.

On the other hand, it can be easily checked that the set $\{t_\lambda(0_T, 1): \lambda$ is a limit ordinal ordinal put in Case 4$\}$ is a stationary anti-chain and hence $T\notin\boldsymbol{AST}$, q. e. d.

THEOREM 15 ($\diamondsuit$). $(\boldsymbol{NS}\setminus\boldsymbol{RE})\setminus\boldsymbol{NCA}\neq\emptyset$.

PROOF. Assume $\diamondsuit$. We can take $T\in(\boldsymbol{NS}\setminus\boldsymbol{RE})\cap(\boldsymbol{NCA}\setminus\boldsymbol{AST})$ (by Theorem 14) and $T'\in\boldsymbol{SAT}\setminus\boldsymbol{NCA}$ ([3, Theorem 5]). Then clearly $T\dotplus T'\in(\boldsymbol{NS}\setminus\boldsymbol{RE})\setminus\boldsymbol{NCA}$.

§4. Proof of Lemma 11.

Let X be an uncountable anti-chain of T. Put:

$$C_0 = \{\alpha<\omega_1 : X\cap T\upharpoonright\alpha = W_i^\alpha \text{ for some } i\in\omega\},$$

$$C_1 = \{\lambda\in\Omega : (\forall y\in T\upharpoonright\lambda)[(\exists x\in X)[y<_T x]\rightarrow(\exists z\in X\cap T\upharpoonright\lambda)[y<_T z]]\},$$

$$C_2 = \{\lambda\in\Omega : (\exists x\in X)[y\gtrless_T x \ \& \ m(x)<q]\rightarrow(\exists z\in X\cap T\upharpoonright\lambda)[y\gtrless_T z \ \& \ m(z)<q]],$$

$$\text{for every } y\in T\upharpoonright\lambda \text{ and every } q\in\boldsymbol{Q}\}.$$

Let C be a club set such that $C\subseteq C_0\cap C_1\cap C_2$.

CLAIM 1. $X\cap T_\lambda = \emptyset$ for every $\lambda\in C$.

PROOF. Suppose $\lambda\in C$ and $x\in X\cap T_\lambda$. Pick $i\in\omega$ so that $X\cap T\upharpoonright\lambda = W_i^\lambda$. Since X is an anti-chain, W_i^λ is an anti-chain of $T\upharpoonright\lambda$. Hence by the assumption of the lemma, (a) or (b) must hold.

Case 1. (a) holds. Pick $y<_T x$ so that $(\forall z\in T\upharpoonright\lambda)[y<_T z\rightarrow z\notin W_i^\lambda]$. Since $W_i^\lambda=X\cap T\upharpoonright\lambda$ and $\lambda\in C_1$, $\neg(\exists x\in X)[y<_T x]$. This contradicts "$y<_T x\in X$".

Case 2. (b) holds. Pick $y\lneqq_T x$ and $q>0$ so that:

$$m(x)\leqq m(y)+q \quad\text{and}\quad (\forall z\in W_i^\lambda)[y\lneqq_T z\rightarrow m(z)\geqq m(y)+2q].$$

Since $X\cap T\upharpoonright\lambda=W_i^\lambda$ and $\lambda\in C_2$, $\neg(\exists x\in X)[y\lneqq_T x\ \&\ m(x)<m(y)+2q]$. This contradicts "$x\in X\ \&\ y\lneqq_T x\ \&\ m(x)\leqq m(y)+q$". Claim 1 is thus proved.

Let $\langle\lambda_\xi:\xi<\omega_1\rangle$ be the monotone enumeration of $C\cup\{0\}$. Let $\langle x_n^\xi: n<\omega\rangle$ be an enumeration of $X\cap T\upharpoonright(\lambda_{\xi+1}\backslash\lambda_\xi)$ such that $x_n^\xi\neq x_m^\xi$ if $n\neq m$, for each $\xi<\omega_1$.

We shall define w_n^ξ for each $\xi<\omega_1$ and each $n<\omega$.

Case 1. $ht(x_n^\xi)\notin\Omega$. w_n^ξ is taken so that $(w_n^\xi, x_n^\xi]$ is a singleton set, i.e. w_n^ξ is the immediate predecessor of x_n^ξ.

Case 2. $ht(x_n^\xi)\in\Omega$. First note that there is a $y<_T x_n^\xi$ such that $(y, x_n^\xi]\cap\bigcup_{j<n}\hat{x}_j^\xi=0$. (To see this, suppose not. Then $\hat{x}_n^\xi\subset\bigcup_{j<n}\hat{x}_j^\xi$ and hence $\hat{x}_n^\xi\subset\hat{x}_j^\xi$ for some $j<n$, which implies $x_n^\xi\leqq_T x_j^\xi$ (Lemma 5-(5)). But it is absurd since $x_n^\xi\neq x_j^\xi$ and X is an anti-chain).

Subcase 2.1. $m(x_n^\xi)=0$. Take y_n^ξ so that:

$$\lambda_\xi<ht(y_n^\xi),\quad y_n^\xi<_T x_n^\xi \quad\text{and}\quad (y_n^\xi, x_n^\xi]\cap\bigcup_{j<n}\hat{x}_j^\xi=0.$$

w_n^ξ is taken so that

$$y_n^\xi\lneqq_T w_n^\xi<_T x_n^\xi \quad\text{and}\quad m(w_n^\xi)>m(y_n^\xi)+1,$$

(this is possible by Lemma 5-(8)).

Subcase 2.2. $m(x_n^\xi)>0$. We can take y_n^ξ so that:

$$ht(y_n^\xi)>\lambda_\xi,\quad (y_n^\xi, x_n^\xi]\cap\bigcup_{j<n}\hat{x}_j^\xi=0 \quad\text{and}\quad y_n^\xi\lneqq_T x_n^\xi.$$

Then w_n^ξ is taken so that

$$y_n^\xi\lneqq_T w_n^\xi\lneqq_T x_n^\xi \quad\text{and}\quad m(x_n^\xi)-m(w_n^\xi)<m(w_n^\xi)-m(y_n^\xi),$$

(this is possible by Lemma 5-(7)).

w_n^ξ is thus defined. Now put:

$$U=\bigcup\{(w_n^\xi, x_n^\xi]:\xi<\omega_1,\ n<\omega\}.$$

This is a nbd of X.

Finally we shall define a nbd V of $T\upharpoonright C$ such that $U\cap V=0$. For this purpose, we shall define v^* for every $v\in T\upharpoonright C$. Let $v\in T\upharpoonright C$ and put $\lambda=ht(v)$. Let i be the number such that $W_i^\lambda=X\cap T\upharpoonright\lambda$. W_i^λ is clearly an anti-chain of $T\upharpoonright\lambda$. So by the assumption of the lemma, Condition (a) or (b) must hold for v (substituted for x).

Case 1. (a) holds. Then we can take $v^* <_T v$ so that

$$(\forall z \in T \upharpoonright \lambda)[v^* <_T z \rightarrow z \notin W_i^\lambda].$$

Case 2. (b) holds. Take $u \in T \upharpoonright \lambda$ and $q>0$ so that $u \lneqq_T v$ and

$$m(v) \leqq m(u)+q \quad \text{and} \quad (\forall z \in W_i^\lambda)[u \lneqq_T z \rightarrow m(z) \geqq m(u)+2q].$$

We may assume that $m(u)>m(v)-1$. (If not so, by Lemma 5-(7), there is u' such that $u \lneqq_T u' \lneqq_T v$ and $m(u')>m(v)-1$. Then take u' and $m(u)+q-m(u')$ instead of u and q.)

CLAIM 2. For at most only one pair $\langle \xi, n \rangle$, $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$.

PROOF. We show first that $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$ implies that (1) $ht(x_n^\xi)<\lambda$ and (2) $ht(x_n^\xi) \in \Omega$. To show (1), suppose not. Then by choice of w_n^ξ, $ht(x_n^\xi)> ht(w_n) \geqq \lambda_\xi \geqq \lambda$. Hence $(u, v] \cap (w_n^\xi, x_n^\xi] = \emptyset$ which contradicts the assumption. To show (2), suppose not. Then $(w_n^\xi, x_n^\xi] = \{x_n^\xi\}$. Hence $x_n^\xi \in (u, v]$, so $u \lneqq_T x_n^\xi$. Note $x_n^\xi \in W_i^\lambda$. (For, $x_n^\xi \in X \cap T \upharpoonright \lambda$ by (1) and $W_i^\lambda = X \cap T \upharpoonright \lambda$ by choice of i.) Hence by the property of q, $m(x_n^\xi) \geqq m(u)+2q>m(u)+q \geqq m(v)$. This is absurd since $x_n^\xi \leqq_T v$. Next we show that $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$ implies that $u \in (y_n^\xi, x_n^\xi]$, where y_n^ξ is as given in the definition of w_n^ξ. (Note that $ht(x_n^\xi) \in \Omega$ by the above.) Suppose that there is $t \in (u, v] \cap (w_n^\xi, x_n^\xi]$. Then $u \lneqq_T t \lneqq_T v$ and $y_n^\xi \leqq_T w_n^\xi <_T t$. So, u and y_n^ξ are comparable. It suffices to show that $y_n^\xi <_T u$. If $m(x_n^\xi)=0$, then $m(y_n^\xi)<m(w_n^\xi)-1<m(t)-1<m(v)-1<m(u)$ and so $y_n^\xi <_T u$. If $m(x_n^\xi)>0$, then $y_n^\xi \lneqq_T w_n^\xi \lneqq_T x_n^\xi$. Hence $u \lneqq_T x_n^\xi$, since $u \lneqq_T t \lneqq_T v$ and $w_n^\xi \lneqq_T t \leqq_T x_n^\xi$. So, by the property of q. $m(x_n^\xi) \geqq m(u)+2q$ and hence

$$m(t)-m(y_n^\xi)>m(w_n^\xi)-m(y_n^\xi)>m(x_n^\xi)-m(w_n^\xi)>m(x_n^\xi)-m(v)$$

$$>(2q+m(u))-(m(u)+q)=q \geqq m(v)-m(u)>m(t)-m(u),$$

which mean $y_n^\xi <_T u$. In both cases, $y_n^\xi <_T u$. Thus $(u, v] \cap (w_n^\xi, x_n^\xi] \neq \emptyset$ implies $u \in (y_n^\xi, x_n^\xi]$. The claim follows from this immediately. For, there is at most only one pair $\langle \xi, n \rangle$ which satisfies $u \in (y_n^\xi, x_n^\xi]$, since the intervals $(y_n^\xi, x_n^\xi]$, $\xi<\omega_1$ and $n<\omega$, have been taken so as to be mutually disjoint. Claim 2 is thus proved.

By this claim we can take v^* so that:

$$v^* <_T v \quad \text{and} \quad (\forall \xi<\omega_1, \forall n<\omega)[(v^*, v] \cap (w_n^\xi, x_n^\xi] = \emptyset].$$

v^* is thus defined for all $v \in T \upharpoonright C$. Clearly $(v^*, v] \cap U = \emptyset$. We set:

$$V = \bigcup \{(v^*, v] : v \in T \upharpoonright C\}.$$

Then V is a nbd of $T \upharpoonright C$ such that $U \cap V = \emptyset$. This completes the proof of

Lemma 11.

§ 5. Remark on Lemma 6.

First note that every ω_1-tree is isomorphic to some P-tree. Concerning Lemma 6 and Corollary 7, it would be natural to ask whether the former is essentially more general than the latter: i.e. whether the following condition (C1) is strictly weaker than (C2) for Aronszajn trees T:

(C1) there is a P-tree T' isomorphic to T such that

$$\{\alpha<\omega_1 : T'_\alpha \cap (T')^0 \text{ is finite}\} \text{ is stationary};$$

(C2) there is a P-tree T'' isomorphic to T such that

$$\{\alpha<\omega_1 : T''_\alpha \cap (T'')^0 = \emptyset\} \text{ is stationary}.$$

The answer is affimative: i.e. the following holds:

PROPOSITION ($\diamondsuit^*$). *There is an Aronszajn tree which satisfies* (C1) *but does not* (C2).

PROOF. Let $\langle \{W_i^\alpha : i<\omega\} : \alpha<\omega_1\rangle$ be a $\diamondsuit^*_{\mathfrak{X}}$-sequence and $\langle Z_\alpha : \alpha<\omega_1\rangle$ be a $\diamondsuit_{\mathfrak{X}}$-sequence. We construct a P-tree T such that $T_\lambda \cap T^0$ has at most one element for every $\lambda \in \Omega$ but (C2) does not hold. We define T_α for $\alpha<\omega_1$ inductively ensuring that:

(1) if $\alpha<\beta<\omega_1$ and $x \in T_\alpha$ and $m(x)<q\in \boldsymbol{Q}$, then there is a $y \in T_\beta$ such that $x \lneqq_T y$ and $m(y)<q$.

Put $T_0=\{0_T\}$ and $T_{\alpha+1}=\{x\cup\{\langle q, \alpha+1\rangle\} : x\in T_\alpha,\ m(x)<q\in Q\}$. Let $\lambda\in\Omega$ and suppose that $T\restriction\lambda$ has been defined. Let $\langle\lambda_n : n<\omega\rangle$ be a sequence such that $\lim_{n<\omega}\lambda_n=\lambda$. For each $x\in T\restriction\lambda$ and each rational $q>m(x)$, we pick x_n inductively so that:

(a) if Z_λ is a cofinal branch of $T\restriction\lambda$, then $x \lneqq_T x_0$ and $x_0 \notin Z_\lambda$ and $m(x_0)<q$; otherwise, $x_0=x$;

(b) $$x_{n+1}>_T x_n,\quad ht(x_{n+1})>\lambda_n \quad\text{and}\quad m(x_{n+1})<q.$$

Put:

$$t_\lambda(x, q)=\bigcup_{n<\omega} x_n \cup \{\langle \sup_{n<\omega} m(x_n), \lambda\rangle\}.$$

Let $K(n)$ mean the number such that $n=2^m(2\cdot K(n)+1)-1$ for some $m\in\omega$. Now, we shall define y_n by induction as follows:

I. if Z_λ is a cofinal branch of $T\restriction\lambda$ then y_0 is taken so that $y_0\notin Z_\lambda$; otherwise, $y_0=0_T$;

II. (a) if $W^{\lambda}_{K(n)}$ is a function from $T\restriction\lambda$ to $[0,\infty)$, then y_{n+1} is taken so that $y_{n+1}>_T y_n$ and $ht(y_{n+1})>\lambda_n$ and one of the following holds:

1°. $W^{\lambda}_{K(n)}(y_{n+1})\geqq n$,

2°. $W^{\lambda}_{K(n)}(y_{n+1})>\sup\{W^{\lambda}_{K(n)}(y): y_n<_T y\in T\restriction\lambda,\ ht(y)>\lambda_n\}-1/n$;

(b) otherwise, y_{n+1} is taken so that $y_{n+1}>_T y_n$ and $ht(y_{n+1})>\lambda_n$.

Put:

$$u_\lambda=\bigcup_{n<\omega} y_n\cup\{\langle r,\lambda\rangle\},$$

where r is taken so that $u_\lambda\in\mathfrak{T}_P$. We set:

$$T_\lambda=\{u_\lambda\}\cup\{t_\lambda(x,q): x\in T\restriction\lambda,\ m(x)<q\in\boldsymbol{Q}\}.$$

Then the tree $T=\bigcup_{\alpha<\omega_1} T_\alpha$ is as required. To see that (C2) is false, take arbitrarily an isomorphic P-tree T' and an isomorphism f from T to T'. Define a function $e: T\to\boldsymbol{R}$ by $e(x)=m(f(x))$. Take club sets $C_0=\{\lambda\in\Omega: (\forall x\in T\restriction\lambda\forall q\in\boldsymbol{Q})[(\exists y\in T)[x<_T y\ \&\ e(y)>q]\to(\exists y\in T\restriction\lambda)[x<_T y\ \&\ e(y)>q]]\}$ and $C_1\subseteq\{\lambda\in\Omega: W^{\lambda}_i=e\restriction(T\restriction\lambda)$ for some $i\}$.

CLAIM. $e(u_\lambda)=0$ for every $\lambda\in C_0\cap C_1$.

PROOF. Suppose $e(u_\lambda)>0$ and $\lambda\in C_0\cap C_1$. Pick $i\in\omega$ so that $W^{\lambda}_i=e\restriction(T\restriction\lambda)$. Then we can take a $v<_T u_\lambda$ such that $f(v)\rightleftarrows_{T'} f(u_\lambda)$. Let t be an immediate successor of u_λ. Pick $n\in\omega$ so that: $\lambda_n>ht(v)$, $n>e(u_\lambda)$, $e(t)-e(u_\lambda)>1/n$ and $K(n)=i$. Recall y_{n+1} in the definition of u_λ. Then 1° or 2° must hold. First notice that $f(v)\rightleftarrows_{T'} f(y_{n+1})\rightleftarrows_{T'} f(u_\lambda)$, since $ht(v)<\lambda_n<ht(y_{n+1})$, $f(v)\rightleftarrows_{T'} f(u_\lambda)$ and $f(y_{n+1})<_{T'} f(u_\lambda)$. And so $e(v)<e(y_{n+1})<e(u_\lambda)<n$.

Case 1. 1° holds. Then $e(y_{n+1})=W^{\lambda}_{K(n)}(y_{n+1})\geqq n$. This is absurd by the above notice.

Case 2. 2° holds. By $\lambda\in C_0$, $e(u_\lambda)>e(y_{n+1})>\sup\{e(y): y_n<y\in T\restriction\lambda\}-1/n=\sup\{e(y): y_n<y\in T\}-1/n\geqq e(t)-1/n$. This is absurd since $e(t)-e(u_\lambda)>1/n$. Claim is thus proved.

It is obvious by the claim that T does not have property (C2). Proposition is thus proved.

References

[1] Baumgartner, J. E., Decompositions and embeddings of trees, Notices A. M. S. 17 (1970), 967.

[2] Devlin, K. J. and Shelah, S., Souslin properties and tree topologies, Proc. London

Math. Soc. **39** (1979), 237-252.
[3] Hanazawa, M., On a classification of Aronszajn trees, to appear.

Department of Mathematics
Faculty of Science
Saitama University
Urawa, Saitama
338 Japan