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ANTILOCALITY AND ONE-SIDED ANTILOCALITY FOR STABLE GENERATORS ON THE LINE

By

Yasushi ISHIKAWA

1. Introduction.

Let X be an open domain in $\boldsymbol{R}^n$. Consider a linear operator $A: C_0^\infty(X)\rightarrow C^\infty(X)$, where $C^\infty(X)$ is the class of infinitely differentiable functions on X and $C_0^\infty(X)$ is the set of functions of $C^\infty(X)$ which have compact support in X. We say A is antilocal if $\operatorname{supp} f\cup\operatorname{supp} Af=X$ for every $f\in C_0^\infty(X)$ such that $f\not\equiv 0$. Equivalently, if $f=Af=0$ in an open subset of X, then $f\equiv 0$ in X.

Antilocality was firstly proved by Reeh-Schlieder [7] for the operator $(m^2I-\Delta)^{1/2}$, where Δ denotes the Laplacian. Subsequently it was extended by Goodman-Segal [1], Masuda [6] and Murata [3] for $(m^2I-\Delta)^\lambda$, $\lambda\in\boldsymbol{C}\backslash\boldsymbol{Z}$. Recently it was extended to the complex powers (z-powers) of elliptic diffetential operators with analytic coefficients of order m such that $mz\notin 2\boldsymbol{Z}$ by Liess [2].

In this paper we study the following operators:

$$(*)\qquad \mathfrak{a}_{p,q}(D)f(x)\equiv\int_{-\infty}^{+\infty}(f(x+y)-f(x))[p1_{\boldsymbol{R}_-}(y)+q1_{\boldsymbol{R}_+}(y)]\frac{dy}{|y|^{1+\alpha}},$$

where $p\geqq 0$, $q\geqq 0$, $p+q=1$, $0<\alpha<1$ and $1_{\boldsymbol{R}_\pm}(y)=1$ or 0 according as $y\in\boldsymbol{R}_\pm$ or not. Here $\boldsymbol{R}_+=(0,+\infty)$ and $\boldsymbol{R}_-=(-\infty,0)$. These operators appear as generators of stable processes on the line with index α in probability theory. So we call them stable generators. In case $p=q$ it is known that the stable generator with index α is $\alpha/2$-power of the constant multiple of $-\Delta$, and therefore it is antilocal by the result mentioned above. However, in case $p\neq q$, the stable generator is not a fractional power of $-\Delta$. Especially, in case $p=0$, $q=1$, this is completely asymmetric. Indeed, the trajectory of stable process with index α moves only to the right only in case $q=1$. Therefore it would not be expected that the antilocality holds for this case, and so we introduce the one-sided antilocality as follows:

DEFINITION. *An operator $T: C_0^\infty(\boldsymbol{R}^1)\rightarrow C^\infty(\boldsymbol{R}^1)$ is antilocal to the right (to the left), if $f\equiv 0$ in $U+\boldsymbol{R}_+$ (resp. $f\equiv 0$ in $U+\boldsymbol{R}_-$) for every $f\in C_0^\infty(\boldsymbol{R}^1)$ such that $f=Tf=0$ in U, where U is an open subset in $\boldsymbol{R}^1$ and $U+\boldsymbol{R}_\pm\equiv\{x+y\in\boldsymbol{R}^1; x\in U, y\in\boldsymbol{R}_\pm\}$. T is*

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simply called antilocal if T is antilocal both to the right and to the left.

In case that T is antilocal to the right (to the left) and is not antilocal, T is called one-sided antilocal to the right (to the left).

Our result is:

THEOREM. *If both p and q are positive, then $\mathfrak{a}_{p,q}(D)$ is antilocal. If $q=1$ ($p=1$), then $\mathfrak{a}_{p,q}(D)$ is one-sided antilocal to the right (resp. to the left).*

As mentioned above, these operators have a probabilitistic meaning. However our proof of Theorem heavily depends on the theory of analytic pseudodifferential operators and is carried out without using probability theory.

In the following part of this paper, we will only treat the case $q=1$ in case $p\cdot q=0$ for simplicity, since the result for the case $p=1$ follows similarly by changing p and q and the signature. And so we say simply one-sided antilocal in place of one-sided antilocal to the right.

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2. Preliminaries I.

In this section we introduce some terminologies and prove a lemma which follows from the Paley-Wiener-Schwartz theorem. For a suitable function f on $\boldsymbol{R}^1$ we denote the Fourier transform of f by $\hat{f}$ or $\mathfrak{F}f$. That is

$$\hat{f}(\xi)=\mathfrak{F}f(\xi)=\int_{R^1} e^{-ix\xi}f(x)dx\,.$$

The celebrated Paley-Wiener-Schwartz theorem states that

The Paley-Wiener-Schwartz theorem. Let f be a temperate distribution on $\boldsymbol{R}^1$. Then the following two conditions are equivalent;

a) $\operatorname{supp} f\subset(-\infty, 0]$ (resp. $[0, +\infty)$)

b) There exists $G(\zeta)$ which is holomorphic in $\{\operatorname{Im}\zeta>0\}$ (resp. in $\{\operatorname{Im}\zeta<0\}$) and satisfies $G(\zeta)=O(e^{\varepsilon|\zeta|})$ in $\{\operatorname{Im}\zeta\geqq\varepsilon\}$ (resp. in $\{\operatorname{Im}\zeta\leqq-\varepsilon\}$) for any $\varepsilon>0$ and such that $\hat{f}(\xi)=G(\xi+i0)$ (resp. $G(\xi-i0)$).

We next note that the operator $\mathfrak{a}_{p,q}(D)$ may be written as

$$\mathfrak{a}_{p,q}(D)f(x)=\mathfrak{F}^{-1}[\mathfrak{a}_{p,q}(\xi)\hat{f}(\xi)](x),$$

where

$$\mathfrak{a}_{p,q}(\xi)=-\frac{\Gamma(1-\alpha)}{\alpha}\left[\cos\frac{\pi\alpha}{2}-i(q-p)sgn(\xi)\sin\frac{\pi\alpha}{2}\right]|\xi|^{\alpha}.$$

See Feller [4] page 541.

Probabilists call $\mathfrak{a}_{p,q}(\xi)$ the exponent of the stable process of index α but we use the term "symbol" of the operator $\mathfrak{a}_{p,q}(D)$ following the terminology of pseudodifferential operators.

The restriction $\mathfrak{a}_{p,q}|_{(0,+\infty)}$ of $\mathfrak{a}_{p,q}$ to $(0,+\infty)$ has an analytic continuation to $\boldsymbol{C}\backslash(-\infty,0]$, which we denote by $\mathfrak{a}_{p,q}(\zeta)$. Choosing the branch from the upper half plane of $\mathfrak{a}_{p,q}(\zeta)$, we extend the domain of $\mathfrak{a}_{p,q}(\zeta)$ to the negative real axis. We denote it by $\mathfrak{a}^{+}_{p,q}(\xi)$. We also choose the branch from the lower half plane on the negative real axis and denote it by $\mathfrak{a}^{-}_{p,q}(\xi)$. That is

$$\mathfrak{a}^{\pm}_{p,q}(\xi)=\begin{cases}-\dfrac{\Gamma(1-\alpha)}{\alpha}\left[\cos\dfrac{\pi\alpha}{2}-i(q-p)\sin\dfrac{\pi\alpha}{2}\right]\xi^{\alpha}, & \xi\in(0,+\infty),\\ -\dfrac{\Gamma(1-\alpha)}{\alpha}\left[\cos\dfrac{\pi\alpha}{2}-i(q-p)\sin\dfrac{\pi\alpha}{2}\right]e^{\pm i\alpha\pi}(-\xi)^{\alpha}, & \xi\in(-\infty,0).\end{cases}$$

Put

$$\mathfrak{a}^{\pm}_{p,q}(D)f(x)=\mathfrak{F}^{-1}[\mathfrak{a}^{\pm}_{p,q}(\xi)\hat{f}(\xi)](x),\qquad f\in C_0^{\infty}(\boldsymbol{R}^1).$$

Then we have

LEMMA 1. *For every function f of $C_0^{\infty}(\boldsymbol{R}^1)$ with* $\operatorname{supp} f\subset(-\infty,0]$ (*resp.* $\operatorname{supp} f\subset[0,+\infty)$), *we have*

$$\mathfrak{a}^{+}_{p,q}(D)f(x)=0 \text{ for } x>0,\qquad (resp.\ \mathfrak{a}^{-}_{p,q}(D)f(x)=0 \text{ for } x<0).$$

PROOF. We only prove the first statement, since the second follows similarly.

We first note that $\mathfrak{a}^{+}_{p,q}(\zeta)$ is holomorphic in $\{\operatorname{Im}\zeta>0\}$. Let $G(\zeta)$ be as in the Paley-Wiener-Schwartz theorem for the given f. Since the order α in ζ of $\mathfrak{a}^{+}_{p,q}(\zeta)$ is at most one, it follows that

$$\mathfrak{a}^{+}_{p,q}(\zeta)G(\zeta)=O(e^{\varepsilon|\zeta|})\quad\text{on } \{\operatorname{Im}\zeta\geqq\varepsilon\}$$

for any $\varepsilon>0$. Thus the statement of the lemma follows directly by the same theorem. Q.E.D.

LEMMA 2. *For every $f\in C_0^{\infty}(\boldsymbol{R}^1)$, there exists a function $G^{\pm}=G^{\pm}_f$ which is holomorphic in* $\{\operatorname{Im}\zeta<0\}$ *such that*

$$G^{\pm}(x-i0)=\mathfrak{a}_{p,q}(D)f(x)-\mathfrak{a}^{\pm}_{\bar{p},q}(D)f(x).$$

PROOF. Since

$$(\mathfrak{a}_{p,q}(D)-\mathfrak{a}^{\pm}_{\bar{p},q}(D))f(x)=\mathfrak{F}^{-1}\Big[1_{R_-}(\xi)\hat{f}(\xi)\Big\{-\frac{\Gamma(1-\alpha)}{\alpha}\Big(\Big(\cos\frac{\pi\alpha}{2}+i(q-p)\sin\frac{\pi\alpha}{2}\Big) -\Big(\cos\frac{\pi\alpha}{2}-i(q-p)\sin\frac{\pi\alpha}{2}\Big)e^{\pm i\alpha\pi}\Big)\Big\}|\xi|^{\alpha}\Big](x),$$

it follows that

$$\mathfrak{F}[(\mathfrak{a}_{p,q}(D)-\mathfrak{a}^{\pm}_{\bar{p},q}(D))f](\xi)=0 \qquad \text{for } \xi>0, \text{ and so}$$
$$\operatorname{supp}\mathfrak{F}^{-1}[(\mathfrak{a}_{p,q}(D)-\mathfrak{a}^{\pm}_{\bar{p},q}(D))f]\subset[0,+\infty).$$

Now we have only to apply the Paley-Wiener-Schwartz theorem. Q.E.D.

REMARK. In case $q=1$, G^+ is identically zero. This fact reflects the one-sided antilocality for $\mathfrak{a}_{0,1}(D)$. See Lemma 3.

3. Preliminaires II.

In this section we prepare some results on analytic pseudodifferential operators, especially in connection with singular spectrum. Subsequently we give a lemma which plays a key role in proving our theorem. For details confer with Kaneko [5] and its references.

A distribution (more generally a hyperfunction) u is said to be micro-analytic at $(x^0, -i\xi dx\infty)$ (resp. $(x^0, i\xi dx\infty)$), denoted by $(x^0, -i\xi dx\infty)\notin$S.S.u (resp. $(x^0, i\xi dx\infty)\notin$ S.S.u), where S.S.u denotes the singular spectrum of u, if u admits the analytic continuation into the half space $\{z\in\boldsymbol{C}; \operatorname{Re}\langle -i\xi, z\rangle>0\}$ (resp. $\{z\in\boldsymbol{C}; \operatorname{Re}\langle i\xi, z\rangle>0\}$) near the point $x^0\in\boldsymbol{R}^1$.

The following theorem plays a key role in the proof of our result.

THEOREM (Kashiwara-Kawai cf. Kaneko [5]).
Let $u(x)$ be a distribution (more generally a hyperfunction) defined on a neighborhood of $0\in\boldsymbol{R}^1$ with $\operatorname{supp} u\subset[0,+\infty)$. If u is micro-analytic at $(0, idx\infty)$ or $(0, -idx\infty)$, then u vanishes on a neighborhood of 0.

We next give a brief explanation of some notations to quote two theorems.
For an open cone $\Gamma\subset\boldsymbol{R}^1\backslash\{0\}$ and $\varepsilon>0$, $\delta>0$, we put

$$\Gamma_{\varepsilon,\delta}\equiv\{\zeta\in\boldsymbol{C}; \operatorname{Re}\zeta\in\Gamma, |\zeta|>\delta, |\operatorname{Im}\zeta|<\varepsilon|\operatorname{Re}\zeta|\}.$$

Let $S^{\mu}(\Gamma)$ (Γ denotes an open cone in $\boldsymbol{R}^1\backslash\{0\}$) be the set of all functions $a(\xi)\in$

$C^\infty(\Gamma)$ such that for every open cone $\Gamma' \Subset \Gamma$ there are $\varepsilon>0$, $\delta>0$, and $c>0$ for which $a(\xi)$ extends to an analytic function on $\Gamma'_{\varepsilon,\delta}$ which satisfies $|a(\xi)| \leqq c(1+|\zeta|)^\mu$ on $\Gamma'_{\varepsilon,\delta}$.

The elements of $S^\mu(\Gamma)$ will be called analytic symbols with constant coefficients of order μ defined on Γ.

We denote by $SF^\mu(\Gamma)$ the set of all formal sums $\sum_{j\geq 0} a_j(\xi)$, $a_j \in S^{\mu-j}(\Gamma)$, with the property below;

> for every cone $\Gamma' \Subset \Gamma$ there are $\varepsilon>0$, $\delta>0$, $c>0$ and $A>0$ such that every $a_k(\xi)$ can be extended as an analytic function on $\Gamma'_{\varepsilon,\delta}$ and satisfies
> $|a_k(\zeta)| \leqq cA^k k!(1+|\zeta|)^{\mu-k}$ on $\Gamma'_{\varepsilon,\delta}$.

For $\sum a_j$, $\sum b_j \in SF^\mu(\Gamma)$, we write $\sum a_j \sim \sum b_j$ in $SF^\mu(\Gamma)$ if for every open cone $\Gamma' \Subset \Gamma$ there exist $\varepsilon>0$, $\delta>0$, $c>0$ and $A>0$ such that

$$\Big|\sum_{j<s}(a_j(\zeta)-b_j(\zeta))\Big| \leqq cA^s s!(1+|\zeta|)^{\mu-s} \quad \text{on } \Gamma'_{\varepsilon,\delta}$$

for every integer $s>0$.

Let $S^\mu_{10}(\boldsymbol{R}^1)$ be the class of classical pseudodifferential operators with constant coefficients of order μ of type $(1,0)$, that is the set of all functions $a(\xi) \in C^\infty(\boldsymbol{R}^1)$ such that for every $j>0$ there exists $C_j>0$ for which $a(\xi)$ satisfies

$$\left|\frac{d^j}{d\xi^j}a(\xi)\right| \leqq C_j(1+|\xi|)^{\mu-j} \quad \text{in } \boldsymbol{R}^1.$$

By $S^\mu(\boldsymbol{R}^1, \Gamma)$ we denote the space of symbols $a \in S^\mu_{10}(\boldsymbol{R}^1)$ such that the restriction $a(\xi)$ to Γ belongs to $S^\mu(\Gamma)$.

Next two theorems are important in the proof.

THEOREM (M. Sato [8], L. Hörmander [9]).

Consider $a \in S^\mu(\boldsymbol{R}^1, \Gamma)$ and suppose there exists $b \in S^{-\mu}(\Gamma)$ such that the restriction (also denoted by a) of a to Γ satisfies $ab \sim 1$ in $SF^0(\Gamma)$ and further $(x^0, i\xi^0 dx\infty) \notin$ S. S. $\mathfrak{a}(D)f$ for some $\xi^0 \in \Gamma$. Then

$$(x^0, i\xi^0 dx\infty) \notin \text{S. S. f}.$$

THEOREM (Analytic pseudolocal property, cf. Liess [2]).

For $a \in S^\mu(\boldsymbol{R}^1, \Gamma)$, if $(x^0, i\xi^0 dx\infty) \notin$ S. S. f, then $(x^0, i\xi^0 dx\infty) \notin$ S. S. $\mathfrak{a}(D)f$ for $(x^0, \xi^0) \in U \times \Gamma$, where U is a domain.

Now we return to our operators $\mathfrak{a}_{p,q}(D)$ and $\mathfrak{a}^\pm_{p,q}(D)$ introduced in § 1 and in § 2 respectively. Let $\omega(\xi)$ be a function of $C^\infty(\boldsymbol{R}^1)$ which is identically one for large $|\xi|$ (e.g. $|\xi| \geqq 1/4$) and vanishes near zero. For a suitable analytic symbol $a(\xi)$, we define a pseudodifferential operator $\mathfrak{a}'(D)$ by

$$\mathfrak{a}'(D)f(x)\equiv\mathfrak{F}^{-1}[\omega(\xi)a(\xi)\hat{f}(\xi)](x).$$

We note that both $\mathfrak{a}_{p,q}(\xi)\omega(\xi)$ and $\mathfrak{a}^{\#}_{p,q}(\xi)\omega(\xi)$ belong to $S^{\alpha}(\boldsymbol{R}^1, \boldsymbol{R}_+)\cap S^{\alpha}(\boldsymbol{R}^1, \boldsymbol{R}_-)$.

REMARK. Consider $\mathfrak{a}(D)f$ and $\mathfrak{a}'(D)f$ defined as above. Then we have

$$\text{S. S. } \mathfrak{a}(D)f\ni(x^0, i\xi^0 dx\infty) \Longleftrightarrow \text{S. S. } \mathfrak{a}'(D)f\ni(x^0, i\xi^0 dx\infty)$$

for $(x^0, \xi^0)\in\boldsymbol{R}^1\times S^0$.

Combining the Theorem (Analytic pseudolocal property) with the above, we see that $\mathfrak{a}_{p,q}(D)f$ is real analytic in $\boldsymbol{R}^1\backslash\text{supp} f$ for every $f\in C_0^\infty(\boldsymbol{R}^1)$.

Indeed, since $\mathfrak{a}(\xi)(1-\omega(\xi))$ is a symbol with constant coefficients of compact support,

$$\mathfrak{a}(D)f-\mathfrak{a}'(D)f=\mathfrak{F}^{-1}[\mathfrak{a}(\xi)(1-\omega(\xi))\hat{f}(\xi)]$$

is an entire function. Thus the first assertion follows directly.

For the proof of the second, note that

$$(\boldsymbol{R}^1\times\boldsymbol{R}_\pm)\cap\text{S. S. } \mathfrak{a}'_{p,q}(D)f\subset\text{S. S. f},$$

since $\omega(\xi)\mathfrak{a}_{p,q}(\xi)$ is in $S^{\alpha}(\boldsymbol{R}^1, \boldsymbol{R}_-)$. That is $\text{S. S. } \mathfrak{a}'_{p,q}(D)f\subset\text{S. S. f}$. So it now follows that

$$\text{S. S. } \mathfrak{a}_{p,q}(D)\subset\text{S. S. f}.$$

Since the analytic singular support of u (A-sing supp u for short) is the projection to $\boldsymbol{R}^1$ of $\text{S. S. u}\subset\boldsymbol{R}^1\times iS_\infty^{*0}$, we have

$$A\text{-sing supp } \mathfrak{a}_{p,q}(D)f\subset A\text{-sing supp} f\subset\text{supp} f.$$

This shows that the second assertion holds.

LEMMA 3. *Let f be in $C_0^\infty(\boldsymbol{R}^1)$. Suppose that $B\cap\text{supp} f\subset[x^0, +\infty)$ for some open neighborhood B of x^0 in $\boldsymbol{R}^1$. Suppose further that there are $\varepsilon>0$ and a real analytic function h in $\{|x-x^0|<\varepsilon\}$ such that*

$$h=\mathfrak{a}_{p,q}(D)f \qquad in\ (x^0-\varepsilon, x^0).$$

Then

$$x^0\notin\text{supp} f.$$

In case $q\neq1$ and only in this case the same conclusion holds even if we replace $\text{supp} f\subset B\cap[x^0, +\infty)$ with $\text{supp} f\subset(-\infty, x^0]\cap B$ and $h=\mathfrak{a}_{p,q}(D)f$ in $(x^0-\varepsilon, x^0)$ with $h=\mathfrak{a}_{p,q}(D)f$ in $(x^0, x^0+\varepsilon)$ respectively.

PROOF. We may assume that $\varepsilon>0$ is sufficiently small so that $\{|x-x^0|<\varepsilon\}\subset B$. We write $f=f_1+f_2$ with $f_1, f_2\in C_0^\infty(\boldsymbol{R}^1)$, such that f_1 is concentrated near x^0 and such that the support of f_2 avoids x^0. In view of the remark, we see that $\mathfrak{a}_{p,q}(D)f_2$ is real analytic near x^0, so that the hypothesis of f is also satisfied for f_1. And so we assume $f=f_1$.

Put

$$u^{\pm}(x)\equiv 1_B(x)(\mathfrak{a}_{p,q}(D)f(x)-\mathfrak{a}^{\pm}_{p,q}(D)f(x)-h(x)).$$

Then $u^{\pm}$ is in $C^\infty(\{|x-x^0|<\varepsilon\})$. Applying Lemma 1 to f and using the assumption of this lemma, we obtain

$$\operatorname{supp} u^{+}\subset\{x\leqq x^0\} \text{ and that } \operatorname{supp} u^{-}\subset\{x\geqq x^0\}.$$

On the other hand, Lemma 2 together with the fact that h is real analytic near x^0 tells us that

$$\mathrm{S.S.}\, u^{\pm}\cap(B\times\boldsymbol{R}^1)=\mathrm{S.S.}\,(\mathfrak{a}_{p,q}(D)f-\mathfrak{a}^{\pm}_{p,q}(D)f\cap(B\times\boldsymbol{R}^1)\not\ni(x^0, idx\infty),$$

where $(x^0, idx\infty)=(x^0, i1dx\infty)$. Hence by the Kashiwara-Kawai theorem we have

$$\mathrm{S.S.}\,(\mathfrak{a}_{p,q}(D)f-\mathfrak{a}^{\pm}_{p,q}(D)f)\not\ni(x^0, -idx\infty).$$

So we have

$$\mathrm{S.S.}\,\mathfrak{F}^{-1}[\omega(\xi)(\mathfrak{a}_{p,q}(\xi)-\mathfrak{a}^{\pm}_{p,q}(\xi))\hat{f}(\xi)]\not\ni(x^0, -idx\infty).$$

Define

$$R^{\pm}(\xi)\equiv 1_{R_-}(\xi)(\mathfrak{a}_{p,q}(\xi)-\mathfrak{a}^{\pm}_{p,q}(\xi))^{-1},$$

in case $0<q<1$. Note that $\mathfrak{a}_{p,q}(\xi)-\mathfrak{a}^{+}_{p,q}(\xi)=0$ in case $q=1$ and so we do not define $R^{+}(\xi)$ and define only $R^{-}(\xi)$. Obviously

$$\omega(\xi)(\mathfrak{a}_{p,q}(\xi)-\mathfrak{a}^{\pm}_{p,q}(\xi))R^{\pm}(\xi)\sim 1 \qquad \text{in } SF^0(\boldsymbol{R}_-).$$

Since $\omega(\xi)(\mathfrak{a}_{p,q}(\xi)-\mathfrak{a}^{\pm}_{p,q}(\xi))\in S^{\alpha}(\boldsymbol{R}^1, \boldsymbol{R}_-)$ and $R^{\pm}(\xi)\in S^{-\alpha}(\boldsymbol{R}_-)$, it follows from the regulality theorem of M. Sato-L. Hörmander that

$$\mathrm{S.S.}\, f\not\ni(x^0, -idx\infty).$$

Applying the Kashiwara-Kawai theorem again, it follows that f must vanish near x^0. Q.E.D.

4. Proof of the Theorem.

Let f be in $C_0^\infty(\boldsymbol{R}^1)$ and U be a bounded open subset in $\boldsymbol{R}^1$. We put $Y=\boldsymbol{R}^1$ in case $p\cdot q>0$ and $Y=\boldsymbol{R}_+$ in case $q=1$. It is sufficient to show that

(4.1) If $f\not\equiv 0$ in $U+Y$ for each connected open $U\subset\boldsymbol{R}^1\backslash\operatorname{supp} f$, then

$$\text{(4.2)} \qquad \operatorname{supp} \mathfrak{a}_{p,q}(D)f \supset U.$$

Indeed, if $f=\mathfrak{a}_{p,q}(D)f=0$ in an open set U_0 then it is easily seen that $f\equiv 0$ in $U_0 + Y$ by (4.2).

Assume (4.1) holds. Let us choose the connected component T of $(U+Y)\setminus \operatorname{supp} f$ with contains U. Clearly $U+Y=T+Y$ and $T\neq U+Y$.

Noting that $\mathfrak{a}_{p,q}(D)f$ is real analytic in T by the remark in § 3, there is no accumulation point of zero's of $\mathfrak{a}_{p,q}(D)f$ in T. In fact, if such a point exists, $\mathfrak{a}_{p,q}(D)f\equiv 0$ in T (hence in $\bar{T}$) and therefore $\bar{T}\cap\partial(\operatorname{supp} f)$ must be empty by Lemma 3 in case $p\cdot q>0$. In case $q=1$, if such an accumulated point exists, there is no right endpoint of $\bar{T}$ by the same reason. Hence $T=U+Y$ and this contradicts to the assumption (4.1).

Hence it follows that

$$\operatorname{supp} \mathfrak{a}_{p,q}(D)f \supset T \supset U. \qquad \text{Q.E.D.}$$

The following example shows that $\mathfrak{a}_{0,1}(D)$ is not antilocal:

Let f be a function in $C_0^\infty(\boldsymbol{R}^1)$ such that

$$f=\begin{cases} 1, & x\in(-3,-2), \\ 0, & x\in(-\infty,-4)\cup(-1,+\infty). \end{cases}$$

Let U be $(-1/2, 1/2)$. Noting

$$\mathfrak{a}_{0,1}(D)f(x)\equiv\int_0^{+\infty}(f(x+y)-f(x))\frac{dy}{|y|^{1+\alpha}},$$

we see that both f and $\mathfrak{a}_{0,1}(D)f$ vanish in U, whereas $f\not\equiv 0$ in $\boldsymbol{R}^1$.

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Yasushi ISHIKAWA
Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki, 305 Japan.