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FRAMED-LINK REPRESENTATIONS OF 3-MANIFOLDS

By

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By Wallace [7] and Lickorish [2], it is proved that every closed orientable connected 3-manifold can be obtained by Dehn surgery along a link in S^3, in other words, every closed orientable connected 3-manifold has a framed-link representation.

The proof of it by Lickorish in [2] is based on the fact that the mapping class group of a closed orientable connected 2-manifold is generated by Dehn twists. However, by Lickorish [3], a special finite set of generators for the mapping class group of a closed orientable connected 2-manifold is found.

These are Dehn twists along the loops shown in Figure 1.

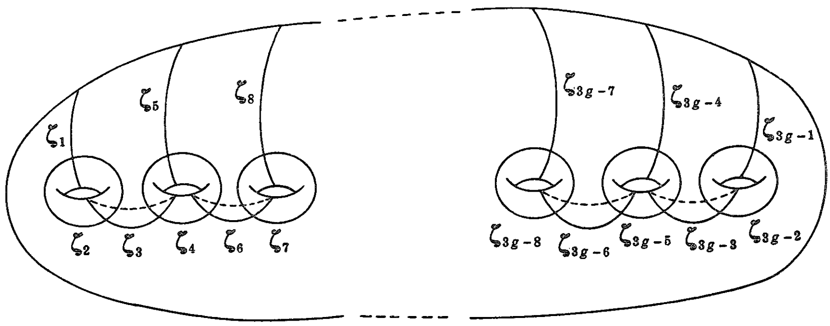

Figure 1.

From this fact, the Wallace-Lickorish theorem can be refined as follows. Let $L_1(n, g)$ be the link illustrated in Figure 2.

THEOREM A. (c. f. Montesinos [4])
Every closed orientable connected 3-*manifold can be obtained by* ± 1 *Dehn surgery along a sublink of* $L_1(n, g)$ *for sufficiently large* n *and* g. *In other words, every closed orientable connected* 3-*manifold can be obtained by* ± 1 *or* ∞ *Dehn surgery along a link* $L_1(n, g)$ *for sufficiently large* n *and* g.

In this paper we shall prove the following more refined theorem.

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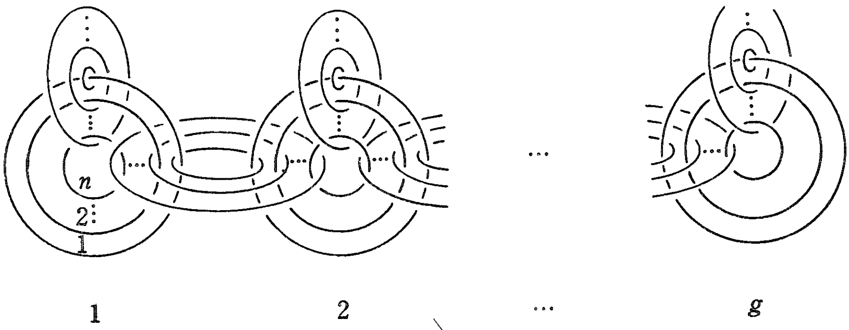
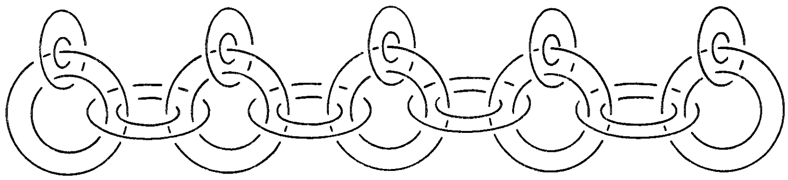
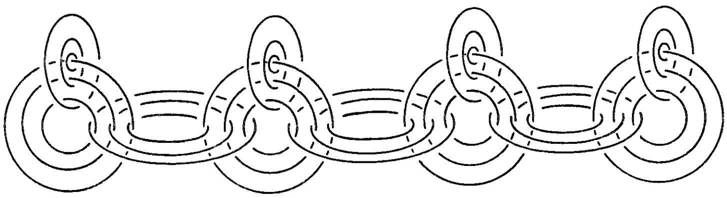


$L_1(n, g)$



$L_1(2, 5)$



$L_1(3, 4)$

Figure 2.

THEOREM 1. *Every closed orientable connected 3-manifold can be obtained by* $+1$ *Dehn surgery only (or* -1 *Dehn surgery only) along a sublink of* $L_1(n, g)$ *for sufficiently large* n *and* g. *In other words, every closed orientable connected 3-manifold can be obtained by* $+1$ *or* ∞ *Dehn surgery (or,* -1 *or* ∞ *Dehn surgery) along a link* $L_1(n, g)$ *for sufficiently large* n *and* g.

PROOF OF THEOREM 1. Let M be a closed orientable connected 3-manifold. Then M has a Heegaard splitting of genus g for sufficiently large g. Consider the closed orientable connected 2-manifold of genus g and the mapping class group G_g of it.

Then M is represented by a word in the generators ζ_i, $i=1, 2, \cdots, 3g-1$ (c. f. Figure 1).

ζ_i corresponds to $+1$ Dehn surgery and ζ_i^{-1} corresponds -1 Dehn surgery. So, it is sufficient to prove that every word is equal to a positive word (a word which does not contain ζ_i^{-1}) in G_g.

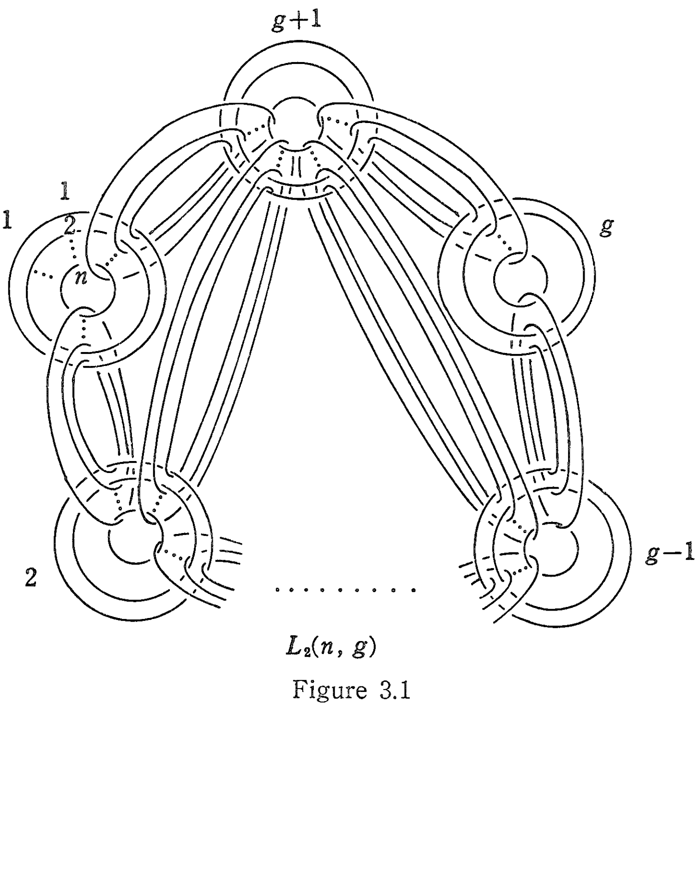
For this it is sufficient to prove that each ζ_i^{-1} is equal to a positive word in G_g. For this it is sufficient to show that there is a positive word w which is equal to 1 in G_g and which includes all the generators ζ_i's, for $w=A\zeta_i B=1$ (A, B are positive words or empty words) implies $\zeta_i^{-1}=BA$, and BA is a positive word. So we shall show the following.

LEMMA. *For $g\geqq 2$, there exists a positive woord w_g such that w_g includes all the Lickorish generators ζ_i of G_g and $w_g=1$ in G_g.*

PROOF. By the induction on g.

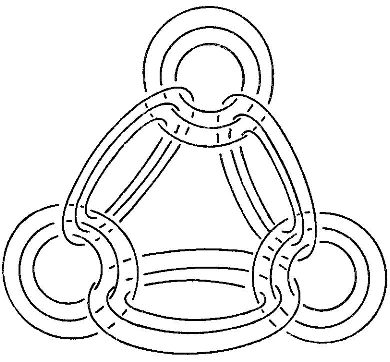
(i) $g=2$. By Birman [1],

$$G_2=\langle \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 \mid \zeta_i\zeta_j=\zeta_j\zeta_i(|i-j|\geqq 2,\ 1\leqq i,\ j\leqq 5)$$
$$\zeta_i\zeta_{i+1}\zeta_i=\zeta_{i+1}\zeta_i\zeta_{i+1}(1\leqq i\leqq 4),\ (\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^6=1,$$
$$(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1)^2=1,$$
$$(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1)\zeta_i=\zeta_i(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1)(1\leqq i\leqq 5)\rangle.$$

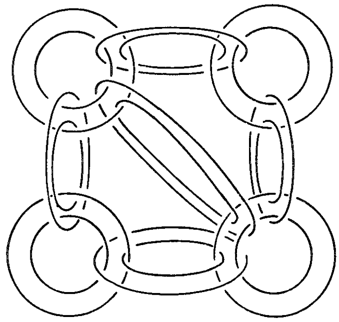


$L_2(n, g)$

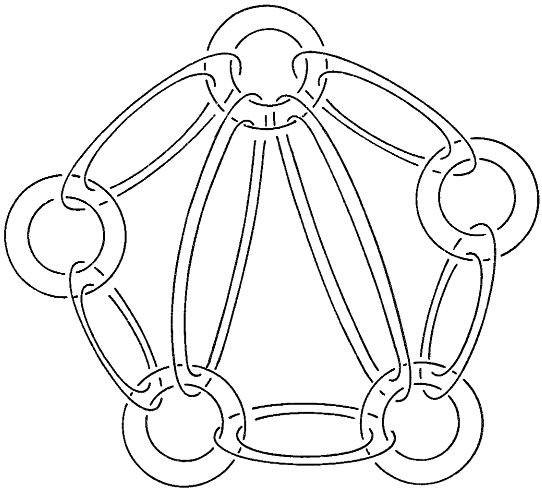
Figure 3.1



$L_2(3, 2)$



$L_2(2, 3)$



$L_2(2, 4)$

Figure 3.2

So, we can take $(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5)^6$ or $(\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5^2\zeta_4\zeta_3\zeta_2\zeta_1)^2$ as w_2.

(ii) $g>2$. By induction hypothesis, w_{g-1} exists. Consider w_{g-1} as a word for genus g. w_{g-1} includes $\zeta_1, \cdots, \zeta_{3g-4}$. w_{g-1} may not be 1 in G_g.

But w_{g-1} is equal to a word v in

$$\zeta_{3g-2} \text{ and } \zeta_{3g-1}, \text{ so, } w_{g-1}v^{-1}=1 \text{ in } G_g.$$

As in the case of genus 2, it is not hard to see

$$u=\{\zeta_1\zeta_2(\zeta_3\zeta_4)(\zeta_6\zeta_7)(\zeta_9\zeta_{10})\cdots(\zeta_{3g-3}\zeta_{3g-2})\zeta_{3g-1}\}^{2(g+1)}=1$$

in G_g. u is a positive word and includes ζ_{3g-2} and ζ_{3g-1}. Hence there exists a positive word $v'=v^{-1}$ in G_g. Now $w_{g-1}v'$ is a positive word and equal to 1 in G_g. So, we can take $w_{g-1}v'u$ as w_g, for $w_{g-1}v'=1$ and $u=1$ in G_g so, $w_{g-1}v'u=1$ and $w_{g-1}v'u$ is a positive word. Moreover w_{g-1} includes $\zeta_1, \zeta_2, \cdots, \zeta_{3g-4}$ and u includes $\zeta_{3g-3}, \zeta_{3g-2}, \zeta_{3g-1}$ and hence $w_{g-1}v'u$ includes all the Lickorish generators of G_g.

This completes the proof of Lemma and hence of Theorem 1.

REMARK. In the Theorem A and Theorem 1, $L_1(n, g)$ can be replaced by $L_2(n, g)$ illustrated in the Figure 3.1.

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