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ON THE MULTIVALENT FUNCTIONS

By

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Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \qquad (p\in N=\{1, 2, 3, \cdots\})$$

which are analytic in the unit disk $U=\{z: |z|<1\}$.

Ozaki, Ono and Umezawa [4, Theorem 1] obtained the following result.

THEOREM A. *Let $f(z)=z+a_2z^2+\cdots$ be analytic in U and suppose that*

$$|f''(z)|<1 \qquad in\ U,$$

then $f(z)$ is univalent in U.

In this paper, we need the following lemmata.

LEMMA 1. *Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r$ at a point z_0, then we can write*

$$z_0w'(z_0)=kw(z_0)$$

where k is a real number and $k\geqq 1$.

We owe this lemma to Jack [1] (also, by Miller and Mocanu [2]).

LEMMA 2. *Let $p\geqq 2$. If $f(z)\in A_p$ and suppose that*

$$Re\frac{f^{(p-1)}(z)}{z}>0 \qquad in\ U.$$

Then $f(z)$ is p-valent in U.

We owe this lemma to Nunokawa [3].

THEOREM 1. *Let $p(z)$ be analytic in U, $p(0)=1$ and suppose that*

(1) $$|p(z)+zp'(z)-1|<2 \qquad in\ U.$$

Then we have

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$$Re\, p(z)>0 \qquad in\ U.$$

PROOF. Let us put

$$p(z)=1+w(z),$$

then we have $w(z)$ is analytic in U and $w(0)=0$.

If we suppose that there exists a point $z_0\in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1,$$

then from Lemma 1, we have

$$z_0w'(z_0)=kw(z_0) \qquad (k\geqq 1).$$

Then we have

$$|p(z_0)+z_0p'(z_0)-1|=|1+w(z_0)+z_0w'(z_0)-1|$$
$$=|w(z_0)+kw(z_0)|=|w(z_0)(1+k)|\geqq 2.$$

This contradicts (1). Therefore we have

$$|w(z)|<1 \qquad in\ U.$$

This shows that

$$Re\, p(z)>0 \qquad in\ U.$$

THEOREM 2. *Let $p\geqq 2$. If $f(z)\in A_p$ and suppose that*

(2) $$|f^{(p)}(z)-p!|<2(p!) \qquad in\ U.$$

Then $f(z)$ is p-valent in U.

PROOF. Let us put

$$p(z)=\frac{f^{(p-1)}(z)}{p!z}, \qquad (p(0)=1).$$

By an easy calculation and from (2), we have

(3) $$|p(z)+zp'(z)-1|=\left|\frac{f^{(p-1)}(z)}{p!\,z}+z\left(\frac{zf^{(p)}(z)-f^{(p-1)}(z)}{p!\,z^2}\right)-1\right|$$
$$=\left|\frac{f^{(p)}(z)}{p!}-1\right|<2 \qquad in\ U.$$

From (3) and Theorem 1, we have

$$Re\,\frac{f^{(p-1)}(z)}{p!z}>0 \qquad in\ U.$$

This shows that

$$Re\,\frac{f^{(p-1)}(z)}{z}>0 \qquad in\ U.$$

From Lemma 2, we have $f(z)$ is p-valent in U.

REMARK. For the case $p\geqq 2$, it is very interesting that $f(z)\in A_p$ continues to be p-valent in U, even if $f^{(p)}(z)$ takes negative real value in U.

THEOREM 3. *Let* $p\geqq 2$. *If* $f(z)\in A_p$ *and suppose that*

$$|f^{(p+1)}(z)|<2(p!) \qquad in\ U.$$

Then $f(z)$ *is* p*-valent in* U.

PROOF. We easily have

$$|f^{(p)}(z)-p!|=\left|\int_0^z f^{(p+1)}(t)dt\right|$$

$$\leqq\int_0^r |f^{(p+1)}(t)||dt|<2(p!)r<2(p!)$$

for $z\in U$ and $|z|=r<1$.

From Theorem 2, we have $f(z)$ is p-valent in U. This completes our proof.

For the case $p\geqq 2$, Theorem 3 is a more excellent result than Theorem A [4, Theorem 1].

References

[1] I.S. Jack, Functions starlike and convex of order α, J. London Math. Soc., **3**, 469-474 (1971).
[2] S.S. Miller and P.T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl., **65**, 289-305 (1978).
[3] M. Nunokawa, On the theory of multivalent functions, Tsukuba J. Math., **11**, 273-286 (1987).
[4] S. Ozaki, I. Ono and T. Umezawa, On a general second order derivative, Sci. Rep. Tokyo Bunrika Daigaku, Sec. A, **5**, 111-114 (1956).

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