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ON OBSERVABLE AND STRONGLY OBSERVABLE HOPF IDEALS

By

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Let $\mathfrak{G}=\mathfrak{Sp}_k A$ be an affine group scheme over a ground field k, $\mathfrak{H}=\mathfrak{Sp}_k A/I$ a closed subgroup scheme and $\bar{k}$ denote an algebraic closure of k. When $\mathfrak{G}$ is algebraic and $\mathfrak{G}(k)$ is dense in $\mathfrak{G}(\bar{k})$, Bialynichi-Birula, Hochschild and Mostow defined in their paper [1] that $\mathfrak{H}(k)$ is observable in $\mathfrak{G}(k)$ if every finite dimensional rational $\mathfrak{H}(k)$-module can be embedded in a rational $\mathfrak{G}(k)$-module as a $\mathfrak{H}(k)$-module, and showed that one of its necessary and sufficient conditions is $\mathfrak{G}(\bar{k})/\mathfrak{H}(\bar{k})$ being a quasi-affine variety. When $\mathfrak{G}$ is algebraic and reduced, and $k=\bar{k}$, Cline, Parshall and Scott defined in their paper [2] that $\mathfrak{H}(k)$ is strongly observable in $\mathfrak{G}(k)$ if every rational $\mathfrak{H}(k)$-module N can be embedded in a rational $\mathfrak{G}(k)$-module M as a $\mathfrak{H}(k)$-module such that $N^{\mathfrak{H}(k)}=M^{\mathfrak{G}(k)}$, and showed that one of its necessary and sufficient conditions is $\mathfrak{G}(k)/\mathfrak{H}(k)$ being an affine variety. Since these concepts are the representation-theoretic ones, we can extend them to a Hopf ideal of an arbitrary (not necessarily commutative) Hopf algebra. In this paper, we characterize a strongly observable Hopf ideal of an arbitrary Hopf algebra; a Hopf ideal I of a Hopf algebra A is strongly observable if and only if A is an injective A/I-comodule. This result and the one of M. Takeuchi [10] give the following equivalent characterizations of a strongly observable k-subgroup $\mathfrak{H}$ of an affine k-group $\mathfrak{G}$; (1) $\mathcal{O}(\mathfrak{G})$ is an injective k-$\mathfrak{H}$-module, (2) $\mathfrak{H}$ is exact in $\mathfrak{G}$ and (3) $\widetilde{\mathfrak{G}/\mathfrak{H}}$ is affine. This is a generalization of a main theorem in [2]. In case of observable Hopf ideals, we do not have such a general characterization except the case when Hopf algebras are commutative. So we can extend some results in [1] to affine k-groups. One of them is that $\widetilde{\mathfrak{G}/\mathfrak{H}}$ is quasi-affine if and only if I is observable in A and $R=A\square_{A/I}k$ contains a simple left coideal M such that $M\subset\sqrt{\mathfrak{A}}$ for any left coideal-ideal $\mathfrak{A}$ in R, where $\square$ denotes the cotensor product (see the first section).

Section 1 contains some preliminary results on cotensor products of comodules and injective comodules. It also contains one main result; for a Hopf ideal I of an arbitrary Hopf algebra A, A is an injective A/I-comodule if and only if A is an injective cogenerator for the category of A/I-comodules (1.6). Section 2 gives a characterization of a strongly observable Hopf ideal. Section 3 gives some results

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of observability, which are generalizations of those in [1] with Hopf algebraic proofs, and a characterization of an observable Hopf ideal by a left coideal subalgebra. Section 4 gives a geometric characterization of an observable k-subgroup of an affine k-group.

In this paper, $\otimes$ means the tensor product over k and we shall use the symbols and the notations in [9] if there is no specification. For a k-coalgebra C, Δ_C and ε_C (simply, Δ and ε) are the comultiplication and the counit of C respectively. The sigma notation, $\Delta(c)=\sum_{(c)}c_{(1)}\otimes c_{(2)}$ $(c\in C)$ is as usual. For a k-Hopf algebra A, m_A, u_A, Δ_A, ε_A and S_A (simply, m, u, Δ, ε and S) are the multiplication, the unit, the comultiplication, the counit and the antipode of A respectively. $\mathbf{Mod}_k$ is the category of k-vector spaces. $\mathbf{Com}_C^r$ (resp. $\mathbf{Com}_C^l$) is the category of right (resp. left) C-comodules, which is simply denoted $\mathbf{Com}_C$ if there is no confusion.

1. Cotensor products and injective comodules

Let C be a k-coalgebra, M a right C-comodule with the structure map ρ_M and N a left C-comodule with the structure map λ_N. We shall denote them by (M, ρ) and (N, λ) respectively, or simply M_C and ${}_CN$ respectively. The cotensor product of M and N over C, $M\square_C N$, is defined by

$$M\square_C N=\ker(M\otimes N \xrightarrow{\rho\otimes 1-1\otimes\lambda} M\otimes C\otimes N).$$

Let D be another k-coalgebra and N both left C-comodule and right D-comodule. We call N a (C, D)-bicomodule if $(\rho_N\otimes 1)\lambda_N=(1\otimes\lambda_N)\rho_N$, which we shall simply denote by ${}_CN_D$. C is a (C, C)-bicomodule if we take $\rho_C=\Delta=\lambda_C$, and is called the regular (C, C)-bicomodule. We shall call $(C, \rho_C=\Delta)$ and $(C, \lambda_C=\Delta)$ the left and the right regular C-comodules respectively.

The following results are fundamental;

(a) for M_C (resp. ${}_CN$), $M\square_C ?$ (resp. $?\square_C N$) is the left exact functor from $\mathbf{Com}_C^l$ (resp. $\mathbf{Com}_C^r$) to $\mathbf{Mod}_k$.

(b) $M_C\simeq M\square_C C$, $m\longmapsto \sum_{(m)}m_{(0)}\otimes m_{(1)}$ and ${}_CN\simeq C\square_C N$, $n\longmapsto \sum_{(n)}n_{(-1)}\otimes n_{(0)}$.

(c) for L_C, ${}_CM_D$, ${}_DN$, We have $(L\square_C M)\square_D N\simeq L\square_C(M\square_D N)$, where $L\square_C M$ is a right D-comodule by $1\square_C\lambda_M$ and $M\square_D N$ is a left C-comodule by $\rho_M\square_C 1$. (In fact, they are subcomodules of $(L\otimes M, 1\otimes\lambda)$ and $(M\otimes N, \rho\otimes 1)$ respectively.)

If I (resp. J) is a left (resp. right) coideal of C, then by (a) and (b) we get the injection $M\square_C I\hookrightarrow M\square_C C\simeq M$ (resp. $J\square_C N\hookrightarrow C\square_C N\simeq N$) by which we shall identify $M\square_C I$ (resp. $J\square_C N$) with the subspace of M (resp. N).

As a matter of convienience, we shall only refer to either right comodules or left comodules. For we shall have the similar definitions and results substituting "right" comodules for "left" comodules and vice versa. Furthermore, if possible, we shall neither refer to them. (For example, in the definitions below, (1) can be changed to the assertion that ${}_CM$ is coflat over C if $?\square_C M$ is exact. In (3), either ${}_CM$ or M_C will do.)

It is well known [12] that there are four definitions;

(1) M_C is C-coflat if $M\square_C ?$ in exact.

(2) M_C is C-faithfully coflat if $M\square_C ?$ is faithfully exact.

(3) M is C-injective if $\mathbf{Com}_C(?, M)$ is exact.

(4) M is a C-injective cogenerator if $\mathbf{Com}_C(?, M)$ is faithfully exact.

The following fundamental properties of these comodules are due to M. Takeuchi. In [12], he called coflat "flat". (For a detailed discussion of C-injective comodules, we refer the reader to Green [5].)

(1.1) THEOREM. ([12], prop. A. 2.1.) The followings are equivalent;

(1) M is C-injective (resp. a C-injective cogenerator),

(2) M is (resp. faithfully) coflat over C.

Since $M_C \simeq M\square_C C$, C is a faithfully coflat C-comodule, and hence it is a C-injective cogenerator. Moreover, for a k-space W, $W\otimes C$ is a C-injective cogenerator taking $\rho_{W\otimes C}=1\otimes\Delta$. That is, the category of C-comodules is enough injective.

Let C and D be k-coalgebras and $\pi: C\rightarrow D$ a coalgebra map. For any M_C, $(M, (1\otimes\pi)\rho)$ is a right D-comodule, which is denoted by M_π. If there is no confusion, we say simply a right D-comodule M the restriction of scalars. C is a (D, C)-bicomodule taking $\rho=\Delta$, $\lambda=(\pi\otimes 1)\Delta$. Hence, for any N_D, $(N\square_D C, 1\square_D\Delta)$ is a right C-comodule, which is denoted by N^π and called the induced comodule. We also call it the extension of scalars to make an induced comodule. The restriction of scalars is a left adjoint functor of the extension of scalars, that is, for M_C and ${}_CN$, we have an isomorphism of k-spaces;

$$\mathbf{Com}_C(M, N^\pi) \xrightarrow{\sim} \mathbf{Com}_D(M_\pi, N),\ F \longmapsto (1\otimes\pi)F$$

whose inverse is given by $f\longmapsto (f\otimes 1)\lambda_M$.

From this adjointness, the induced comodule N with the D-comodule map $1\otimes\varepsilon$: $(N^\pi)_\pi \xrightarrow{1\square_D\pi} N\square_D D \xrightarrow{\sim} N$ has the universal mapping property as usual.

Let C, D and π be as above in (1.2) and (1.3).

(1.2) LEMMA. The functor $?\square_D C$ is faithful if and only if $1\otimes\varepsilon: M\square_D C \longrightarrow M$ is surjective for any right D-comodule M.

PROOF. Remarking that the mapping

$$\mathbf{Com}_D(M, N) \xrightarrow{\text{can.}} \mathbf{Com}_C(M^\pi, N^\pi) \xrightarrow{\sim} \mathbf{Com}_D((M^\pi)_\pi, N)$$

sends f to $(id_N \otimes \pi) f^\pi = f(id_M \otimes \varepsilon)$, "only if" part is obvious. If $M \square_D C \xrightarrow{1 \otimes \varepsilon} M$ is not surjective for some M, take $N = M/\operatorname{Im}(1 \otimes \varepsilon)$ and $f: M \longrightarrow N$ as the canonical non-zero D-comodule map. Then f is mapped to zero, which contradicts the faithfulness of the functor. Q.E.D.

(1.3) PROPOSITION. Assume that π is surjective. Then C is a D-injective cogenerator if and only if the D-subcomodule $\ker \pi$ of C is D-injective.

PROOF. Since D is a D-injective cogenerator, "if" part is obvious. Assume that C is an injective cogenerator for $\mathbf{Com}_D$. Let $J = \ker \pi$. Given any short exact sequence of right D-comodules;

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

we get a commutative diagram;

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \square_D J \arrow[r] \arrow[d] & M \square_D J \arrow[r] \arrow[d] & N \square_D J \arrow[d] & \\
0 \arrow[r] & L \square_D C \arrow[r] \arrow[d] & M \square_D C \arrow[r] \arrow[d] & N \square_D C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where all columns and rows are exact. In fact the exactness of the first row follows from the left exactness of the cotensor functor, that of the second row does from the injectivity of C and that of columns does from (1.2). The standard argument (9-lemma) shows that $M \square_D J \rightarrow N \square_D J$ is surjective. Therefore $? \square_D J$ is exact, which means that J is injective as a left D-comodule by (1.1). Q.E.D.

It is easy to make examples such that C is D-injective but not a D-injective cogenerator. But if C and D are Hopf algebras and π is a Hopf algebra map, then the fact that C is D-injective is equivalent to that C is a D-injective cogenerator. This is a main result in this section.

First, we shall review comodules over a Hopf algebra. Let A be a Hopf algebra. Given two right A-comodules M and N, we give $M \otimes N$ a structure of a right A-comodule taking $\rho_{M\otimes N}=(1\otimes 1\otimes m_A)(1\otimes T\otimes 1)(\rho_M\otimes\rho_N)$, where T is the twisting map. We call it the tensor product comodule of M and N.

The following is well known (Cf. [4], prop. 9, Cor. 2).

(1.4) THEOREM. Let M and N be right A-comodules. If N is A-injective, then the tensor product comodule $M\otimes N$ is A-injective.

It is convienient to give a remark here.

(1.5) REMARK. A right (resp. left) comodule over a coalgebra C is a left (resp. right) comodule over its opposite coalgebra C^{op}. If $\phi: C\to D$ is an anti-coalgebra map, then it becomes a coalgebra map $\phi: C^{op}\to D$. We have the composite of functors,

$$F_\phi:\mathbf{Com}^r_C \xrightarrow{\simeq} \mathbf{Com}^l_{C^{op}} \longrightarrow \mathbf{Com}^l_D \text{ (or } \mathbf{Com}^l_C \xrightarrow{\simeq} \mathbf{Com}^r_{C^{op}} \longrightarrow \mathbf{Com}^r_D),$$

where the second one is the restriction of scalars. Since the antipode S of a Hopf algebra A is an anti-coalgebra map, we have functors,

$$\mathbf{Com}^r_A \underset{F_S}{\overset{F_S}{\rightleftarrows}} \mathbf{Com}^l_A$$

If S is bijective, then F_S is an isomorphism whose inverse is $F_{S^{-1}}$. In particular, if S is involutive, $S^2=id_A$, then so is F_S. If B is another Hopf algebra and $\pi: A\to B$ a Hopf algebra map, then

$$F_{S_B}(A_\pi) \xrightarrow{S_A} {}_\pi A$$

is a left B-comodule map. Therefore, if S_A and S_B are bijective, it is an isomorphim of left B-comodules. It follows from this that ${}_\pi A$ is B-injective if and only if A_π is B-injective and so forth.

Let A, B and π be as above. Let M be a right A-module with the structure map, $\omega_M: M\otimes A\to M$. M is called a right (A, B)-Hopf module if it is also a right B-comodule and ω_M is a B-comodule map, where $M\otimes A$ is regarded as the tensor product comodule. We can similarly define a left (A, B)-Hopf module. Note that the right regular A-module A is a right (A, B)-Hopf module regarding it as a right B-comodules by A_π, and $J=\ker\pi$ is a (A, B)-sub-Hopf module of A.

(1.6) THEOREN. Let A, B and π be as above. Assume that π is surjective. The following conditions are equivalent:

(1) A_π is B-injective.

(2) There is a B-comodule map $\phi: B\to A$ with $\phi(1)=1$.

(3) Every right (A, B)-Hopf module is B-injective.

From (1.3), we get,

COROLLARY. A is B-injective if and only if A is a B-injective cogenerator.

PROOF OF THEOREM. It is obvious that (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). For the exact sequence of B-comodules, $0 \to k \xrightarrow{u_B} B$, we get the exact sequence of k-spaces, $\mathbf{Com}_B(B, A) \xrightarrow{\mathbf{Com}_B(u_B, 1)} \mathbf{Com}_B(k, A) \to 0$. One of the pre-images of u_A is the required one.

(2) $\Longrightarrow$ (3). Let M be a right (A, B)-Hopf module and $\iota: M \to M \otimes B$ a k-linear map such that $\iota(x) = x \otimes 1$ for any $x \in M$. Regarding $M \otimes A$ and $M \otimes B$ as tensor product comodules, $\iota, 1 \otimes \phi$ and ω_M are B-comodule maps. Since $\omega_M(\phi \otimes 1)\iota = id_M$, M is a direct summand of $M \otimes B$ as a B-comodule. By (1.4), $M \otimes B$ is B-injective, and so is M. Q.E.D.

(1.7) DEFINITIONS. Let A, B and and π be as above, and $I = \ker \pi$. If the condition (1) (resp. (2)) below holds, we shall say that I is (resp. strongly) observable in A, or that B is A-(resp. strongly) observable with respect to right comodules;

(1) for any right B-comodule N of finite dimension, there exists a right A-comodule M such that N is a B-subcomodule of M_π,

(2) for any right B-comodule N, there exists a right A-comodule M such that N is a B-subcomodule of M_π and $N \square_B k = M \square_A k$.

It is obvious that (2) implies (1). We remark that (1) is equivalent to the following each condition:

(3) for any left B-comodule W of finite dimension, $\varepsilon \otimes 1 : A \square_B W \to W$ is surjective.

(4) for any left B-comodule W, $\varepsilon \otimes 1 : A \square_B W \to W$ is surjective.

(5) the functor $A \square_B ?$ is faithful.

In fact, the linear dual of a finite dimensional right B-comodule has the left B-comodule structure which is called the transpose structure. (cf. [9] p. 99). Taking the dual, (1) $\Longleftrightarrow$ (3) follows from the univesal mapping property of induced comodules. Since a comodule is locally finite, that is, a union of the subcomodules of finite dimension, we get (3) $\Longleftrightarrow$ (4). It follows from (1.2) that (4) $\Longleftrightarrow$ (5).

If the antipodes are bijective, (strong) observability with respect to right comodules is equivalent to that with respect to left comodules by (1.5). Recall that the antipode of a commutative Hopf algebra is involutive. So we shall simply say (strongly) observable in that case.

For later use, we shall review some notions. For detailed discussions of these notions, we refer the reader to Green [5], where he called a comodule "a module ".

(1.8) (1) Invariant matrices. Let (M, ρ) be a right C-comodule and $\{x_i\}_{i\in I}$ be a k-basis of M. For any $j\in J$, we have an equation

$$\rho(x_j)=\sum_{i\in I} x_i \otimes c_{ij}$$

in which the c_{ij} are well-defined elements of C. The $I\times I$ matrix (c_{ij}) is called the invariant matrix for C afforded by the k-basis $\{x_i\}_{i\in I}$ of M. Note that the entries c_{ij} of column j are determined by the above equation. From the coassociativity and the counit law of a coalgebra, we have

$$\Delta(c_{ij})=\sum_{h\in I} c_{ih}\otimes c_{hj} \text{ and } \varepsilon(c_{ij})=\delta_{ij} \quad (i, j\in I)$$

where δ_{ij} is the Kronecker symbol.

Similarly, for a left C-comodule (N, λ) with a k-basis $\{y_i\}_{i\in I}$, we can define the invariant matrix afforded by the basis. Note that its entries d_{ij} of row j are determined by the equation

$$\lambda(y_j)=\sum_{i\in I} d_{ji}\otimes y_i.$$

We also have

$$\Delta(d_{ji})=\sum_{h\in I} d_{jh}\otimes d_{hi} \text{ and } \varepsilon(d_{ji})=\delta_{ji} \quad (i, j\in I).$$

(2) Coefficient spaces. The coefficient space $cf(M)$ of M is defined to be the k-subspace of C generated by the c_{ij} $(i, j\in I)$ in (1). It does not depend on the choice of a k-basis of M, and is a subcoalgebra of C. Remark that, if D is any subcoalgebra of C containing $cf(M)$, then M is a D-comodule and $M=M\square_C D$ by the definition of a cotensor product.

(3) Socles. A non-zero comodule M is called to be simple if it has no subcomodule except M and (0). M is simple if and only if $cf(M)$ is a simple subcoalgebra. For an arbitrary M, the socle $\mathrm{Soc}(M)$ of M is defined to be the sum of all simple subcomodules of M. M is said to be completely reducible if every subcomodule is a direct summand. M is completely reducible if and only if $M=\mathrm{Soc}(M)$. The coradical $R(C)$ of C is the socle of the right and the left regular C-comodule C. We have the following fundamental properties;

(a) $\mathrm{Soc}(M)=M\square_C R(C)$.

(b) If D is a simple subcoalgebra of C and $M\square_C D\neq(0)$, then $M\square_C D$ is completely reducible and $cf(M\square_C D)=D$. Remark that $\mathrm{Soc}(M)\square_C D=M\square_C R(C)\square_C D=M\square_C D$.

(c) Let $f: M\to N$ be a comodule map. If f is injective on $\mathrm{Soc}(M)$. then f is injective.

(d) Let M and N be C-injective. $M \simeq N$ if and only if $\mathrm{Soc}(M) \simeq \mathrm{Soc}(N)$. Note that M is an injective envelope of $\mathrm{Soc}(M)$.

(4) Transpose comodules and Dual comodules. Let (M, ρ) be a right C-comodule of finite dimension, $\{x_i : 1 \leq i \leq n\}$ a k-basis of M and (c_{ij}) is the invariant matrix afforded by the basis. We define its transpose ${}^tM$ to be the left C-comodule whose underlying k-space is the dual k-space of M and whose structure map is

$$\lambda(x_j^*) = \sum_{i=1}^{n} c_{ji} \otimes x_i^* \quad (1 \leq j \leq n)$$

where $\{x_i^*\}$ is the dual k-basis of $\{x_i\}$ (Cf. [9], 5.1.4). (c_{ij}) is also the invariant matrix afforded by the dual k-basis of ${}^tM$.

If C is a Hopf algebra, then we can define the dual comodule of M. It is the right C-comodule ${}^dM$ whose underlying k-space is also the dual k-space of M and whose structure map is

$$\rho(x_j^*) = \sum_{i=1}^{n} x_i^* \otimes S(c_{ji}) \quad (1 \leq j \leq n)$$

where S is the antipode of C. That is, ${}^dM = F_S({}^tM)$ where F_S is in (1.5).

2. Strong observability.

(2.1) THEOREM. Let A and B be Hopf algebras, and $\pi : A \to B$ a surjective Hopf algebra map. The followings are equivalent:

(1) B is A-strongly observable with respect to right comodules.

(2) A_π is B-injective.

(3) A_π is a B-injective cogenerator.

(4) A_π is B-coflat.

(5) A_π is B-faithfully coflat.

REMARK. Cline-Parshall-Scott [2] showed that, if A is a reduced, finitely generated and commutative algebra over an algebraically closed field k, then (1), (2) and (4) are equivalent.

PROOF. From (1.1) and (1.6), (2), (3), (4) and (5) are equivalent. The idea of proof is similar to that in [2]. (1) $\Longrightarrow$ (2). For the right regular B-comodule B, there exists a right A-comodule M such that B is a B-subcomodule of M and $M \square_A k = B \square_B k = k$. Since A is A-injective the A-comodule map $k \to A$ can be extended to an A-comodule map $M \to A$. Then the composite $B \to M \to A$ is a B-comodule map sending 1 to 1. Hence by (1.6), A is B-injective.

(2) $\sim$ (5) $\Longrightarrow$ (1). First, we shall show that a simple right B-comodule N can

be embedded in a simple right A-comodule M such that $N\square_B k=M\square_A k$. Recall that, since k is a trivial (B, B)-sub-bicomodule of B, $N\square_B k$ is a right B-subcomodule of $N\square_B B=N$. Since N is simple, we have either $N\square_B k=0$ or $N=N\square_B k\simeq k$. If $N\simeq k$, take $M=N$ which is considered as a trivial A-comodule. Hence $N\square_B k=N=M=M\square_A k$. Suppose that $N\square_B k=0$. Since $A\square_B ?$ is faithful, B is observable with respect to right comodules by (1.7). Hence there exists a right A-comodule which contains N as a B-subcomodule and is of finite dimension. Among such A-comodules, take M of the smallest dimension. Let M' be a simple A-subcomodule of M. If $M'\cap N=0$, then the composite $N\to M\to M/M'$ is an injective B-comodule map. This contradicts to the choice of M. Hence $M'\cap N=N$ by the simplicity of N. By the choice of M, we have $M=M'$. Since M is simple, we have $M\square_A k=0$ or $M\square_A k=M\simeq k$. If $M\simeq k$, then $N=M$. Therefore, if $N\square_B k=0$, then $M\square_A k=0$.

Let N_B be given arbitrarily. From the above, we have a completely reducible right A-comodule M which contains the socle of N, $\mathrm{Soc}(N)$, as a B-subcomodule and $\mathrm{Soc}(N)\square_B k=M\square_A k$. Let $E(M)$ be an injective envelope of M, then it is also B-injective by Prop. 5 in [4]. Hence a B-comodule map $\mathrm{Soc}(N)\to M\to E(M)$ can be extended to a B-comodule map $\phi: N\to E(M)$. Since the restriction of ϕ on $\mathrm{Soc}(N)$ is injective, ϕ is injective. By (1.8), we get $N\square_B k=\mathrm{Soc}(N)\square_B k=M\square_A k=E(M)\square_A k$. Q.E.D.

In the rest of this section, we assume that A is a commutative Hopf algebra. A subalgebra R of A is called a left coideal subalgebra if it is also a left coideal. The following theorem is due to M. Takeuchi.

(2.2) THEOREM ([10] & [11])

(1) Consider the following sets:

$\mathcal{R}=\{R; R$ is a left coideal subagebra of A and A is faithfully flat over $R\}$

$\mathcal{I}=\{I;\ I$ is a Hopf ideal of A and A is A/I-faithfully coflat$\}$

$\mathcal{H}=\{R; R$ is a sub-Hopf algebra of $A\}$

$\mathcal{I}_n=\{I; I$ is a normal Hopf ideal of $A\}$

The correspondence $R\longmapsto R^+A=$the ideal of A generated by $\ker\varepsilon\cap R=R^+$ and the correspondence $I\longmapsto A\square_{A/I}k$ are mutually inverse correspodences between $\mathcal{R}$ and $\mathcal{I}$. Under these, the subset $\mathcal{H}$ of $\mathcal{R}$ correspnds to the subset $\mathcal{I}_n$ of $\mathcal{I}$.

(2) If A is pointed, then A is faithfully coflat over A/I for any Hopf ideal I.

(3) Let $\mathfrak{G}=\mathfrak{Sp}_k A$, $\mathfrak{H}=\mathfrak{Sp}_k A/I$ and $\tilde{\tilde{\mathfrak{G}/\mathfrak{H}}}$ a dur k-sheaf of right cosets ([3] p. 353). The followings are equivalent:

(a) $\tilde{\tilde{\mathfrak{G}/\mathfrak{H}}}$ is an affine scheme.

(b) A is faithfully coflat over A/I.

Combining (2.1) and (2.2), we get:

(2.3) THEOREM. Let $\mathfrak{G}$ be an affine k-group scheme and $\mathfrak{H}$ a closed k-subgroup scheme. Then the followings are equivalent:

(1) $\mathfrak{H}$ is strongly observable in $\mathfrak{G}$.

(2) $\mathcal{O}(\mathfrak{G})$ is an injective k-$\mathfrak{H}$-module.

(3) $\mathfrak{H}$ is an exact subgroup of $\mathfrak{G}$, which means that $\mathcal{O}(\mathfrak{G})\square_{\mathcal{O}(\mathfrak{H})}?$ is exact.

(4) $\widetilde{\mathfrak{G}/\mathfrak{H}}$ is an affine k-scheme.

REMARK. If $\mathfrak{G}$ is reduced and algebraic over an algebraically closed field k, then this is a main theorem for a strongly observable subgroup in [2].

Finally, we shall give one criterion for strong observability. An ideal $\mathfrak{a}$ of a left coideal subalgebra R is called a left coideal-ideal if $\mathfrak{a}$ is also a left coideal of A.

(2.4) PROPOSITION. Let R be a left coideal subalgebra of a Hopf algebra A. The followings are equivalent:

(1) R has no non-trivial left coideal-ideals.

(2) A is faithfully flat over R.

PROOF. (2) $\Longrightarrow$ (1). Let $\mathfrak{a}$ be a left coideal-ideal in R. If $\mathfrak{a}\neq 0$, then $\varepsilon(\mathfrak{a})\neq 0$. In fact, if $\varepsilon(\mathfrak{a})=0$, $(1\otimes\varepsilon)\Delta(\mathfrak{a})=0$, for $\Delta(\mathfrak{a})\subset A\otimes\mathfrak{a}$. But $(1\otimes\varepsilon)\Delta=id$. Take $a\in\mathfrak{a}$ satisfying with $\varepsilon(a)=1$. Then $1=\varepsilon(a)=\sum_{(a)}S(a_{(1)})a_{(2)}\in\mathfrak{a}A$. This contradicts the fact that A is faithfully flat over R.

(1) $\Longrightarrow$ (2). We may assume that k is algebraically closed and both A and R are finitely generated.

(a) Suppose A is reduced. Let $i: R\to A$ be the inclusion map, then

$$f=\mathfrak{Sp}_k i(k): \mathfrak{Sp}_k A(k) \longrightarrow \mathfrak{Sp}_k R(k)$$

is a $\mathfrak{Sp}_k A(k)$-equivariant morphism. From the generic flatness and homogeneity of $\mathfrak{Sp}_k A(k)$, A is flat over R. Therefore f is an open map. If $\mathfrak{a}$ is the definition ideal of the closed set $\mathfrak{Sp}_k R(k)-f(\mathfrak{Sp}_k A(k))$, $\mathfrak{a}$ is $\mathfrak{Sp}_k A(k)$-stable, and hence it is a left coideal-ideal. By the assumption, $\mathfrak{a}=0$. This means f is surjective, and hence A is faithfully flat over R.

(b) Suppose A is not reduced. Since a Hopf algebra over a field of characteristic zero is reduced, k has positive characteristic p. Then $A^{(p^n)}=\{a^{p^n}; a\in A\}$ and $T=R^{(p^n)}$ are reduced for a natural number n being large enough. T is also a left coideal subalgebra of $A^{(p^n)}$ and has no non-trivial left coideal-ideals. Since $A^{(p^n)}$ is faithfully flat over T by (a) and A is faithfully flat over $A^{(p^n)}$ by (2.2), A if faithfully flat over T. Since R and A are (T, A)-Hopf module, we have A-module isomorphisms,

$$A\otimes_T R \xrightarrow{\sim} A\otimes R/T^+R,\ a\otimes r \longmapsto \sum_{(r)} ar_{(1)}\otimes(r_{(2)} \bmod T^+R),$$

and

$$A\otimes_T A \xrightarrow{\sim} A\otimes A/T^+A,\ a\otimes b \longmapsto \sum_{(b)} ab_{(1)}\otimes(b_{(2)} \bmod T^+A)$$

by the structure theorem of (T, A)-Hopf modules ([7], th. 1). Since $A\otimes_T R \to A\otimes_T A$ is injective, $R/T^+R \to A/T^+A$ is injective and it is easy to see that R/T^+R is a left coideal subalgebra of a Hopf algebra A/T^+A and $(R/T^+R)^+$ is nilpotent. Hence the lemma below shows that A/T^+A is a free R/T^+R-module. In particular, $A\otimes_T A$ is faithfully flat over $A\otimes_T R$. Therefore A is faithfully flat over R. Q.E.D.

Lemma. Let R be a left coideal subalgebra of A. If R^+ is nilpotent, then A is a free R-module.

Proof. Choose $a_j \in A (j\in J)$ such that $a_j \bmod R^+A$ is a k-basis of A/R^+A. Since R^+ is nilpotent, A is spanned by the a_j as an R-module by Nakayama's lemma. Hence the R-module map, $\otimes_J R \to A$ sending the j-th basis element to a_j, is surjective. Tensoring with A on the left, we get

$$\begin{tikzcd} \oplus_J R \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ A\otimes_R(\oplus_J R)=\oplus_J A \arrow[r] & A\otimes_R A\ . \end{tikzcd}$$

The map $\oplus_J R \to \oplus_J A$ is injective; hence the top row must be injective if the bottom one is. Note that the map sends the i-th basis element of $\oplus_J A$ to $1\otimes a_i$. Let M be the kernel of the bottom map. Since $A\otimes_R A \simeq A\otimes A/R^+A$,

$$0 \longrightarrow M \longrightarrow \oplus_J A \longrightarrow A\otimes A/R^+A \longrightarrow 0$$

is a split exact sequence of A-modules; hence R-modules. Apply $R/R^+\otimes_R ?$, and we get an exact sequence

$$0 \longrightarrow M/R^+M \longrightarrow \oplus_J A/R^+A \longrightarrow A/R^+A\otimes A/R^+A \longrightarrow 0.$$

Since $\oplus_J A/R^+A \to A/R^+A\otimes A/R^+A$ is A/R^+A-linear and sends the i-th basis element to $1\otimes(a_i \bmod R^+A)$, it is an isomorphism. Hence $M/R^+M=0$, which means $M=0$ by Nakayama's lemma. Q.E.D.

3. Observability.

In this section we assume that all Hopf algebras are commutative. Given any Hopf algebra A, we denote the set of all group-like elements in A by $G(A)$ and the coradical of A by $R(A)$ as usual.

First, we shall give some results which are easily deduced from the definition.

(3.1) Given the following commutative diagram of Hopf algebras,

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B_1 \arrow[d] \\ A_2 \arrow[r] & B_2 \end{tikzcd}$$

where all the maps are Hopf algebra maps and surjective. We get,

(a) (transitivity of observability) if B_1 is A_1-observable and B_2 is B_1-observable, then B_2 is A_1-observable, and

(b) if B_2 is A_1-observable, then B_2 is A_2-observable.

A fundamental theorem of observability is due to [1].

(3.2) THEOREM. ([1], th. 1) Let A and B be Hopf algebras, and $\pi: A \to B$ a surjective Hopf algebra map. Assume that, for any $g \in G(B)$ such that $A\square_B kg \neq 0$, we have $A\square_B kg^{-1} \neq 0$, where kg and kg^{-1} are one dimensional subcoalgebra of B spanned by g and g^{-1} respectively. Then B is A-observable. In particular, if $A\square_B kg \neq 0$ for any $g \in G(B)$ then B is A-observable.

REMARKS. (1) Since $A\square_B kg$ is a left coideal of A, $\varepsilon(A\square_B kg) \neq 0$ if it is not zero. Hence the canoncal map $A\square_B kg \to kg$ is surjective. For $a \in A$, $a \in A\square_B kg$ iff $(1\otimes\pi)\Delta(a) = a\otimes g$. Hence $\pi(a) = \varepsilon(a)g$ for any $a \in A\square_B kg$.

(2) If N is a B-comodule of dimension one, then $cf(N) = kg$ for some $g \in G(B)$ and $N \simeq kg$ as a B-comodue. We call g the weight of N.

It is easy to see that, for $g \in G(B)$, the following conditions are equivalent; (a) $A\square_B kg \neq 0$, (b) $kg^{-1}\square_B A \neq 0$, and (c) any right (resp. left) B-comodule N of dimension one whose weight is g (resp. g^{-1}) can be embedded in a right (resp. left) A-comodule as a B-subcomodule.

By (c), we see that this theorem is a Hopf algebraic version of the theorem 1 in [1].

COROLLARY. Let A, B and π be as above.

(1) For any field extension K of k, the A-observability of B is equivalent to the $A\otimes K$-observability of $B\otimes K$.

(2) B is A-observable if either of the following condition holds;

$$\text{(a)} \quad \pi(G(A)) = G(B) \quad \text{or} \quad \text{(b)} \ \pi(R(A)) = R(B).$$

In particular, we get

(i) if $A\otimes K$ is pointed for some extension field K of k, then every Hopf ideal of A is observable, and

(ii) if $G(B) = \{1\}$, then B is A-observable.

PROOF (i) Suppose that $B\otimes K$ is $A\otimes K$-observable. For any $g\in G(B)$, we have $(A\square_B kg)\otimes K\simeq(A\otimes K)\square_{B\otimes K}(kg\otimes K)$, which is easily deduced from the definition of cotensor products. Hence $A\square_B kg\neq 0$. Suppose the contrary. For any $g\in G(B\otimes K)$, we have only to show that a right $B\otimes K$-comodule Kg can be embedded in a right $A\otimes K$-comodule as a $B\otimes K$-subcomodule by the remark (2) above. Express $g=\sum b_i\otimes\alpha_i$, b_i, $\in B$, $\alpha_i\in K$. Let N be a right coideal of B generated by the b_i. Since B is A-observable, N can be embedded in a right A-comodule M as a B-subcomodule. The composite $Kg\to N\otimes K\to M\otimes K$ is the required embedding, where the first map is defined to send g to $\sum b_i\otimes\alpha_i$.

(2) In case of (a), there is nothing to prove. Suppose (b) holds. For any $g\in G(B)$, we get $R(B)=kg\oplus C$ as a coalgebra and

$$R(A)=R(A)\square_{R(B)}R(B)=(R(A)\square_{R(B)}kg)\oplus(R(A)\square_{R(B)}C)$$

as a left A-comodule. If $R(A)\square_{R(B)}kg=0$, then there is no element a in $R(A)$ with $\pi(a)=g$, which contradicts to (b). Since $R(A)\square_{R(B)}kg=R(A)\square_B kg\subset A\square_B kg$, we have $A\square_B kg\neq 0$.

The homomorphic image of a pointed Hopf algebra is pointed. Hence (i) follows from (1) and (a). (ii) follows from $k\subset A\square_B k$. Q.E.D.

(3.3) PROPOSITION. Suppose that k is perfect.

(1) I is an observable Hopf ideal in A if and only if $\sqrt{I}$ is observable in A.

(2) B is A-observable if and only if B_{red} is A_{red}-observable, where $B_{\mathrm{red}}=B/\sqrt{(0)}$ and $A_{\mathrm{red}}=A/\sqrt{(0)}$.

PROOF. Recall that, if I is a Hopf ideal, so is $\sqrt{I}$, for k is perfect. If k has characteristic zero, every Hopf algebra is reduced, hence there is nothing to prove. Suppose k has positive characteristic p.

(1) Let $B=A/I$; then $B_{\mathrm{red}}=A/\sqrt{I}$. Suppose B is A-observable. From (3.1) (b), it suffices to show that B_{red} is B-observable. Let $g\in G(B_{\mathrm{red}})$ with $B\square_{B_{\mathrm{red}}}kg\neq 0$ and choose $b\in B\square_{B_{\mathrm{red}}}kg$ with $\varepsilon(b)=1$. Take $f\in B$ such that $f \bmod \sqrt{0}=g^{-1}$; then $f^{p^n}\in G(B)$ for some positive integer n. We get an element $b^{p^n-1}f^{p^n}$ of $B\square_{B_{\mathrm{red}}}kg^{-1}$ which is not zero, for $\varepsilon(b^{p^n-1}f^{p^n})=1$. Suppose the contrary. Let $g\in G(B)$ with $A\square_B kg\neq 0$ and choose $a\in A\square_B k$ with $\varepsilon(a)=1$. Since $\bar{g}=g \bmod \sqrt{0}$ is in $G(B_{\mathrm{red}})$ and $\sqrt{I}$ observable in A, $A\square_{B_{\mathrm{red}}}kg^{-1}\neq 0$. Take its element f with $\varepsilon(f)=1$; then $f\otimes g^{-1}=\sum_{(f)}f_{(1)}\otimes(f_{(2)} \bmod I)$ modulo $A\otimes\sqrt{0}$. Hence

$$f^{p^n}\otimes(g^{-1})^{p^n}=\sum_{(f)}f_{(1)}^{p^n}\otimes(f_{(2)} \bmod I)^{p^n}$$

for some positive integer n. We get a non-zero element $a^{p^n-1}f^{p^n}$ of $A\square_B kg^{-1}$.

(2) It is obvious from (1) and (3.1). Q.E.D.

Let $\pi: A \to B$ be a surjective Hopf algebra map, A_1 a sub-Hopf algebra of A, $B_1 = \pi(A_1)$, $A_0 = A/I(A_1)$ and $B_0 = B/I(B_1)$, where $I(A_1) = A_1^+A$ and $I(B_1) = B_1^+B$. From (2.1) and (2.2), A_0 is A-strongly observable and B_0 is B-strongly observable. Hence if B is A-observable, B_0 is A-observable by (3.1). Note that $A_1 = A\square_{A_0}k = k\square_{A_0}A$ and $B_1 = B\square_{B_0}k = k\square_{B_0}B$ ([10]).

(3.4) PROPOSITION. Under the situation as above, if B_1 has a non-zero left integral τ, we get that B is A-observable if and only if B_0 is A_0-observable.

REMARKS. (1) From the definition of a left integral ([9], p. 91), it is easy to see that τ is a left B_1-comodule map from B_1 to k, where k is regarded as a trivial comodule.

(2) A finite dimensional Hopf algebra always has a non-zero left integral ([9], ch. 5, 5.1.6.). Let A be finitely generated as a k-algebra and take $A_1 = \pi_0(A) = \{a \in A; k[a]$ is a separable k-algebra.$\}$ in the above situation; then A_1 is of finite dimension and so is B_1. Therefore the result of the proposition holds.

PROOF. Suppose B_0 is A_0-observable. Then B_0 is A-observable by transitivity. Hence $A\square_{B_0}kg \neq 0$ for any $g \in G(B)$, viewing kg as a B_0-comodule by the restriction of scalars. Note that it is an A_1-module, where A_1 operates on it by the multiplication on the factor A. Let B_1g be a B_1-submodule of B generated by g. It is also a subcoalgebra. Define a k-linear map $\phi: A\square_{B_0}kg \to B_1g$ by $\phi(a\otimes g)\pi(a)$. It is easy to show that ϕ is a B-comodule map and an A_1-module map, where B_1g is regarded as an A_1-module via $\pi: A_1 \to B_1$. We claim that ϕ is surjective. In fact, $\mathrm{Im}\,\phi$ is a non-zero B_1-submodule and a left coideal of B_1g. We can choose $bg \in \mathrm{Im}\,\phi$ ($b \in B_1$) with $\varepsilon(bg) = 1$. Since $u\varepsilon = m(S\otimes 1)$, we get $1 = \sum_{(b)} S(b_{(1)})b_{(2)}$; hence $g = \sum_{(b)} S(b_{(1)})b_{(2)}g \in \mathrm{Im}\,\phi$. Therefore $\mathrm{Im}\,\phi = B_1g$. The composite of B-comodule maps ϕ and $B_1g \to kg$, $bg \to \tau(b)g$, is a non-zero map; hence we get $A\square_B kg \neq 0$ by the universality of induced comodules. Q.E.D.

REMARMK. Let B_1 be any sub-Hopf algebra of B, $B_0 = B/I(B_1)$ and $g \in G(B)$. Then we get

(i) $B\square_{B_0}kg \ni b\otimes g \Longleftrightarrow g\otimes b \in kg\square_{B_0}B$; hence $B\square_{B_0}kg$ is a subcoalgebra of $B\otimes kg$, and

(ii) $B\square_{B_0}kg \simeq B_1\otimes kg \simeq B_1g$ as coalgebras, where $B\square_{B_0}kg$ and $B_1\otimes kg$ are subcoalgebras of $B\otimes kg$, and B_1g is a subcoalgebra of B. They are also isomorphic as left B-comodules where $\lambda_{B\square_{B_0}kg} = \Delta\square_{B_0}1$, $B_1\otimes kg$ is the tensor product comodule and $\lambda_{B_1g} = \Delta$, and also isomorphic as B_1-modules, where B_1 operates respectively on $B\square_{B_0}kg$ and $B_1\otimes kg$ by the multiplication on the factors B and B_1g, and B_1g is a B_1-submodule of B.

PROOF. (i) The proof of this assersion is the same as the one of $B\square_{B_0}k = k\square_{B_0}B$.

(ii) It is obvious that if $b\otimes g\in B\square_{B_0}kg$, then $bg^{-1}\in B_1$. It is easy to show that mappings $B\square_{B_0}kg \longrightarrow B_1\otimes kg, b\otimes g\longmapsto bg^{-1}\otimes g$ and $B_1\otimes kg \longrightarrow B_1 g, b\otimes g\longmapsto bg$ give us the required isomorphisms. Q.E.D.

COROLLARY (of proof). Let A, B and π be as above, B_1 a finite dimensional sub-Hopf algebra of B and $B_0=B/I(B_1)$. Then the A-observabilities of B and B_0 are equivalent.

PROOF. Since $R=A\square_B B_1=A\square_{B_0}k$ is a sub-(B, B)-bicomodule of A and

$$\phi: A\square_B B_1 \xrightarrow{\pi\square_B 1} B\square_B B_1 \xrightarrow{\simeq} B_1$$

is a (B, B)-bicomodule map, $B_2=\operatorname{Im}\phi$ is a subcoalgebra of B_1. Since ϕ is also a k-algebra map, B_2 is a subalgebra; hence a sub-bialgebra of B_1. Since B_1 is of finite dimension, B_2 is really a sub-Hopf algebra. Let τ be a non-zero left integral of B_2.

Now suppose B_0 is A-observable and take any $g\in G(B)$ with $A\square_B kg\neq 0$; then $A\square_{B_0}kg^{-1}\neq 0$. Note that it is an R-module, where R operates on it by the multiplication on the factor A. Let ψ be the composite;

$$A\square_{B_0}kg^{-1} \xrightarrow{\pi\square_{B_0}1} B\square_{B_0}kg^{-1} \xrightarrow{\simeq} B_1 g^{-1}.$$

If we show $\operatorname{Im}\psi\subset B_2 g^{-1}$, then R, B_2, ψ and τ play the same roles as A_1, B_1, ψ and τ in the proof of the proposition respectively, so we shall get $A\square_B kg^{-1}\neq 0$.

Choose $a\in A\square_B kg$ with $\varepsilon(a)=1$. Then $\pi(a)=g$. Given any $a'\otimes g^{-1}\in A\square_{B_0}kg^{-1}$, $\pi(a')g=\pi(aa')\in B_1$; hence $\sum_{(a')} a'_{(1)}a\otimes\pi(a'_{(2)})g\in A\square_B B_1=R$. Therefore $\phi(\sum_{(a')} a'_{(1)}a\otimes \pi(a'_{(2)})g)=\pi(a')g\in B_2$ and $\psi(a')=\pi(a')\in B_2 g^{-1}$. Q.E.D.

(3.5) PROPOSITION. Let A and B be as in (3.2), B_1 a pointed irredecible sub-Hopf algebra of B, and $B_0=B/I(B_1)$. Then the A-observabilities of B and B_0 are equivalent.

PROOF. Suppose that B_0 is A-observable. Then $A\square_{B_0}kg\neq 0$ for any $g\in G(B)$. We have $A\square_{B_0}kg\simeq A\square_B(B\square_{B_0}kg)\simeq A\square_B B_1 g$. Since $B_1 g$ is a pointed irreducible coalgebra with the unique simple coalgebra kg and $A\square_B B_1 g$ is a right $B_1 g$-comodule, we have $\operatorname{Soc}(A\square_B B_1 g)=A\square_B B_1 g\square_{B_1 g}kg=A\square_B kg$, where $\operatorname{Soc}(A\square_B B_1 g)$ is the socle of a $B_1 g$-comodule $A\square_B B_1 g$. Since $A\square_B B_1 g\neq 0$, we get $A\square_B kg\neq 0$. Q.E.D.

A Hopf algebra A is split pointed if $A\simeq kG(A)\otimes A^1$ as Hopf algebras, where $kG(A)$ is the group algebra of the group $G(A)$ and A^1 is the pointed irreducible component of A containing k.

COROLLARY. Let A and B be as above. If $B\otimes K$ is split pointed for some extension field K of k, then B is A-observable.

PROOF. By the corollary in (3.2), we may assume that B is split pointed. Since $kG(B)$ is co-semi-simple, that is, every $kG(B)$-comodule is an injective comodule, A is $kG(B)$-injective, where A is a $kG(B)$-comodule through the Hopf algebra map, $A \to B \to B/I(B^1) \simeq kG(B)$. Hence $kG(B)$ is A-strongly observable. Therefore the proposition assures that B is A-observable. Q.E.D.

The next theorem was asserted in [7], prop. 3, but its proof is incorrect (see the appendix at the last page), so we shall give a new proof here.

For any left coideal subalgebra R of A, we denote by $I(R)$ or $I_A(R)$ the ideal of A generated by $R^+ = R \cap \ker \varepsilon$. Remark that it is a Hopf ideal.

(3.6) THEOREM. If R is a left coideal subalgebra A, then $I(R)$ is observable in A.

PROOF. We shall reduce to proving the following problem: let A be a finitely generated Hopf domain over an algebraically closed field and R a finitely generated as a k-algebra, then $I(R)$ is observable in A.

In fact, it is obvious that we may assume that k is algebraically closed. Let $B = A/I(R)$ and $\pi : A \to B$ be the canonical map. Given any $g \in G(B)$ with $A \square_B kg \neq 0$, we have only to show $A \square_B kg^{-1} \neq 0$. Let $\mathfrak{c}$ be a simple left coideal in $A \square_B kg$. If $\dim \mathfrak{c} = 1$, then there is nothing to prove. Let $a_i \in A (1 \leqq i \leqq n)$ be a k-basis of $\mathfrak{c}$ with $\varepsilon(a_i) = \delta_{1i}$ where δ_{1i} is the Kronecker symbol. Then $\pi(a_i) = \sum_{(a_i)} \varepsilon\pi(a_{i(1)})\pi(a_{i(2)}) = \varepsilon\pi(a_i)g = \delta_{1i}g$. Hence $\pi(a_1) = g$ and $a_i \in I(R)$ for $i = 2, \cdots, n$. Expressing

$$a_i = \sum_{j=1}^{m} r_{ij}h_j, \quad r_{ij} \in R, \quad h_j \in A \ (i = 2, \cdots, n),$$

put R' the left coideal subalgebra generated by the r_{ij}, and A' the sub-Hopf algebra generated by the a_i and R'. Since $\Delta(a_1) - a_1 \otimes a_1 \in A' \otimes \mathfrak{c}^+ \subset A' \otimes I_{A'}(R')$, $\bar{a}_1 = a_1 \bmod I(R')$ is a group-like element and $a_1 \in A' \square_{B'} k\bar{a}_1$, where $B' = A'/I(R')$. If we have $A' \square_{B'} k\bar{a}_1 \neq 0$, then we get $A \square_B kg^{-1} \neq 0$. Hence we can reduce to the case that A and R are finitely generated as k-algebras.

Let $q : A \to A_{\mathrm{red}}$ be the canonical map. We have $A_{\mathrm{red}}/\sqrt{I(q(R))} = A/\sqrt{I(R)} = B_{\mathrm{red}}$. By (3.4), if $\sqrt{I(q(R))}$ is observable in A_{red}, $I(R)$ is observable in A. Hence we can reduce to the case that A is reduced. Finally, let A_0, B_0 be as the ones in the remarks (2) of (3.4), and $q : A \to A_0$ the canonical map. It is easy to show that $A_0/I(q(R)) \simeq B_0$. Hence by (3.3) and (3.4), if $\sqrt{I(q(R))}$ is observable in A_0, $I(R)$ is observable in A. Hence the reduction to the problem that is stated at the beginning, can be done.

Now let $\mathfrak{G}$ be an affine algebraic group defined by A, $\mathfrak{K}$ a closed subgroup defined by $B = A/\sqrt{I(R)}$, and $\mathfrak{X}$ an affine variety defined by R. The commutative diagrams

$$\begin{tikzcd} R \arrow[r, "J"] \arrow[d, "\Delta"'] & A\otimes R \arrow[d, "1\otimes J"] \\ A\otimes R \arrow[r, "\Delta\otimes 1"] & A\otimes A\otimes R \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[rr, "\Delta"] \arrow[dr, "\wr"'] & & A\otimes R \arrow[dl, "\varepsilon\otimes 1"] \\ & k\otimes R & \end{tikzcd}$$

induce the commutative diagrams

$$\begin{tikzcd} \mathfrak{X} & \mathfrak{G}\times\mathfrak{X} \arrow[l] \\ \mathfrak{G}\times\mathfrak{X} \arrow[u] & \mathfrak{G}\times\mathfrak{G}\times\mathfrak{X} \arrow[l] \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{X} & & \mathfrak{G}\times\mathfrak{X} \arrow[ll] \\ & \mathfrak{e}\times\mathfrak{X} \arrow[ul, "\wr"] \arrow[ur] & \end{tikzcd}$$

where $\mathfrak{e}$ is the unit group.

Hence the inclusion $R \hookrightarrow A$ induces a $\mathfrak{G}$-equivariant dominant morphism of varieties $\mathfrak{p}: \mathfrak{G} \to \mathfrak{X}$. Given any $g \in G(B)$ with $A\square_B kg \neq 0$, we choose an element a of $A\square_B kg$ with $\varepsilon(a)=1$ and take $\mathfrak{Z}$ as the closed set of zeros of a. Then we get (1) $\mathfrak{K}\mathfrak{Z}=\mathfrak{Z}$, (2) $\mathfrak{p}^{-1}(\mathfrak{p}\mathfrak{Z})=\mathfrak{Z}$, and (3) $\mathfrak{p}$ is an open map. In fact, it is easy to show (1) and (2). (3) may be well-known (cf. [8], p. 58, Prop. 1). From (3), $\mathfrak{p}\mathfrak{Z}$ is closed in $\mathfrak{p}(\mathfrak{G})$; hence $\mathfrak{p}^{-1}(\overline{\mathfrak{p}\mathfrak{Z}})=\mathfrak{p}^{-1}(\mathfrak{p}\mathfrak{Z})$, where $\overline{\mathfrak{p}\mathfrak{Z}}$ is the closure of $\mathfrak{p}\mathfrak{Z}$ in $\mathfrak{X}$. Since $\varepsilon(a)=1$, $\mathfrak{p}(\mathfrak{e}) \in \overline{\mathfrak{p}\mathfrak{Z}}$. Therefore there exists $r \in R$ such that $r(\mathfrak{p}(\mathfrak{e}))=1$ and $r=0$ on $\overline{\mathfrak{p}\mathfrak{Z}}$. Viewing r as a regular function on $\mathfrak{G}$, we get that $r=0$ on $\mathfrak{Z}$. Hence $r \in \sqrt{(a)}$. Expressing $r^n=aa'$ for some $a' \in A$ and for some positive integer n, we have $(a\otimes g)(a'\otimes g^{-1})=a^n\otimes 1=(a\otimes g)(\sum_{(a')} a'_{(1)}\otimes \pi(a'_{(2)}))$ in $A\otimes B$. Since A is an integral domain, B is reduced and g is an invertible element, we have $a'\otimes g^{-1}=\sum_{(a')} a'_{(1)}\otimes\pi(a'_{(2)})$, which means $a' \in A\square_B kg^{-1}$. Since $\varepsilon(r)=\varepsilon(a)=1$, $a' \neq 0$. Q.E.D.

(3.7) Theorem ([7]) If I is an observable Hopf ideal in A, then $I=I(A\square_{A/I}k)=I(k\square_{A/I}A)$.

Corollary. If I_1 and I_2 are observable Hopf ideals in A, then so is I_1+I_2.

Proof. By the above theorem, I_i is determined by a left coideal subalgebra $R_i (i=1, 2)$. We claim that $I_1+I_2=I(R_1R_2)$, where R_1R_2 is a left coideal subalgebra generated by R_1 and R_2. Since I_i is generated by $R_i^+=\ker\varepsilon\cap R_i$, $I_1+I_2 \subseteq I(R_1R_2)$. Let $x_j \in R_1, y_j \in R_2$ $(j=1, \cdots, n)$ and $\sum x_jy_j \in (R_1R_2)^+$. Then

$$\sum_j x_jy_j=\sum x_j(y_j-\varepsilon(y_j))+\sum_j (x_j-\varepsilon(x_j))\varepsilon(y_j) \in I_1+I_2. \qquad \text{Q.E.D.}$$

The following two propositions show the relation between observable closed k-subgroup schemes and stabilizer k-subgroup schemes of an affine k-group scheme.

(3.8) Proposition. Let $\mathfrak{G}=\mathfrak{S}\mathfrak{p}_kA$ be an affine k-group scheme, V a left A-comodule and $v \in V$. Then the stabilizer group $\mathfrak{G}_v$ of v is observable in $\mathfrak{G}$.

PROOF. Let W be any subcomodule of V such that it contains v and is of finite dimension. Let ${}^tW$ be the transpose comodule (1.8) (4). The symmetric algebra $S({}^tW)$ on ${}^tW$ has naturally the structure of right A-comodule induced by ${}^tW$. The map $\phi_v: S({}^tW) \longrightarrow A, f \longmapsto \sum_{(f)} f_{(0)}(v)f_{(1)}$ is both the k-algebra one and the A-comodule one. Hence $\mathrm{Im}\,\phi_v$ is a right coideal subalgebra of A. By (3.6), $B=A/I(\mathrm{Im}\phi_v)$ is A-observable. We claim that $\mathfrak{G}_v=\mathfrak{Sp}_k B$. In fact, it is enough to show that $\mathfrak{G}_v(T)=\mathfrak{Sp}_k B(T)$ for any commutative k-algebra T. Let $v=v_1, v_2, \cdots, v_n$ be a k-basis of W and $X_1, \cdots, X_n$ its dual basis of ${}^tW$. If (a_{ij}) is the invariant matrix afforded by the k-basis $\{v_i\,; 1 \leqq i \leqq n\}$, that is, $\lambda_W(v_i)=\sum_{j=1}^{n} a_{ij}\otimes v_j$, then it is also the invariant matrix afforded by $\{X_i\,; 1 \leqq i \leqq n\}$. Hence $\phi_v(X_i)=\sum_{j=1}^{n} X_j(v)a_{ij}=a_{1i}$. We get

$$\mathrm{Im}\,\phi_v=k[a_{11}, a_{12}, \cdots, a_{1n}],$$
$$(\mathrm{Im}\,\phi_v)^+=(a_{11}-1, a_{12}, \cdots, a_{1n})$$

and

$$B=A/(a_{11}-1, a_{12}, \cdots, a_{1n})A.$$

Recall that $\mathfrak{G}_v(T)=\{g\in\mathfrak{Sp}_k A(T)\,; g.v=v\}$. Since

$$g.v=\sum g(v_{(-1)})\otimes v_{(0)}=\sum_{j=1}^{n} g(a_{ij})\otimes v_j\in T\otimes V,$$

we have $g.v=v \Longleftrightarrow g(a_{ij})=\delta_{ij} \Longleftrightarrow \ker g\supset(a_{11}-1, a_{12}, \cdots, a_{1n})A \Longleftrightarrow g\in\mathfrak{Sp}_k B(T)$. Therefore $\mathfrak{G}_v(T)=\mathfrak{Sp}_k B(T)$ Q.E.D.

Remark that $I(\mathrm{Im}\,\phi_v)$ is finitely generated. We have its converse.

(3.9) PROPOSITION. If a Hopf ideal I is observable in A and finitely generated, then there exists a left A-comodule V and its element v such that $(\mathfrak{Sp}_k A)_v=\mathfrak{Sp}_k(A/I)$.

PROOF. Since I is observable, we have $I=I(R)$ and $R=k\square_{A/I}A$. Hence we can express

$$I=(r_1, \cdots, r_n), \quad r_i\in R\ (1\leqq i\leqq n).$$

Let V_i be a left A-comodule generated by r_i, $V=\bigoplus_{i=1}^{n} V_i$ and $v=r_1+\cdots+r_n$. Then we claim that $(\mathfrak{Sp}_k A)_v=\mathfrak{Sp}_k A/I$. As in the proof of (3.8), we have

$$(\mathfrak{Sp}_k A)_v=\mathfrak{Sp}_k A/I(\mathrm{Im}\,\phi_v),$$

where

$$\phi_v: S({}^tV) \longrightarrow A, f \longmapsto \sum_{(f)} f_{(0)}(v)f_{(1)}.$$

Hence it is enough to show that $I=I(\mathrm{Im}\,\phi_v)$. For each i, let $v(i)_1=r_i, v(i)_2, \cdots, v(i)_{N(i)}$

be a k-basis of V_i, $X(i)_1, \cdots, X(i)_{N(i)}$ its dual basis of ${}^tV_i$, and $(a(i)_{jm})$ $(1\leqq j, m\leqq N(i))$ the invariant matrix of V_i. Then $\Delta(r_i)=\sum_{m=1}^{N(i)} a(i)_{1m}\otimes v(i)_m$. Since $(1\otimes\varepsilon)\Delta=id$ and $\varepsilon(v(i)_1)=0$, we get

$$r_i=\sum_{m=2}^{N(i)} a(i)_{1m}\varepsilon(v(i)_m). \tag{1}$$

Since $v(i)_1=r_i\in R$, we get $\sum\pi(a(i)_{1m})\otimes v(i)_m=1\otimes v(i)_1$ where $\pi: A\to A/I$ is the canonical Hopf algebra map. Hence

$$a(i)_{11}-1, a(i)_{1m}\in I\ (2\leqq m\leqq N(i)). \tag{2}$$

On the other hand, since $\phi_v(X(i)_j)=\sum_{m=1}^{N(i)}\langle X(i)_m, v\rangle a(i)_{mj}=a(i)_{1j}$, we get

$$\operatorname{Im}\phi_v=k[a(i)_{1j}\,;1\leqq i\leqq n, 1\leqq j\leqq N(i)]$$

and

$$(\operatorname{Im}\phi_v)+=(a(i)_{11}-1, a(i)_{1j}\,;1\leqq i\leqq n, 2\leqq j\leqq N(i)). \tag{3}$$

From (1), (2) and (3), we get $I=I(\operatorname{Im}\phi_v)$. Q.E.D.

4. A geometric characterization.

We shall give a geometric characterization of observable Hopf ideals. We shall use notations and technical terms in [3].

Let A be a Hopf algebra, R a left coideal subalgebra of A, $\mathfrak{G}=\mathfrak{Sp}_kA$ and $\mathfrak{X}=\mathfrak{Sp}_kR$. Then $\mathfrak{G}$ operates on $\mathfrak{X}$ via $\mathfrak{Sp}_k\lambda_R$, that is, $(gx)(r)=\sum g(r_{(1)})x(r_{(2)})$ for any $T\in\boldsymbol{M}_k$, any $g\in\mathfrak{G}(T)$, any $x\in\mathfrak{X}(T)$ and any $r\in R$. If $i: R\to A$ is the inclusion map, then $\mathfrak{Sp}_ki$ is exactly the orbit map $\mathfrak{p}_\omega:\mathfrak{G}\to\mathfrak{X}$, where $\omega=\varepsilon_A|_R\in\mathfrak{X}(k)$, that is, $\mathfrak{p}_\omega(g)=g|_R$ for any $T\in\boldsymbol{M}_k$ and any $g\in\mathfrak{G}(T)$.

Now we shall give some results which are easily proved.

(1) $\mathfrak{Cent}_{\mathfrak{G}}(\omega)=\mathfrak{Sp}_kA/I(R)$. In fact, recalling that $I(R)$ is the ideal of A generated by $R^+=R\cap\ker\varepsilon_A=\ker\omega$, it is easy to see that $\mathfrak{Cent}_{\mathfrak{G}}(\omega)(T)=\mathfrak{Sp}_kA/I(R)(T)$ for any $T\in\boldsymbol{M}_k$.

(2) If a Hopf ideal I contains $I(R)$, then we have by (1) and ([3], III, §3, 1.6.) that there is a unique morphism $\mathfrak{r}:\widetilde{\widetilde{\mathfrak{G}/\mathfrak{H}}}\to\mathfrak{X}$ such that $\mathfrak{p}_\omega=\mathfrak{r}\cdot\tilde{\tilde{\mathfrak{p}}}_{\mathfrak{G},\mathfrak{H}}$, where $\mathfrak{H}=\mathfrak{Sp}_kA/I$ and $\tilde{\tilde{\mathfrak{p}}}_{\mathfrak{G},\mathfrak{H}}:\mathfrak{G}\to\widetilde{\widetilde{\mathfrak{G}/\mathfrak{H}}}$ is the canonical morphism. $\mathfrak{r}$ is a monomorphism if and only if $\mathfrak{H}=\mathfrak{Cent}_{\mathfrak{G}}(\omega)$, that is, $I=I(R)$. Notice that $\mathfrak{r}$ is induced from the morphism

$$\mathfrak{G}(T)/\mathfrak{H}(T)\longrightarrow\mathfrak{X}(T),\ g\mathfrak{H}(T)\longmapsto g|_R,\ T\in\boldsymbol{M}_K$$

and the origin of $\mathfrak{G}/\mathfrak{H}, \bar{\mathfrak{e}}$, is mapped to ω by $\mathfrak{r}$.

(3) If A is finitely generated as a k-algebra, then $\widetilde{\mathfrak{G}/\mathfrak{Cent}_{\mathfrak{G}}(\omega)}$ is quasi-affine. In fact, take generators $r_1, \cdots, r_n$ of $I(R)$ from R as an ideal. Then the left coideal

subalgebra R' generated by the r_i is finitely generated as a k-algebra. By the construction, $I(R)=I(R')$. Hence we get $\mathfrak{Cent}_{\mathfrak{G}}(\omega)=\mathfrak{Cent}_{\mathfrak{G}}(\omega')$ by (1), where $\omega'=\varepsilon_A|_{R'}\in\mathfrak{Sp}_kR'(k)$. It follows from ([3], III, §3, 5. 2) that $\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)}=\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega')}$ is an algebraic k-scheme and $\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega')}\to\mathfrak{Sp}_kR'$ is an immersion. Therefore $\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)}$ is a separated, quasi-compact, open subscheme of an affine scheme, which means that it is quasi-affine.

Since $\mathcal{O}(\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)})$ can be identified with $A\square_{A/I(R)}k$ (Cf. [3], III, §3, 7.8.) and $\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)}$ is quasi-affine, it follows from ([5], II, 5.1.2.) that the canonical morphism $\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)}\to\mathfrak{Sp}_k\mathcal{O}(\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)})\to\mathfrak{Sp}_k(A\square_{A/I}k)$ is an open immersion. Notice that it is the same morphism that $\mathfrak{r}$ taking $R=A\square_{A/I}k$. In paticular, A is flat over $A\square_{A/I}k$, since the canonical morphism $\mathfrak{G}\to\mathfrak{G}/\widetilde{\mathfrak{Cent}_{\mathfrak{G}}(\omega)}$ is faithfully flat. This assertion is also valid when A is not a finitely generated k-algebra. In fact, $A=\cup A_i$, where the A_i are sub-Hopf algebras of A and finitely generated as k-algebras. Let $R_i=A_i\cap R$ be a left coideal subalgebra of A_i. It is obvious that

$$A_i\square_{A_i/I(R_i)}k=A_i\cap(A\square_{A/I(R)}k).$$

Since A_i is flat over $A_i\square_{A_i/I(R_i)}k$, A is flat over $A\square_{A/I(R)}k$.

(4) Let I be a Hopf ideal of A and $R=A\square_{A/I}k$. It is obvious that $I(R)\subset I$. If $\mathfrak{a}$ is a non-zero left coideal-ideal of R, then $\mathfrak{r}:\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}\to\mathfrak{Sp}_kR$ in (2) factors through $(\mathfrak{Sp}_kR)_{\mathfrak{a}}$. In fact, since $\mathfrak{r}$ is the dur-sheafication of $\mathfrak{r}':\mathfrak{G}/\mathfrak{H}\to\mathfrak{Sp}_kR$, it is enough to show that $\operatorname{Im}\mathfrak{r}'(T)\subset(\mathfrak{Sp}_kR)_{\mathfrak{a}}(T)$ for any $T\in\boldsymbol{M}_k$. Recall that for any $g\in\mathfrak{G}(T)$, $\mathfrak{r}'(T)(g\mathfrak{H}(T))=g|_R$. Since $\mathfrak{a}A=A$, we get $1\in g(\mathfrak{a}A)\subset g|_R(\mathfrak{a})T$. Therefore $g|_R\in(\mathfrak{Sp}_kR)_{\mathfrak{a}}(T)$.

THEOREM. Let A be a Hopf algebra, I a Hopf ideal of A, $R=A\square_{A/I}k$, $\mathfrak{G}=\mathfrak{Sp}_kA$, $\mathfrak{H}=\mathfrak{Sp}_kA/I$ and $\mathfrak{X}=\mathfrak{Sp}_kR$. The other notations are similar to the above.

(i) I is observable if and only if $\mathfrak{r}:\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}\to\mathfrak{X}$ is a monomorphism.

(ii) The following conditions are equivalent;

(a) I is observable and R has a simple left coideal M such that, if $\mathfrak{b}$ is a non-zero left coideal-ideal of R, then $M\subset\sqrt{\mathfrak{b}}$,

(b) $\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}$ is a quasi-affine k-scheme,

(c) $\mathfrak{r}:\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}\to\mathfrak{X}$ is an open immersion.

(iii) I is observable and finitely generated if and only if $\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}$ is a quasi-affine algebraic k-scheme.

PROOF. (i) Suppose I is observable. Since $I=I(R)$, $\mathfrak{r}$ is a monomorphism by (2). Suppose the contrary. The diagram

$$\begin{tikzcd}
\mathfrak{H} \arrow[r, "incl."] \arrow[d, "can."'] & \mathfrak{G} \arrow[d, "\tilde{\tilde{\mathfrak{p}}}_{\mathfrak{G},\mathfrak{H}}"] \\
\mathfrak{e}_k \arrow[r, "\bar{\bar{\mathfrak{e}}}^{\#}"] & \mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}
\end{tikzcd}$$

where $\bar{\bar{\mathfrak{e}}}^{\#}=\mathfrak{p}_{\mathfrak{G},\mathfrak{H}}\circ\mathfrak{Sp}_k\varepsilon_A$, is cartesian. (cf. [3], III, § 3, 1.5) Since $\mathfrak{r}$ is a monomorphism, the diagram of affine k-schemes

$$\begin{tikzcd}
\mathfrak{H} \arrow[r, "incl."] \arrow[d, "can."'] & \mathfrak{G} \arrow[d, "\mathfrak{r}\cdot\tilde{\tilde{\mathfrak{p}}}_{\mathfrak{G},\mathfrak{H}}"] \\
\mathfrak{e}_k \arrow[r, "\omega^{\#}"] & \mathfrak{X}
\end{tikzcd}$$

where $\omega^{\#}=\mathfrak{p}_\omega\circ\mathfrak{Sp}_k\varepsilon_A$, is cartesian. Hence $A/I\simeq k\otimes_R A\simeq R/R^+\otimes_R A\simeq A/I(R)$. Since $I\supset I(R)$, we get $I=I(R)$. Therefore I is observable.

(ii) (c) $\Rightarrow$ (a). There is an ideal $\mathfrak{a}$ of R such that $\sqrt{\mathfrak{a}}=\mathfrak{a}$ and

$$\mathfrak{r}:\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}\xrightarrow{\sim}(\mathfrak{Sp}_kR)_\mathfrak{a}\hookrightarrow\mathfrak{X}.$$

Given any non-zero left coideal-ideal $\mathfrak{b}$, it follows from (4) that it factors through $(\mathfrak{Sp}_kR)_\mathfrak{b}$. Hence $\mathfrak{a}\subset\sqrt{\mathfrak{b}}$. Since $\lambda_R^{-1}(A\otimes\mathfrak{a})$ is a left coideal-ideal contained in $\mathfrak{a}$, it is enough to show $\lambda_R^{-1}(A\otimes\mathfrak{a})\neq 0$. In fact, if M is a simple left coideal contained in $\lambda_R^{-1}(A\otimes\mathfrak{a})$, then $M\subset\sqrt{\mathfrak{b}}$. Now we claim $\mathfrak{a}A=A$. Given any maximal ideal $\mathfrak{m}$ of A, take the residue field $K=A/\mathfrak{m}$. Let $g\in\mathfrak{G}(K)$ be the canonical map $g:A\to K$ and $e_K\in\mathfrak{G}(K)$ the unit element. Then $\mathfrak{p}_\omega(e_K)=\omega_K\in(\mathfrak{Sp}_kR)_\mathfrak{a}(K)$. Since $\mathfrak{G}/\tilde{\tilde{\mathfrak{H}}}\simeq(\mathfrak{Sp}_kR)_\mathfrak{a}$ is $\mathfrak{G}$-stable, $g\omega_K\in(\mathfrak{Sp}_kR)_\mathfrak{a}(K)$, that is, $g\omega_K(\mathfrak{a})\neq 0$. On the other hand, we have $g\omega_K(r)=\sum_{(r)}g(r_{(1)})\varepsilon_A(r_{(2)})=g(r)$ for any $r\in R$. Hence $g(\mathfrak{a})\neq 0$, or $\mathfrak{a}\not\subset\ker g=\mathfrak{m}$. This means $\mathfrak{a}A=A$. From the lemma below, we get $\lambda_R^{-1}(A\otimes\mathfrak{a})\neq 0$.

(a) $\Rightarrow$ (b) or (c). Let $\mathfrak{a}$ be the ideal generated by M. Notice that $\mathfrak{a}$ is a left coideal-ideal. If $f_1,\cdots,f_n$ form a k-basis of M, then $\mathfrak{a}=(f_1,\cdots f_n)$. We claim $\mathfrak{G}/\mathfrak{H}\simeq(\mathfrak{Sp}_kR)_\mathfrak{a}=\bigcup_{i=1}^{n}\mathfrak{Sp}_kR_{f_i}$, from which (b) and (c) follow. It is enough to show that the sequence in $\widetilde{\widetilde{M_kE}}$

$$\mathfrak{G}\times\mathfrak{H}\underset{pr_1}{\overset{\mathfrak{Sp}_k(1\otimes\pi)\Delta}{\rightrightarrows}}\mathfrak{G}\xrightarrow{\mathfrak{p}_\omega}(\mathfrak{Sp}_kR)_\mathfrak{a}$$

where $\pi:A\to A/I$ is the canonical map, is exact.

The remaining proof is to show that $\mathfrak{p}_\omega$ is faithfully flat. Since $\mathfrak{p}_\omega:\mathfrak{G}\to(\mathfrak{Sp}_kR)_\mathfrak{a}$ is flat by (3), it is enough to show that given any maximal ideal $\mathfrak{m}$ of R such that

$\mathfrak{m}\not\supset\mathfrak{a}$, we have $\mathfrak{m}A\neq A$. Suppose $\mathfrak{m}A=A$. By the lemma below, $\mathfrak{b}=\lambda_R^{-1}(A\otimes\mathfrak{m})\neq 0$. By the assumption on M, we get $\mathfrak{a}\subset\sqrt{\mathfrak{b}}$. Since $\mathfrak{b}\subset\mathfrak{m}, \mathfrak{a}\subset\mathfrak{m}$, which is the contradiction.
(b) $\Rightarrow$ (c). It follows from (3).

(iii) ($\Rightarrow$) Let $I=(r_1, \cdots, r_n)$, R' a left coideal subalgebra generated by $r_1, \cdots, r_n$, A' a sub-Hopf algebra generated by R' and $I'=I_{A'}(R')=R'^{+}A'$. Since A is faithfully flat over A' (2.2), $I'=I\cap A'$. Since $I_A(R')=I$, $\mathfrak{H}=\mathfrak{Cent}_{\mathfrak{G}}(\omega')$ where $\omega'=\varepsilon_A|_{R'}\in(\mathfrak{Sp}_k R')(k)$. Since $I_{A'}(R')=I'$, $\mathfrak{H}'=\mathfrak{Sp}_k A'/I'=\mathfrak{Cent}_{\mathfrak{G}}(\omega')$. Since $\mathfrak{G}'$ and $\mathfrak{H}'$ are affine algebraic k-schemes, $\mathfrak{G}'\widetilde{\widetilde{/}}\mathfrak{H}'\simeq\mathfrak{G}'\widetilde{/}\mathfrak{H}'$ is an algebraic k-scheme. We get the commutative diagram,

$$\begin{tikzcd}
\mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H} \arrow[rr, "mono."] \arrow[dr, "can."'] & & \mathfrak{Sp}_k R' \\
& \mathfrak{G}'\widetilde{/}\mathfrak{H}' \arrow[ur, "open\ immersion"'] &
\end{tikzcd}$$

Hence the canonical morphism $\mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H}\to\mathfrak{G}'\widetilde{/}\mathfrak{H}'$ is a monomorphism. On the other hand, we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{G} \arrow[r, "can."] \arrow[dr, "\tilde{\tilde{\mathfrak{p}}}_{\mathfrak{G},\mathfrak{H}}"'] & \mathfrak{G}' \arrow[r, "\tilde{\mathfrak{p}}_{\mathfrak{G}',\mathfrak{H}'}"] & \mathfrak{G}'\widetilde{/}\mathfrak{H}' \\
& \mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H} \arrow[ur, "can."'] &
\end{tikzcd}$$

where $\mathfrak{G}\to\mathfrak{G}'$ is faithfully flat and quasi-compact, and $\mathfrak{G}'\to\mathfrak{G}'\widetilde{/}\mathfrak{H}'$ is faithfully flat and of finite presentation ([3], III, §3, 2.5). By ([3], III, §1, 2.10 & 3.3), we get the top morphism is an epimorphism. Then $\mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H}\xrightarrow{can.}\mathfrak{G}'\widetilde{/}\mathfrak{H}'$ is an epimorphism. Hence $\mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H}\simeq\mathfrak{G}'\widetilde{/}\mathfrak{H}'$. By (3), $\mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H}$ is quasi-affine algebraic.

($\Leftarrow$). $\mathfrak{r}$ can be expressed as

$$\mathfrak{G}\widetilde{\widetilde{/}}\mathfrak{H}\xrightarrow{\ \sim\ }\bigcup_{i=1}^{n}\mathfrak{Sp}_k R_{f_i}\hookrightarrow\mathfrak{Sp}_k R$$

and each R_{f_i} is a finitely generated k-algebra. Let

$$R_{f_i}=k[r_{i1}/f_i^{n_1}, \cdots, r_{im(i)}/f_i^{n_{m(i)}}], \quad r_{ij}\in R, (1\leq i\leq n, 1\leq j\leq m(i))$$

and R' is the left coideal subalgebra generated by $r_{i1}, \cdots r_{im(i)}, f_i$ $(1\leq i\leq n)$. Note that R' is finitely generated as a k-algebra. It is obvious that $R_{f_i}=R'_{f_i}$. Since $\mathfrak{H}=\mathfrak{Cent}_{\mathfrak{G}}(\omega)=\mathfrak{Cent}_{\mathfrak{G}}(\omega')$, $I=I(R')$. Therefore I is finitely generated. Q.E.D.

LEMMA. Let R be a left coideal subalgebra of a Hopf algebra A and $\mathfrak{a}$ a non-zero ideal of R such that $\sqrt{\mathfrak{a}}=\mathfrak{a}$. If $\mathfrak{a}A=A$, then $\lambda_R^{-1}(A\otimes\mathfrak{a})\neq 0$.

Proof. Recall that $\lambda_R^{-1}(A\otimes\mathfrak{a})$ is the largest left coideal of R contained in $\mathfrak{a}$. It is easy to show that $(\lambda_R\otimes\bar{k})^{-1}(A\otimes\sqrt{\mathfrak{a}\otimes\bar{k}})\neq 0$ implies $\lambda_R^{-1}(A\otimes\mathfrak{a})\neq 0$, where $\bar{k}$ is an algebraic closure of k. Hence we may assume k is an algebraically closed field. Let $\sum_{i=1}^{n} r_i a_i=1, r_i\in\mathfrak{a}$ and $a_i\in A$. If R' is a left coideal subalgebra generated by $r_1,\cdots,r_n$ and A' a sub-Hopf algebra generated by $r_1,\cdots,r_n,a_1,\cdots,a_n$, then they are finitely generated as k-algebras and R' is a left coideal subalgebra of A'. If $\mathfrak{a}'=\mathfrak{a}\cap R'$, then $\sqrt{\mathfrak{a}'}=\mathfrak{a}'$ and $\mathfrak{a}'A'=A'$. Since $\lambda_{R'}^{-1}(A'\otimes\mathfrak{a}')\subset\lambda_R^{-1}(A\otimes\mathfrak{a})$, we may assume that A and R are finitely generated as k-algebras. We may also assume that A is reduced. In fact, let $\phi: A\to A/\sqrt{(0)}$ be the canonical Hopf algebra map, then $\phi(A),\phi(R)$ and $\phi(\mathfrak{a})$ satisfy the assumptions. If $\lambda_{\phi(R)}^{-1}(\phi(A)\otimes\phi(\mathfrak{a}))\neq 0$, then there is its element $\phi(r)$ such that $\varepsilon(\phi(r))=1$. Hence $\sum r_{(1)}^{p^n}\otimes r_{(2)}^{p^n}\in A\otimes\mathfrak{a}$ for a natural number n being large enough, where p is the characteristic exponent of k. (Recall that, if $p=1$, then A is always reduced.) Therefore $r^{p^n}\in\lambda_R^{-1}(A\otimes\mathfrak{a})$. Since $\varepsilon(r)=1, r^{p^n}\neq 0$.

Now, if $\mathfrak{a}=R$, then the assertion is trivial. If $\mathfrak{a}$ is a proper ideal then the proof of (2.4) shows that the defining ideal $\mathfrak{b}$ of the closed set Spec $R-|\mathfrak{p}_\omega|$(Spec A) is a non-zero left coideal. Given any $\mathfrak{p}\in\mathrm{Spec}\,R, \mathfrak{p}\supset\mathfrak{a}$ implies $\mathfrak{p}A=A$, hence $\mathfrak{p}\supset\mathfrak{b}$. We get $\mathfrak{a}\supset\mathfrak{b}$. Therefore $\mathfrak{b}$ is contained in $\lambda_R^{-1}(A\otimes\mathfrak{a})$. Q.E.D.

Appendix: Correction to my paper "A correspondence between observable Hopf ideals and left coideal subalgebras" [7].

In my paper [7], prop. 3, it was claimed that let A be any left coideal subalgebra of a Hopf algebra H; then $H\otimes_A H$ is a right H-comodule via $1\otimes\Delta$ where $H\otimes H$ is viewed as an A-module through $A\to H\otimes A$ and $H\otimes H\otimes H$, through $A\to H\otimes H\otimes A$. This is false because $H\otimes_A H$ can not be a right H-comodule via $1\otimes\Delta$. So I have given in (3.6) a new proof of prop. 3 of [7]. The theorem 4 in [7] is also incorrect; it was claimed in its proof that $k[G/K]=S^{-1}A$. This is false because they are not always equal. Therefore it must be deleted the theorem 4 and its corollary in the paper [7].

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