ON THE CONCRETE CONSTRUCTION OF HYPERBOLIC STRUCTURES OF 3-MANIFOLDS

By

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Contents

§ 1. Introduction.

In [3] Chapter 4, Thurston constructed a hyperbolic structure of figure eight knot complement by glueing together the faces of two ideal 3-simplexes which are in 3-dimensional Poincaré model of hyperbolic geometry. In this paper we shall show by illustration that this construction can also be applied to other knot complements and even to more general 3-manifolds whose Heegaad diagrams are given.

In § 1, we shall define the notion of nice triangulations of 3-manifolds. This definition is made entirely under the category of combinatorial topology, irrespective of geometric structure. The Nice Triangulation Theorem which asserts that every compact 3-manifold with boundary has a nice triangulation, shows that this notion is quite general.

In § 2, we shall illustrate, for 5_2-knot, how to construct a nice triangulation of the complement of a given knot, and then how Thurston's method to construct hyperbolic structure in [3] Chapter 4 can be applied to the nice triangulation obtained thus.

In § 3, we shall give the results for some other knots $(6_1, 6_2, 6_3)$ by the same method as above.

In § 4, we shall illustrate how to construct a nice triangulation from a given Heegaad diagram which does not necessarily give a knot complement.

In § 5, we shall give other examples of the construction of nice triangulations with 2- or 3-simplexes and of hyperbolic structures.

In § 6, we shall give a method to construct the (non-commutative) representations of the fundamental group $\pi_1(M)$ into $PSL(2, C)$, from a given nice triangulation of M. It would be interesting that non-commutative relators of the presentation of $\pi_1(M)$ turns to the corresponding algebraic equations of commutative field C, and any non-trivial solution of these equations corresponds to an equivalence class of the representations.

In § 7, we shall give other types of special concrete construction of a hyperbolic closed manifold and hyperbolic manifolds with totally geodesic boundary.

In § 8, we shall give a rigorous proof of The Nice Triangulation Theorem stated in § 1, although the method illustrated in § 4 already gives a sketch of the proof of the theorem.

§ 1. The nice triangulations.

Let $K=\{\varDelta_1, \varDelta_2, \cdots, \varDelta_n\}$ be a set of disjoint 3-simplexes. Suppose that one of the ways of glueing pairwise the $4n$ faces of these n simplexes is specified. This means, (i) $2n$ pairs of the faces to be glued together are specified, and (ii) for each of these pairs, a correspondence between the vertices of the two faces is specified. (This correspondence induces a linear glueing map between the two faces.) We call the K with this specification a *nice complex*.

Now let $M'(K)$ be the topological space obtained from K by glueing pairwise the faces of the 3-simplexes by the specified way. Set theoretically, $M'(K)$ is obtained from the disjoint union $A=\varDelta_1\cup\varDelta_2\cup\cdots\cup\varDelta_n$ by taking the quotient space $\bar{A}=A/\equiv$, where $\equiv$ is the equivalence relation induced by the glueing map. Moreover, the topological structure of $\bar{A}$ is given by:

$$U\subseteq\bar{A} \text{ is open} \Longleftrightarrow p^{-1}(U)\subseteq A \text{ is open,}$$

where p is the natural map from A onto $\bar{A}$. $M'(K)$ is obviously compact.

Next let $\varGamma$ be the set of those points in $M'(K)$ which are the images of the vertices of $\varDelta_1, \varDelta_2, \cdots, \varDelta_n$. $\varGamma$ is a finite set. Let

$$M(K) = M'(K) - \Gamma ,$$

and

$$\tilde{M}(K) = M'(K) - [\text{a small regular neighborhood of } \Gamma] .$$

$M(K)$ is an open subset of $M'(K)$ and $\tilde{M}(K)$ is a closed (and hence compact) subset of $M'(K)$.

We first observe the condition for $M(K)$ to be a 3-manifold. By the natural map $p : A \to \bar{A}$, the interiors of $\Delta_i's$ are mapped homeomorphically. The faces of $\Delta_i's$ are glued together pairwise. So $\bar{A} - [\text{1-skelton}]$ is a 3-manifold. So we have only to consider the images of the edges of $\Delta_i's$.
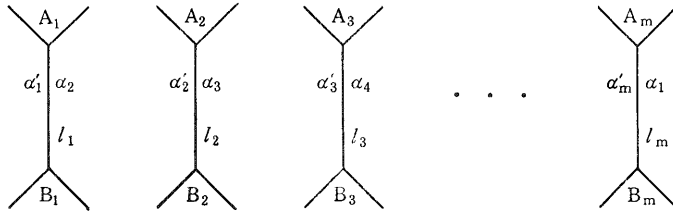
Fig. 1.

Let l_1 be an edge of one of the 3-simplexes. Suppose that l_1 is the intersection of two adjacent faces α_1' and α_2. Now α_2 is glued to a face α_2', and let l_2 be that edge of α_2' which corresponds to l_1 by the glueing. Let α_3 be another face which contains l_2. α_3 is glued to a face α_3', and so on. Since the number of the 3-simplexes of K is finite, there is an m such that the edge l_m is the intersection of the faces α_m' and α_1, and α_1 is glued to α_1' so that l_m corresponds to l_1. (We assume that m is chosen to be the smallest possible.) Now let A_1 and B_1 be the vertices of l_1, and let A_2 and B_2 be the vertices of l_2 such that A_1 coresponds to A_2 and B_1 corresponds to B_2 by the glueing of α_2 and α_2'. Similarly, A_i and B_i $(i = 1, 2, \cdots, m)$ are defined. (See Figure 1.)

Now there are two possibilities: By the glueing of α_1 and α_1',

(i) A_m corresponds to A_1 and B_m corresponds to B_1,

(ii) A_m corresponds to B_1 and B_m corresponds to A_1.

If the case (ii) happens (for some edge l_1), then $M(K)$ fails to be a 3-manifold. For, the glueing map induces a series of linear maps between $l_i's$:

$$l_1 \xrightarrow{\pi_1} l_2 \xrightarrow{\pi_2} l_3 \xrightarrow{\pi_3} l_4 \xrightarrow{\pi_4} \cdots \xrightarrow{\pi_{m-1}} l_m \xrightarrow{\pi_m} l_1 ,$$

and the composition $\sigma = \pi_m \circ \pi_{m-1} \circ \cdots \circ \pi_2 \circ \pi_1$ is a linear map of l_1 onto itself which is orientation-reversing (since σ maps A_1 to B_1 and B_1 to A_1). So the middle point C of l_1 does not have a neighborhood homeomorphic to R^3. Next suppose that

the case (ii) does not happen arround any edge and hence the case (i) always happens, then the topological structure arround any edge is as shown on Figure 2 and hence each interior point of the edge has a neighborhood homeomorphic to R^3. So $M(K)$ is a 3-manifold.
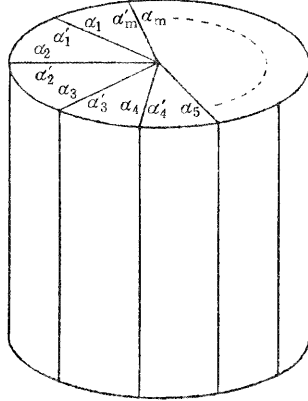

Fig. 2.

We call this combinatorial condition for K that the case (i) always happens, the *local orientability condition*. Thus $M(K)$ is a 3-manifold if and only if K satisfies the local orientability condition.

Next we consider the condition for $M(K)$ to be orientable. We call the following condition for K the *orientability condition*: Each of $\Delta_1, \Delta_2, \cdots, \Delta_n$ is oriented and each glueing map between the faces is orientation-reversing. Since the glueing map is determined by the correspondence between the vertices of the faces, this condition is also combinatorial. It is not hard to see that the orientability condition implies the local orientability condition. So, if K satisfies the orientability condition, then $M(K)$ is a manifold and it is easy to see that $M(K)$ is orientable. The orientability condition is not a necessary condition for $M(K)$ to be orientable. However, if K satisfies the local orientability condition and $M(K)$ is orientable, then by changing the orientation of some of $\Delta_1, \Delta_2, \cdots, \Delta_n$, we can assume that K satisfies the orientability condition.

Hereafter we only consider the nice complexes which satisfies the local orientability condition. Then, $\tilde{M}(K)$ is obviously a compact 3-manifold with boundary and $M(K)$ is homeomorphic to the interior of $\tilde{M}(K)$.

Let M be a 3-manifold and K be a nice complex. We say that K is a nice triangulation of M if M is homeomorphic either to $M(K)$ or to $\tilde{M}(K)$. For the existence of a nice triangulation of a 3-manifold, we have the following theorem, the proof of which will be carried out in § 8.

[Nice Triangulation Theorem]. Every compact 3-manifold with boundary has a nice triangulation.

REMARK. The nice triangulations of a given 3-manifold are not unique. There are infinitely many nice triangulations of a given compact 3-manifold with boundary.

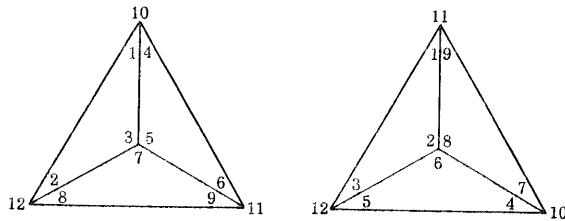EXAMPLE 1. (Thurston) The complement of figure eight knot has the following nice triangulation:

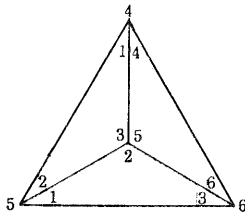
Fig. 3.

EXAMPLE 2. A nice triangulation of the 3-disk:

Fig. 4.

EXAMPLE 3. A nice triangulation of the solid torus (or the complement of the trivial knot):
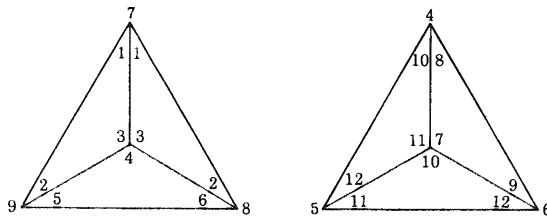
Fig. 5.

EXAMPLE 4. A nice triangulation of the trefoil knot complement:
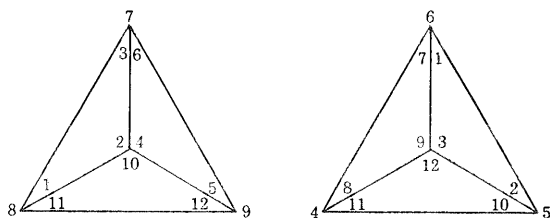
Fig. 6.

§ 2. The nice triangulation and hyperbolic structure of 5_2-knot complement.

The following method to construct hyperbolic structures is useful: Given a compact 3-manifold with toral boundary, first construct its nice triangulation and then construct its hyperbolic structure by the method described in Chapter 4 of Thurston's Lecture Note [3]. Probably this method would be applicable for every hyperbolizable 3-manifold with (or even without) toral boundary.

In this and the next sections we shall illustrate how to construct a nice triangulation of a given manifold practically and how to construct a hyperbolic structure from it. We have deviced two practical methods to construct a nice triangulation of a given 3-manifold: One is applicable for knot (or link) complements and the other is applicable for all compact 3-manifolds with boundary whose Heegaard diagrams are given. In this section we shall illustrate the former method with 5_2-knot complement as an example, and in the next section we shall illustrate the latter method with a rather simple example.

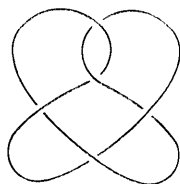Now 5_2-knot is illustrated in the Figure 7.

Fig. 7.

In general, given a regular projection of a knot, we can span the following (mutually dual) two surfaces A, B with this knot as boundary, illustrated in the Figure 8A and 8B. (These surfaces may be orientable or not, incompressible or or not. This does not matter.)
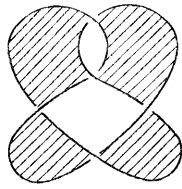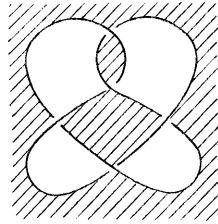
Fig. 8A. Fig. 8B.

If we span these two surfaces at the same time, some intersection arises naturally. Removing redundant intersection we find that the intersection consists of several segments, one segment near each crossing point of the projection, as shown in the Figure 9 and arround each crossing point, the two surfaces intersect as shown in the Figure 10, where l and k are parts of the knot and Figure 10 shows this side of l. (The same also for the other side of l.)
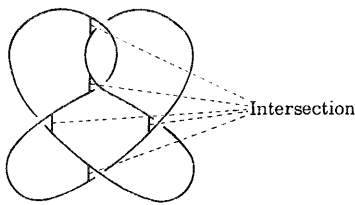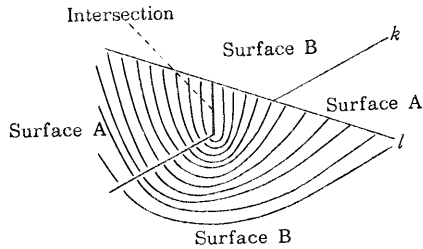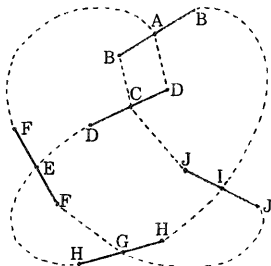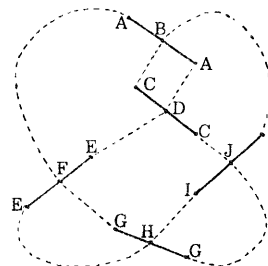
Fig. 9. Fig. 10.

Now we see that these two surfaces A and B divide S^3 into two domains D^+ (which is over the surfaces) and D^- (which is under the surfaces). D^+ and D^- are obviously open 3-disks. The boundary of these 3-disks are considered as 2-spheres with some identification of points.

∂D^+ (D^+ is over this paper)

Fig. 11A.

∂D^- (D^- is under this paper)

Fig. 11B.

The fragments of the knot are denoted by the dotted arcs. (Since we are considering the knot complement, the knot is not there.) We contract these dotted arcs to points on ∂D^+ and ∂D^-. Then we obtain the charts on ∂D^+ and ∂D^- illustrated in the Figure 12.
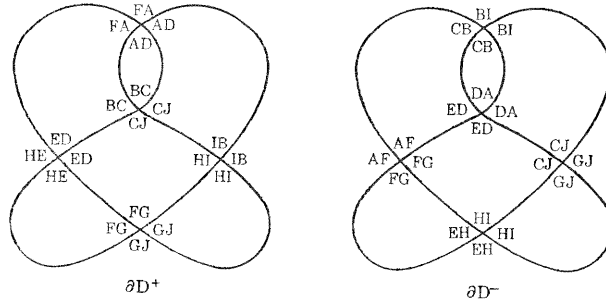

Fig. 12.

In Figure 12, the points AD indicates that the dotted arc AD in Figure 11 is contracted to this point. In order to simplify the chart we relabel the points as follows: $AD=DA=1$, $BC=CB=2$, $FA=AF=3$, $ED=DA=4$, $HE=EH=5$, $FG=GF$ $=6$, $GJ=JG=7$, $HI=IH=8$, $IB=BI=9$, $CJ=JC=10$. Then we obtain the Figure 13.
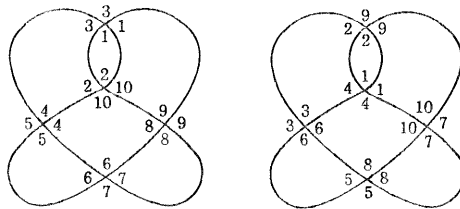

Fig. 13.

Here we remark that the graph of the Figure 13 is just the projection of the given knot to the plane. This fact is true for any alternating knot. For non-alternating knots, the graph becomes different from the projection of the knot. But it does not matter.

Returning to the case of 5_2-knot, we easily see from our construction that the 5_2-knot complement is homeomorphic to the result of glueing the faces of the graphs drawn on ∂D^+ and ∂D^- as in the Figure 13 so that the points with the same label coincide.

Next step is to delete "2-gons", (see Figure 14.) which obstruct to get triangulations of the 3-disks.
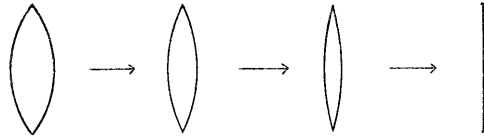
Fig. 14.

The process of the deletion of a 2-gon is to thin it gradually until it becomes a segment. (See Figure 14.) For most cases but not always, this process is possible (without changing the glueing results up to homeomorphism). In order to observe when it is possible, we first check the identification of edges. They split into five groups +, ‡, ⧻, ⧺, ⧾. (See Figure 15.)
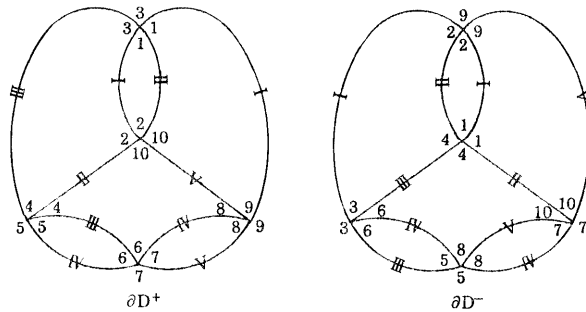
Fig. 15.

Now consider for instance the following 2-gon in ∂D^+ in the Figure 16.

Fig. 16.

The two edges of the 2-gon are not identified. It is not hard to see that the process of the deletion of the 2-gon is possible in this case. After the process the edges + and ‡ are identified and we have the following reduction of the graph (Figure 17).
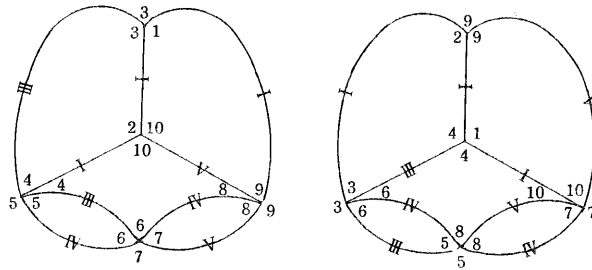
Fig. 17.

Moreover, for the left-hand-side 2-gon in the lower part of the Figure 17, two edges are not identified, so that the reduction is again possible. After the reduction ⧓ and ⧔ are identified but ⧔ and ⧒ are not identified. So the reduction of the right-hand-side 2-gon in the lower part of the Figure 17 is also possible. Thus we were able to delete all the 2-gons and we obtain the graphs of Figure 18.
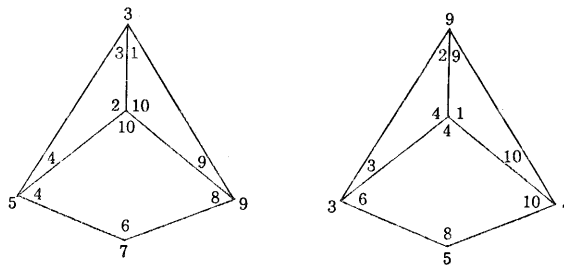

Fig. 18.

In order to get the triangulation with as few 3-simplexes as possible, we first glue the faces with labeled verticies 3-5-7-9. Then we obtain a single 3-disk with the graph of Figure 19 on the boundary.
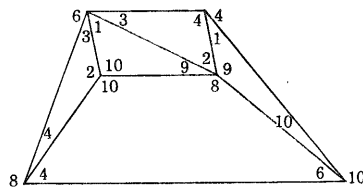

Fig. 19.

Now we draw arcs connecting the labeled vertices 6 and 10 to obtain a triangulation of the boundary of the 3-disk. This is shown in the Figure 20 with relabeled vertices.
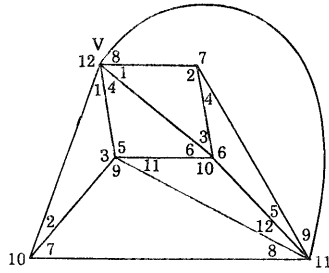
Fig. 20.

From this triangulation of the boundary of the 3-disk, we can easily obtain a triangulation of the whole 3-disk without adding new vertices. One way to obtain such a triangulation is to view Figure 20 as a cone with the top vertex V and three triangles (7-8-9), (10-11-12), (4-5-6) as the base. Finally we obtain a nice triangulation of 5_2knot complement. (Figure 21.)
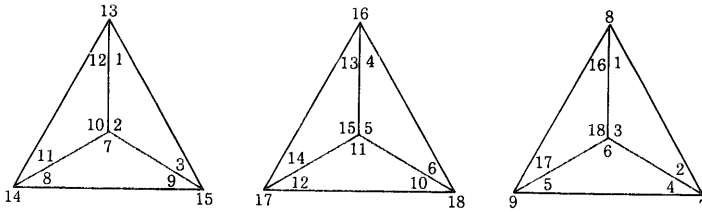

Fig. 21.

Next we shall construct a hyperbolic structure of 5_2-knot complement from this nice triangulation, using the method of Thurston [3] §4.

First of all we regard each of the 3-simplexes as an ideal 3-simplex (with vertices at infinity) of the hyperbolic 3-space. According to [3], an ideal 3-simplex is determined up to isometry by the dihedral angles $\alpha, \beta, \gamma (\alpha+\beta+\gamma=\pi)$ as in the Figure 22.

Fig. 22.
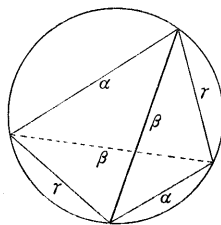
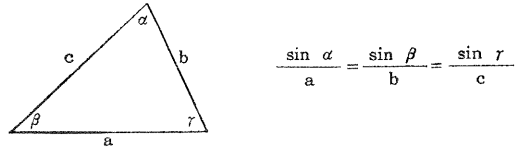In other words it is determined by the similarity type of a triangle in Euclidean plane. (Figure 23.)

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Fig. 23.

If we define the complex numbers $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ by

$$\tilde{\alpha} = \frac{b}{c} e^{i\alpha} = \frac{\sin \beta}{\sin \gamma} (\cos \alpha + i \sin \alpha),$$

$$\tilde{\beta} = \frac{c}{a} e^{i\beta} = \frac{\sin \gamma}{\sin \alpha} (\cos \beta + i \sin \beta),$$

$$\tilde{\gamma} = \frac{a}{b} e^{i\gamma} = \frac{\sin \alpha}{\sin \beta} (\cos \gamma + i \sin \gamma),$$

then we have the identities

$$\tilde{\beta} = \frac{1}{1-\tilde{\alpha}}, \qquad \tilde{\gamma} = \frac{1}{1-\tilde{\beta}}, \qquad \tilde{\alpha} = \frac{1}{1-\tilde{\gamma}},$$

$$(0) \quad \tilde{\alpha} = 1 - \frac{1}{\tilde{\beta}}, \qquad \tilde{\beta} = 1 - \frac{1}{\tilde{\gamma}}, \qquad \tilde{\gamma} = 1 - \frac{1}{\tilde{\alpha}},$$

$$\tilde{\alpha}\tilde{\beta}\tilde{\gamma} = -1.$$

(Only two among these equations are independent.)
Moreover we have

$$\alpha = \arg \tilde{\alpha}, \qquad \beta = \arg \tilde{\beta}, \qquad \gamma = \arg \tilde{\gamma}.$$

Thus the similarity type of a triangle is determined by any one of $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$.

Now returning to the case of the nice triangulation of 5_2-knot complement we let the dihedral angles of the three ideal 3-simplexes as in Figure 24, where $\alpha_i + \beta_i + \gamma_i = \pi$ $(i=1,2,3)$.
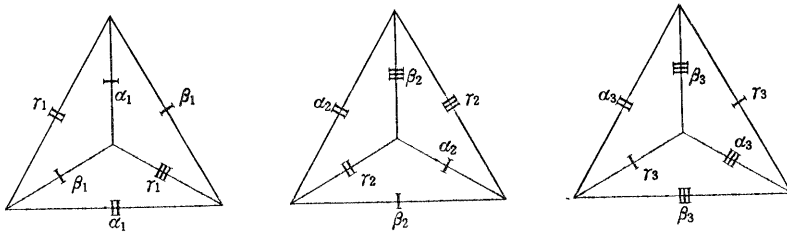

Fig. 24.

By the glueing identification the edges are divided into three groups $+$, $\neq$, $\equiv$. According to [3] Chapter 4, in order that these ideal simplexes make a hyperbolic

structure by glueing, the following equations must be satisfied.

$(0)_i$ (the equations (0) with suffix i, $(i=1,2,3)$.)

$(+)$ $\tilde{\alpha}_1\tilde{\beta}_1\tilde{\alpha}_2\tilde{\beta}_2\tilde{\gamma}_3^2=1$,

$(\mp)$ $\tilde{\alpha}_1\tilde{\gamma}_1\tilde{\alpha}_2\tilde{\gamma}_2\tilde{\alpha}_3=1$,

$(\equiv)$ $\tilde{\beta}_1\tilde{\gamma}_1\tilde{\beta}_2\tilde{\gamma}_2\tilde{\alpha}_3\tilde{\beta}_3^2=1$.

Under $(0)_i$, $(i=1,2,3)$, $(+)$, $(\mp)$, $(\equiv)$ are not independent, beause the product of these three equations is

$$\tilde{\alpha}_1^2\tilde{\beta}_1^2\tilde{\gamma}_1^2\tilde{\alpha}_2^2\tilde{\beta}_2^2\tilde{\gamma}_2^2\tilde{\alpha}_3^2\tilde{\beta}_3^2\tilde{\gamma}_3^2=1 ,$$

which is a consequence of $(0)_i$, $(i=1,2,3)$. Thus we have only to consider $(+)$ and $(\mp)$ besides $(0)_i$ $(i=1,2,3)$. By $(0)_i$, $(i=1,2,3)$, $(+)$, $(\mp)$ are equivalent to

$$\tilde{\gamma}_3^2=\tilde{\gamma}_1\tilde{\gamma}_2 , \qquad (1)$$
$$\tilde{\alpha}_3=\tilde{\beta}_1\tilde{\beta}_2 , \qquad (2)$$

If we glue 3 ideal simplexes which satisfy these equations together with

$$(*) \quad \begin{cases} \alpha_i+\beta_i+\gamma_i=\pi , \\ 0<\alpha_i,\beta_i,\gamma_i<\pi , \end{cases}$$

we obtain a hyperbolic structure of 5_2-knot complement. But it is not necessarily complete. As in [3], the completeness condition is obtained by the following developing map of the link of the vertex.
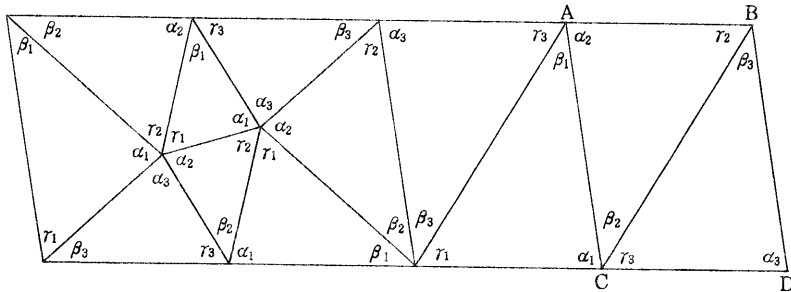

Fig. 25.

The completeness condition is $\overrightarrow{AB}=\overrightarrow{CD}$, that is, $\tilde{\gamma}_2^{-1}\tilde{\gamma}_3=1$ or

$$\tilde{\gamma}_2=\tilde{\gamma}_3 . \qquad (3)$$

We solve the simultaneous equations (0_i) $(i=1,2,3)$, (1), (2),' (3) to obtain the complete hyperbolic structure of 5_2-knot complement. First, from (1) and (3) we have $\tilde{\gamma}_3=\tilde{\gamma}_1$. Thus,

$$\tilde{\gamma}_1=\tilde{\gamma}_2=\tilde{\gamma}_3 \ (=\tilde{\gamma}, \text{ say}) .$$

Therefore, by $(0)_i$ $(i=1, 2, 3)$ we have

$$\tilde{\alpha}_1 = \tilde{\alpha}_2 = \tilde{\alpha}_3 \ \ (= \tilde{\alpha}, \text{ say}),$$
$$\tilde{\beta}_1 = \tilde{\beta}_2 = \tilde{\beta}_3 \ \ (= \tilde{\beta}, \text{ say}).$$

Then, (2) becomes $\tilde{\alpha} = \tilde{\beta}^2$. Since $\tilde{\alpha} = 1 - 1/\tilde{\beta}$, we have $1 - 1/\tilde{\beta} = \tilde{\beta}^2$, or

$$\tilde{\beta}^3 - \tilde{\beta} + 1 = 0.$$

This cubic equation has one real root and two conjugate imaginary roots. We cannot obtain a hyperbolic structure from the real root. The conjugate imaginary roots corresponds essentially the same hyperbolic structure. So we only consider the case of the root with positive imaginary part:

$$\tilde{\beta} \doteqdot 0.66235898 + 0.56227951i.$$

Then,

$$\tilde{\alpha} \doteqdot 0.12256117 + 0.74486177i,$$
$$\tilde{\gamma} \doteqdot 0.78492015 + 1.30714128i,$$
$$\arg(\tilde{\alpha}) = \alpha_1 = \alpha_2 = \alpha^3 \doteqdot 80.656154°,$$
$$\arg(\tilde{\beta}) = \beta_1 = \beta_2 = \beta_3 \doteqdot 40.328077°,$$
$$\arg(\tilde{\gamma}) = \gamma_1 = \gamma_2 = \gamma_3 \doteqdot 59.015770°.$$

If we glue the three 3-simplexes with these dihedral angles in the specifien way, we finally obtain the complete hyperbolic structure of 5_2-knot complement. The developing map arround the cusp is illustrated in Figure 26.
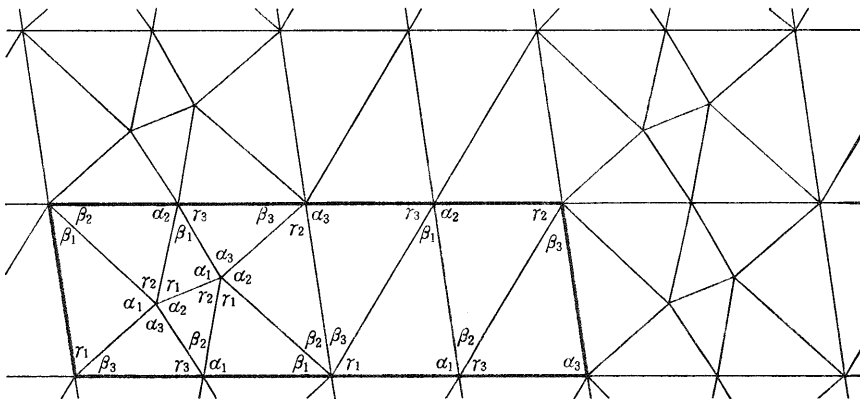

Fig. 26.

REMARK. There are some other known methds to obtain the complete hyperbolic structure of 5_2-knot complement.

1. By Thurston's general theorem (not yet published). See also Sullivan [4].)

2. (Hyperbolic) Dehn surgery along one component of Whitehead link.

3. Riley's parabolic representation ([1]).

§3. The construction of hyperbolic structures of the complements of some other knots.

In this section we present briefly the same results as in the preceding section for the three knots 6_1, 6_2, 6_2.

6_1 - knot

(Stevedores knot)
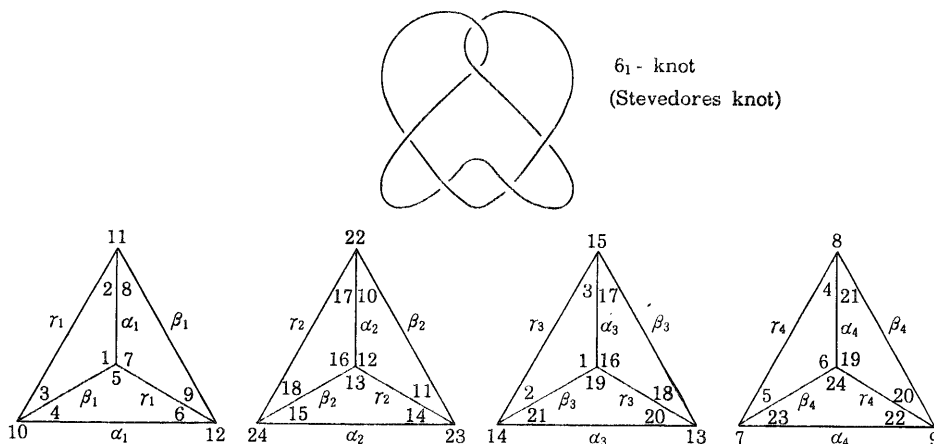
A nice triangulation of 6_1 - knot complement

Fig. 27.

Equations:
$$\tilde{\beta}_1 \tilde{\gamma}_2 \tilde{\alpha}_3 \tilde{\beta}_4 = 1, \qquad \tilde{\gamma}_1^2 \tilde{\alpha}_2 \tilde{\beta}_2 \tilde{\gamma}_3 \tilde{\alpha}_4 \tilde{\beta}_4 = 1,$$
$$\tilde{\alpha}_1^2 \tilde{\beta}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\beta}_3 \tilde{\alpha}_4 \tilde{\gamma}_4 = 1, \qquad \tilde{\beta}_2 \tilde{\gamma}_2 \tilde{\beta}_3 \tilde{\gamma}_3 \tilde{\gamma}_4 = 1.$$

The completeness condition: $\tilde{\gamma}_1 = \tilde{\gamma}_4$.

$\tilde{\beta}_1$ is a solution of the algebraic equation

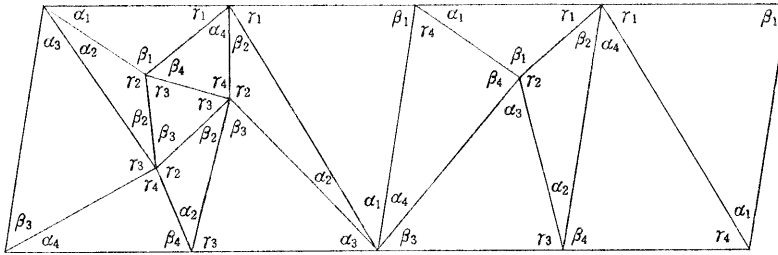$$\tilde{\beta}_1^4 - \tilde{\beta}_1^3 + 3\tilde{\beta}_1^2 - 2\tilde{\beta}_1 + 1 = 0.$$

This equation has the following 4 roots:

$$0.10486618 \pm 1.5524918i,$$
$$0.395124 \pm 0.5068431i,$$

among which the desired (or excellent) solution is

$$\tilde{\beta}_1 \doteqdot 0.10486618 + 1.5524918i.$$
$$\alpha_1 = \alpha_4 \doteqdot 33.8312°,$$
$$\beta_1 = \beta_4 \doteqdot 86.1353°,$$
$$\gamma_1 = \gamma_4 \doteqdot 60.0335°,$$
$$\alpha_2 \doteqdot 13.1014°,$$

$$\beta_2 \doteqdot 26.1020° ,$$
$$\gamma_2 \doteqdot 140.7966° ,$$
$$\alpha_3 \doteqdot 46.9328° ,$$
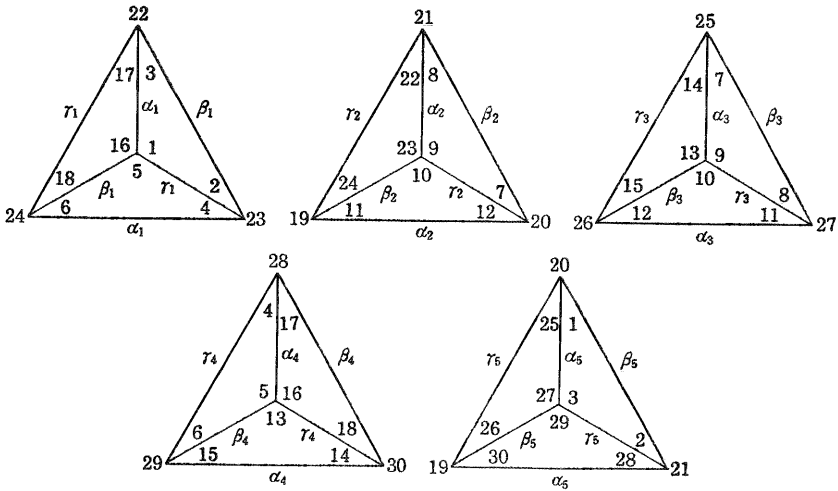$$\beta_3 \doteqdot 52.3041° ,$$
$$\gamma_3 \doteqdot 80.7631° .$$

Developing map arround the cusp

Fig. 28.

2. 6_2-knot complement.
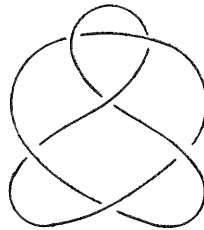
6_2- knot

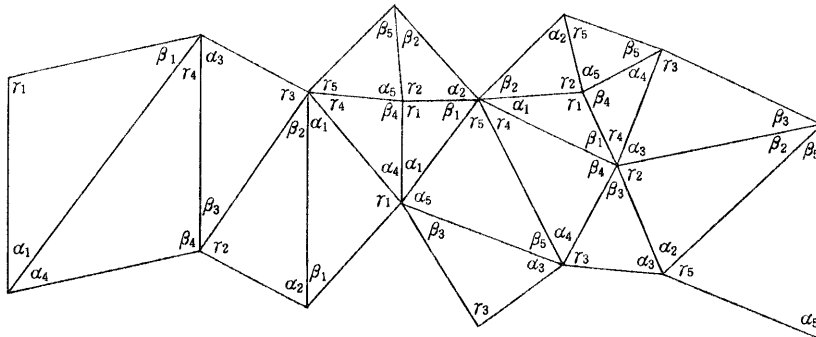A nice triangulation of 6_2- knot complement

Fig. 29.

Equations:

$$\tilde{\alpha}_1\tilde{\gamma}_1\tilde{\beta}_2\tilde{\beta}_3\tilde{\alpha}_4\tilde{\alpha}_5\tilde{\beta}_5=1\ ,$$

$$\tilde{\beta}_1\tilde{\gamma}_2\tilde{\alpha}_3\tilde{\beta}_3\tilde{\beta}_4\tilde{\gamma}_4=1\ ,$$

$$\tilde{\alpha}_1\tilde{\beta}_1\tilde{\alpha}_2\tilde{\beta}_2\tilde{\gamma}_3\tilde{\gamma}_4\tilde{\gamma}_5=1\ ,$$

$$\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\beta}_4\tilde{\gamma}_5=1\ ,$$

$$\tilde{\alpha}_2\tilde{\alpha}_3\tilde{\gamma}_3\tilde{\alpha}_4\tilde{\beta}_5\tilde{\gamma}_5=1\ .$$

The completeness condition: $\tilde{\beta}_1=\tilde{\alpha}_4$.

$$\tilde{\gamma}_1^5-2\tilde{\gamma}_1^4+3\tilde{\gamma}_1^3-2\tilde{\gamma}_1^2+2\tilde{\gamma}_1-1=0\ .$$

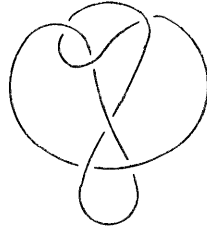The excellent solution: $\tilde{\gamma}_1\fallingdotseq-0.18607840+0.87464646i$.

$\alpha_1=\gamma_4\fallingdotseq36.406072°$,

$\beta_1=\alpha_4\fallingdotseq41.583489°$,

$\gamma_1=\beta_4\fallingdotseq102.01044°$,

$\alpha_2=\alpha_3\fallingdotseq60.426950°$,

$\beta_2=\beta_3\fallingdotseq32.802138°$,

$\gamma_2=\gamma_3\fallingdotseq86.770867°$,

$\alpha_5\fallingdotseq69.208256°$,

$\beta_5\fallingdotseq45.187378°$,

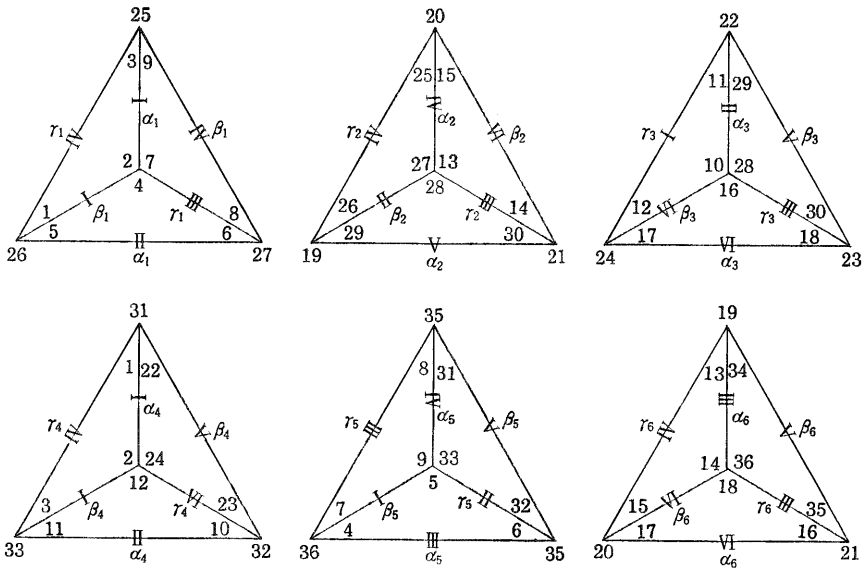$\gamma_5\fallingdotseq65.604367°$,

Developing map arround the cusp

Fig. 30.

3. 6_3-knot complement.

6_3- knot

A nice triangulation of 6_3- knot complement

Fig. 31.
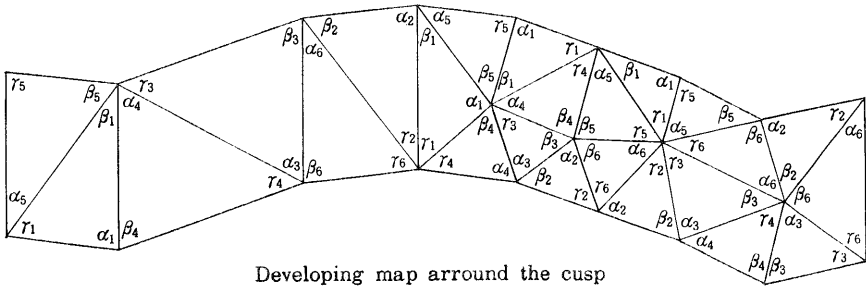
Equations :

$$\tilde{\alpha}_1\tilde{\beta}_1\tilde{\gamma}_8\tilde{\alpha}_4\tilde{\beta}_4\tilde{\beta}_5=1\,,$$

$$\tilde{\alpha}_1\tilde{\beta}_2\tilde{\alpha}_3\tilde{\alpha}_4\tilde{\gamma}_5=1\,,$$

$$\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3\tilde{\alpha}_5\tilde{\gamma}_5\tilde{\alpha}_6\tilde{\gamma}_6=1\,,$$

$$\tilde{\beta}_1\tilde{\gamma}_1\tilde{\alpha}_2\tilde{\gamma}_2\tilde{\gamma}_4\tilde{\alpha}_5\tilde{\gamma}_6=1\,,$$

$$\tilde{\alpha}_2\tilde{\beta}_3\tilde{\beta}_4\tilde{\beta}_5\tilde{\beta}_6=1\,,$$

$$\tilde{\beta}_2\tilde{\alpha}_3\tilde{\beta}_3\tilde{\gamma}_4\tilde{\alpha}_6\tilde{\beta}_6=1\,.$$

The completeness condition : $\tilde{\beta}_2=\tilde{\gamma}_6\,.$

The excellent solution : $\tilde{\alpha}_1\doteqdot0.15993610+1.2001426i\,.$

$$\alpha_1=\alpha_2=\gamma_5=\beta_6\doteqdot82.460850°\,,$$

$$\beta_1=\gamma_2=\alpha_5=\alpha_6\doteqdot42.565249°\,,$$

$$\gamma_1 = \beta_2 = \beta_5 = \gamma_6 \fallingdotseq 54.973901° \, ,$$
$$\alpha_3 = \beta_3 = \alpha_4 = \beta_4 \fallingdotseq 70.052199° \, ,$$
$$\gamma_3 = \gamma_4 \fallingdotseq 39.895602° \, .$$

Developing map arround the cusp

Fig. 32.

§ 4. The construction of hyperbolic structures from Heegaad diagrams.

In this section we shall investigate a method of construction of hyperbolic structure from a given Heegaad diagram of a 3-manifold.

Let V be a solid torus of genus 2. Let us consider the following loop l on the boundary of V, illustrated in the Figure 33.

V
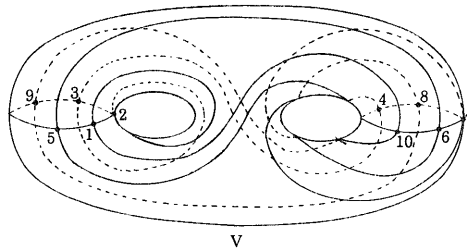
Fig. 33.

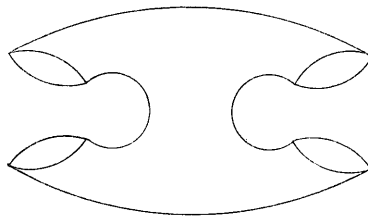We can exhibit this loop as a graph, as follows. That is, we first cut V as in the Figure 34.

Fig. 34.

Then we obtain a 3-disk D^3 and a graph on ∂D^3, illustrated in the Figure 35.
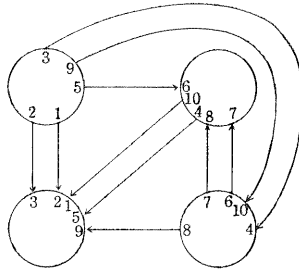

Fig. 35.

Let N be the 3-manifold obtained from V by attaching a 2-handle along l.[*] ∂N is a torus. From the Figure 35 we see that

$$\pi_1(N) \cong \langle a, b \,|\, a^3 bab^{-3} ab = 1 \rangle .$$

We shall construct a complete hyperbolic structure of $\mathring{N} = N - \partial N$ with finite volume.

First we shall find a nice triangulation of N as follows. Consider the dual graph of the Figure 35. (See Figure 36.)
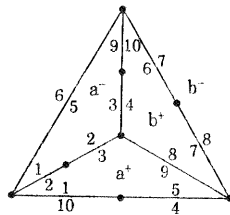

Fig. 36.

We think of this graph as drawn on ∂D^3. If we glue the faces a^+ and a^- so that the edges with the same label coincide, and glue the faces b^+ and b^- similarly, and then delete all the vertices (denoted by $\cdot$ in the Figure 36), then we obtain an open 3-manifold N'. Now we shall show that N' is homeomorphic to $\mathring{N}$.

The loop l is divided into 10 arcs in the graph of Figure 35. Corresponding to these arcs, we devide the 2-handle to be attached to V into 10 "thickened fans". (See Figure 37.)
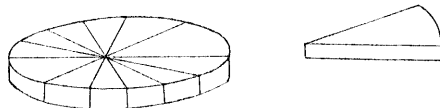

Fig. 37.

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[*] This manifold was considered in [2].

If we glue these fans along the corresponding arcs of the graph of the Figure 35, then we obtain the Figure 38.
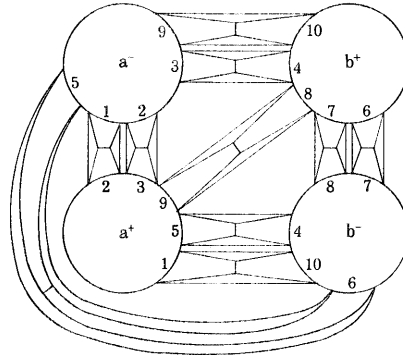

Fig. 38.

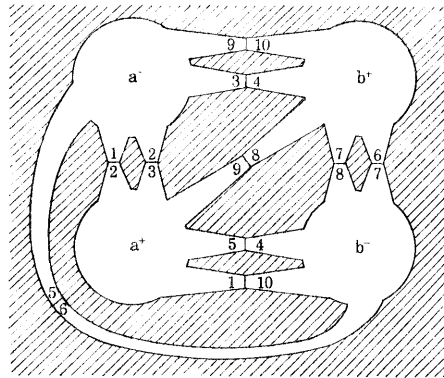In the Figure 39 the shaded area is those parts which are not glued and remain as the boundary.

Fig. 39.

Each component of the shaded area can be contracted to one point. After the contraction we obtain the graph of the Figure 36. This proves that N' and $\mathring{N}$ are homeomorphic. We construct a nice triangulation of N' and hence of N from the graph of Figure 36. We wish to find the simplest possible one. For this, we first construct a triangulation of ∂D^3 by adding some edges to the graph of Figure 36 but without adding any vertices. The Figure 40 shows an example. Note that the added edges are glued together pairwise by the glueing of the faces.
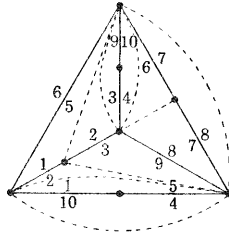
Fig. 40.

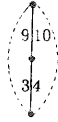Consider the part shown in the Figure 41.

Fig. 41.

If we span a 2-disk in D^3 with the dotted curve as boundary and cut D^3 along it, then we obtain the thing shown in the leftmost of the Figure 42, and this can be pushed down as shown in Figure 42.
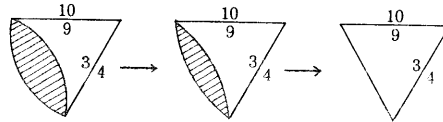

Fig. 42.

A similar process can be performed for the part shown in the Figure 43.

Fig. 43.

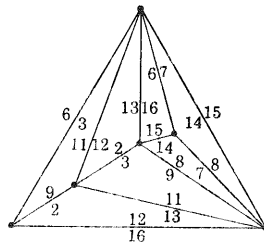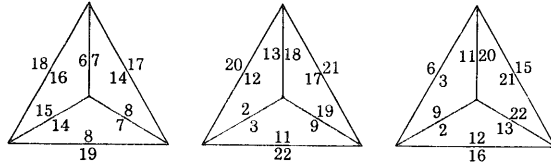After these reductions the triangulation of ∂D^3 and their glueing map become as in the Figure 44.

Fig. 44.

From this we can finally obtain a nice triangulation with three 3-simplexes. (Figure 45.)

A nice triangulation of N.

Fig. 45.

Now that we obtain a nice triangulation of N, we can construct a hyperbolic structure of $\mathring{N}$ by the same method as in §2, due to Thurston.

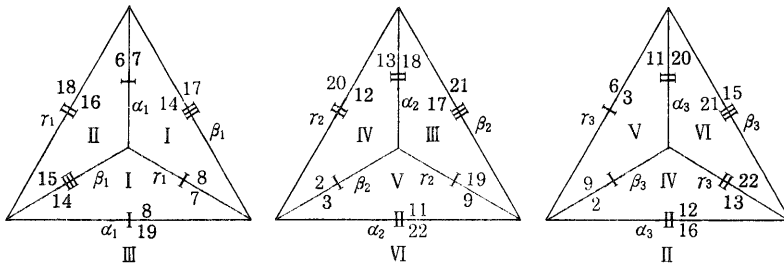That is, we construe the three simplexes as ideal simplexes with dihedral angles as shown in the Figure 46.

Fig. 46.

Then we obtain the following equations:

$$(+) \quad \tilde{\alpha}_1\tilde{\gamma}_1\tilde{\alpha}_1\tilde{\gamma}_2\tilde{\beta}_3\tilde{\beta}_2\tilde{\gamma}_3 = 1,$$
$$(\mp) \quad \tilde{\gamma}_1\tilde{\alpha}_2\tilde{\gamma}_3\tilde{\alpha}_2\tilde{\alpha}_3\tilde{\gamma}_2\tilde{\alpha}_3 = 1,$$
$$(\equiv) \quad \tilde{\beta}_1\tilde{\beta}_1\tilde{\beta}_2\tilde{\beta}_3 = 1,$$

or equivalently,

$$\tilde{\alpha}_1^2\tilde{\gamma}_1 = \tilde{\alpha}_2\tilde{\alpha}_3, \qquad (1)$$
$$\tilde{\gamma}_1\tilde{\alpha}_2^2\tilde{\gamma}_2\tilde{\alpha}_3^2\tilde{\gamma}_3 = 1, \qquad (2)$$
$$\tilde{\beta}_1^2\tilde{\beta}_2\tilde{\beta}_3 = 1. \qquad (3)$$

Since one of them is redundant we use (1) and (3). The completeness condition is $\tilde{\alpha}_2 = \tilde{\alpha}_3$. Hence we have

$$\tilde{\beta}_2 = \tilde{\beta}_3, \qquad \tilde{\gamma}_2 = \tilde{\gamma}_3.$$

Then (1) and (3) become

$$\tilde{\alpha}_1^2 \tilde{\gamma}_1 = \tilde{\alpha}_2^2, \qquad (1)'$$

$$\tilde{\beta}_1^2 \tilde{\beta}_2^2 = 1. \qquad (3)'$$

From (3)' we have $\tilde{\beta}_1 \tilde{\beta}_2 = \pm 1$. However, in order to obtain a hyperbolic structure we reguire a solution with

$$2 \arg \tilde{\beta}_1 + 2 \arg \tilde{\beta}_2 = 2\pi,$$

or

$$\arg \tilde{\beta}_1 + \arg \tilde{\beta}_2 = \pi.$$

Thus we must have

$$\tilde{\beta}_1 \tilde{\beta}_2 = -1. \qquad (4)$$

REMARK. From a solution with $\tilde{\beta}_1 \tilde{\beta}_2 = 1$ we can not obtain a hyperbolic structure but can obtain a representation of $\pi_1(N)$. (See § 6.)

Now, from (4) we have

$$\tilde{\beta}_2 = -\frac{1}{\tilde{\beta}_1}$$

and

$$\tilde{\alpha}_2 = 1 - \frac{1}{\tilde{\beta}_2} = 1 + \tilde{\beta}_1.$$

Moreover

$$\tilde{\alpha}_1 = -\frac{1}{\tilde{\beta}_1}, \qquad \tilde{\gamma}_1 = \frac{1}{1 - \tilde{\beta}_1}.$$

Hence from (1)', we have

$$\left(1 - \frac{1}{\tilde{\beta}_1}\right)^2 \frac{1}{1 - \tilde{\beta}_1} = (1 + \tilde{\beta}_1)^2$$

or

$$\tilde{\beta}_1^4 + 2\tilde{\beta}_1^3 + \tilde{\beta}_1^2 + \tilde{\beta}_1 - 1 = 0. \qquad (5)$$

The excellent solution is

$$\tilde{\beta}_1 \doteqdot -0.29342445 \pm 1.0014412i.$$

Thus

$$\beta_1 \doteqdot 106.33070°,$$
$$\gamma_1 \doteqdot 37.749030°,$$
$$\alpha_1 \doteqdot 35.920271°,$$

$$\beta_2 = \beta_3 \doteqdot 73.669301° \,,$$
$$\alpha_2 = \alpha_3 \doteqdot 54.794786° \,,$$
$$\gamma_2 = \gamma_3 \doteqdot 51.535914° \,.$$

We obtain a complete hyperbolic structure of $\mathring{N}$ with the following developing map arround the cusp.
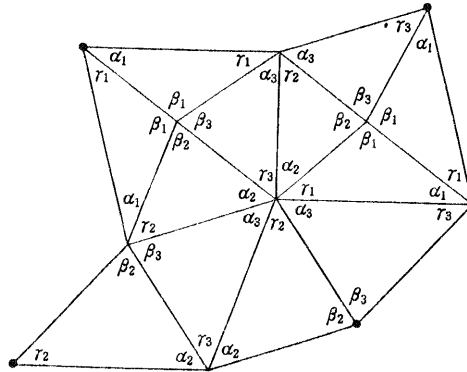

Fig. 47.

§ 5. Some more examples of the construction by the method of § 4.

In this section we present ten more examples of the construction by the method of § 4. As in § 3, we omit the process of construction. One example has a nice triangulation with two 3-simplexes, just like the figure eight knot complement, but the glueing is different. These two manifolds are distinguished by the first homology group.

Each of the other nine examples has a nice triangulation with three 3-simplexes. Some of these examples have the same types of the ideal 3-simplexes and, in particular, have the same volume.

In showing examples we only exhibit presentations of the fundamental groups (with two generators and one relator) rather than Heegaad diagrams, because they can easily be constructed from the presentations.

EXAMPLE 1.
$$\pi_1(M_1) \cong \langle a, b \,|\, a^2bab^{-2}ab = 1 \rangle$$
$$\cong \langle c, d \,|\, cd^2c^2dc^2d^2 = 1 \rangle \,.$$
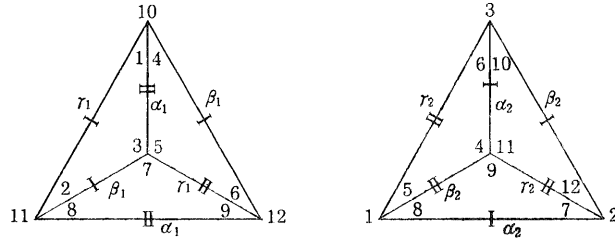$$(H_1(M_1) \cong \mathbf{Z} \times \mathbf{Z}_5.)$$

Fig. 48.

$$\alpha_1 = \beta_1 = \gamma_1 = \alpha_2 = \beta_2 = \gamma_2 = 60° ,$$
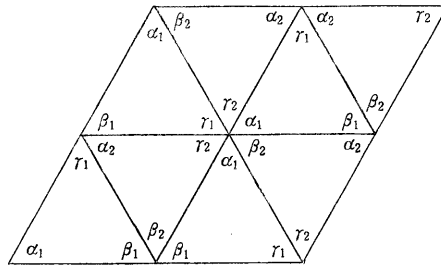$$\text{volume} \fallingdotseq 2.02988 .$$

Fig. 49.

REMARK.   In contrast to this example, a presentation of the fundamental group of the figure eight knot complement is

$$\langle a, b \,|\, a^{-3}bab^{-2}ab = 1 \rangle .$$

EXAMPLE 2. $\pi_1(M_2) \cong \langle a, b \,|\, a^{-2}b^2ab^{-1}ab^2 = 1 \rangle .$
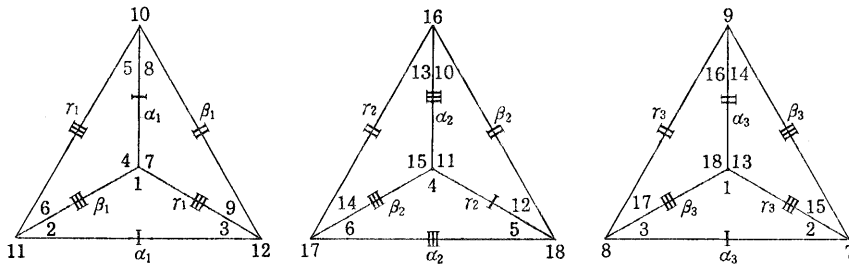$$(H_1(M_2) \cong \mathbf{Z} \times \mathbf{Z}_3.)$$

Fig. 50.

$$\tilde{\alpha}_1^2 \tilde{\gamma}_2 \tilde{\alpha}_3 = 1 ,$$
$$\tilde{\beta}_1 \tilde{\beta}_2 \tilde{\gamma}_2 \tilde{\alpha}_3 \tilde{\gamma}_3 = 1 ,$$
$$\tilde{\beta}_2 \tilde{\gamma}_1^2 \tilde{\alpha}_2^2 \tilde{\beta}_2 \tilde{\beta}_3^2 \tilde{\gamma}_3 = 1 .$$

The completeness condition : $\tilde{\alpha}_2 = \tilde{\beta}_3$.

$$2\tilde{\alpha}_2^3 - \tilde{\alpha}_2^2 - \tilde{\alpha}_2 + 1 = 0 .$$
$$\tilde{\alpha}_2 \fallingdotseq 0.6647418 + 0.4011273i .$$
$$\alpha_1 \fallingdotseq 81.219632° ,$$
$$\beta_1 \fallingdotseq 62.216354° ,$$
$$\gamma_1 \fallingdotseq 36.564014° ,$$
$$\alpha_2 = \beta_3 \fallingdotseq 31.108177° ,$$
$$\beta_2 = \gamma_3 \fallingdotseq 50.111455° ,$$
$$\gamma_2 = \alpha_3 \fallingdotseq 98.780368° ,$$
$$\text{volume} \fallingdotseq 2.56897 .$$
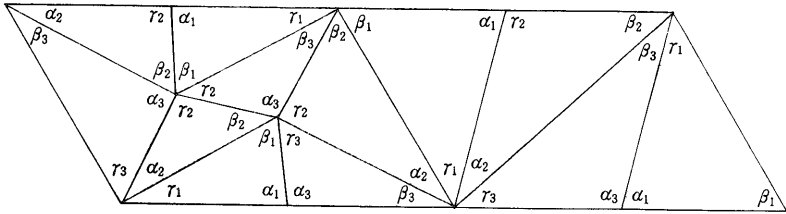
Fig. 51.

EXAMPLE 3. $\pi_1(M_3) \cong \langle a, b \, | \, a^{-2}b^2abab^2 = 1 \rangle$
$$\cong \langle c, d \, | \, c^3dcd^3cd = 1 \rangle .$$
$$(H_1(M_3) \cong \mathbf{Z} \times \mathbf{Z}_5.)$$
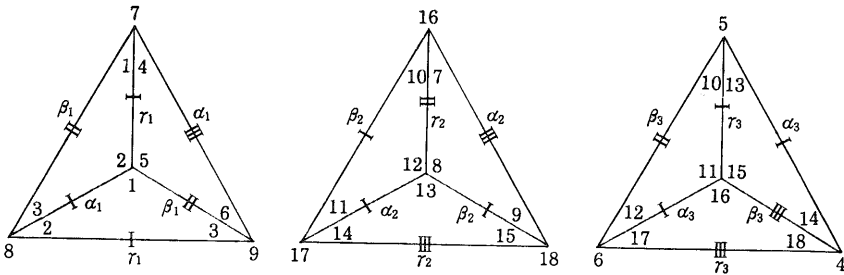

Fig. 52.

$$\tilde{\alpha}_1 \tilde{\gamma}_1^2 \tilde{\alpha}_2 \tilde{\beta}_2^2 \tilde{\alpha}_3^2 \tilde{\gamma}_3 = 1 ,$$
$$\tilde{\beta}_1^2 \tilde{\gamma}_2 \tilde{\beta}_3 = 1 ,$$
$$\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\gamma}_2 \tilde{\beta}_3 \tilde{\gamma}_3 = 1 .$$

The completeness condition : $\tilde{\beta}_2 = \tilde{\alpha}_3$

$$\tilde{\beta}_2^3 - \tilde{\beta}_2^2 - \tilde{\beta}_2 + 2 = 0 .$$
$$\tilde{\beta}_2 \fallingdotseq 1.1027847 + 0.6654570i .$$

$$\alpha_1 \doteqdot 62.216354°,$$
$$\beta_1 \doteqdot 81.219632°,$$
$$\gamma_1 \doteqdot 36.564014°,$$
$$\alpha_2 = \gamma_3 \doteqdot 50.111455°,$$
$$\beta_2 = \alpha_3 \doteqdot 31.108177°,$$
$$\gamma_2 = \beta_3 \doteqdot 98.780368°,$$
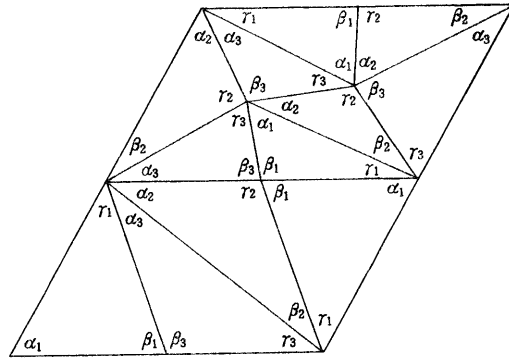$$\text{volume} \doteqdot 2.56897.$$

Fig. 53.

EXAMPLE 4. $\pi_1(M_4) \cong \langle a, b \,|\, a^4bab^{-2}ab=1\rangle$.
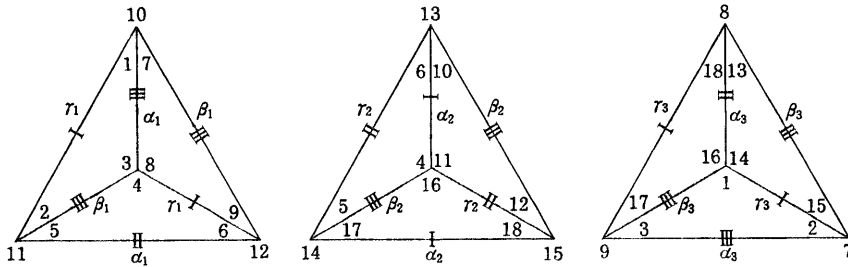
$(H_1(M_4) \cong \mathbf{Z} \times \mathbf{Z}_6.)$

Fig. 54.

$$\tilde{\gamma}_1^2 \tilde{\alpha}_2^2 \tilde{\gamma}_3^2 = 1,$$
$$\tilde{\alpha}_1 \tilde{\gamma}_2^2 \tilde{\alpha}_3 = 1,$$
$$\tilde{\alpha}_1 \tilde{\beta}_1^2 \tilde{\beta}_2^2 \tilde{\alpha}_3 \tilde{\beta}_3^2 = 1.$$

The completeness condition: $\tilde{\gamma}_1 = \tilde{\beta}_2 \tilde{\beta}_3$.

$$\tilde{\gamma}_1^2 - \tilde{\gamma}_1 + 2 = 0.$$
$$\tilde{\gamma}_1 = \frac{1 + \sqrt{-7}}{2}.$$

$$\alpha_1 = \gamma_1 = \alpha_3 = \gamma_3 \doteqdot 69.295189°,$$
$$\beta_1 = \alpha_2 = \beta_3 \doteqdot 41.409622°,$$
$$\beta_2 \doteqdot 27.885567°,$$
$$\gamma_2 \doteqdot 110.704811°,$$
$$\text{volume} \doteqdot 2.66675.$$

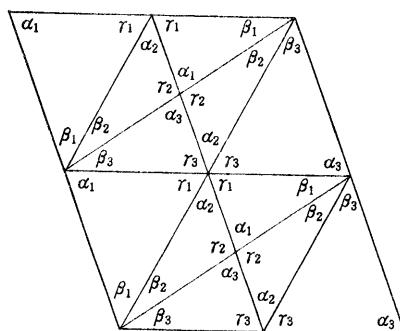

Fig. 55.

EXAMPLE 5. $\pi_1(M_5) \cong \langle a, b \mid ab^3a^2ba^2b^3 = 1 \rangle$

$$\cong \langle c, d \mid c^3d^3cd^2cd^3 = 1 \rangle.$$

$(H_1(M_5) \cong \mathbf{Z}.)$

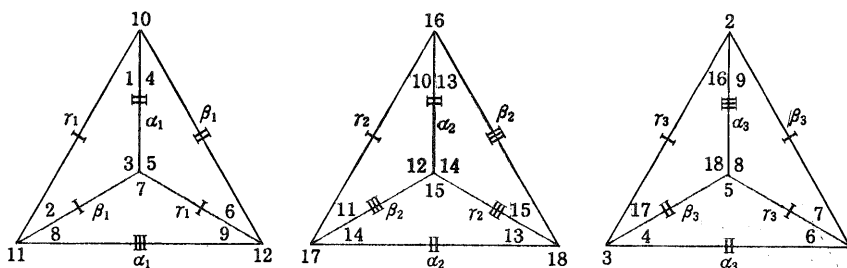Fig. 56.

$$\tilde{\beta}_1\tilde{\gamma}_1^2\tilde{\gamma}_2\tilde{\beta}_3\tilde{\gamma}_3^2 = 1,$$
$$\tilde{\alpha}_1\tilde{\beta}_1\tilde{\alpha}_2^2\tilde{\alpha}_3\tilde{\beta}_3 = 1,$$
$$\tilde{\alpha}_1\tilde{\beta}_2^2\tilde{\gamma}_2\tilde{\alpha}_3 = 1.$$

The completeness condition: $\tilde{\alpha}_1 = \tilde{\alpha}_3$.

$$\tilde{\alpha}_1^4 - \tilde{\alpha}_1^3 - 1 = 0.$$
$$\tilde{\alpha}_1 \doteqdot 0.219447 + 0.914474i.$$
$$\alpha_1 = \beta_2 = \alpha_3 \doteqdot 76.505819°,$$
$$\beta_1 = \gamma_2 = \beta_3 \doteqdot 53.976723°,$$
$$\gamma_1 = \alpha_2 = \gamma_3 = 49.517458°,$$

$$\text{volume} \doteq 2.9441 \, .$$
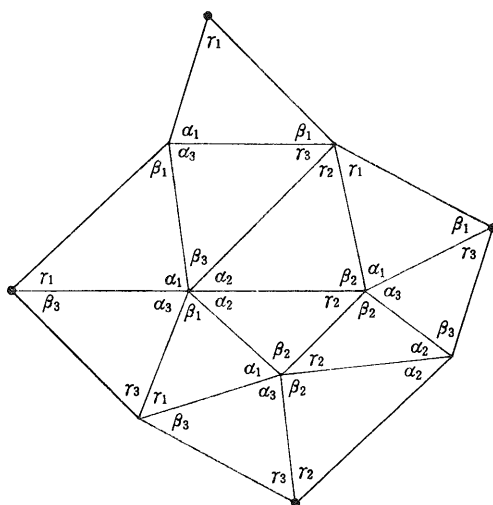
Fig. 57.

EXAMPLE 6. $\pi_1(M_6) \cong \langle a, b \,|\, ab^3 a^2 b^2 a^2 b^3 = 1 \rangle$

$$\cong \langle c, d \,|\, c^3 d^3 cdcd^3 = 1 \rangle \, .$$
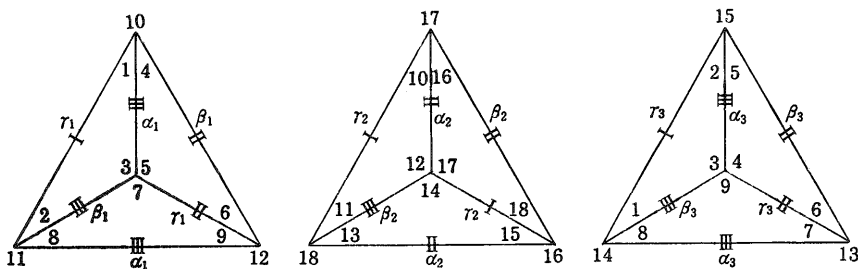
$(H_1(M_6) \cong \boldsymbol{Z}.)$

Fig. 58.

$$\tilde{\gamma}_1 \tilde{\gamma}_2^2 \tilde{\gamma}_3 = 1 \, ,$$

$$\tilde{\beta}_1 \tilde{\gamma}_1 \tilde{\alpha}_2^2 \tilde{\beta}_2 \tilde{\beta}_3 \tilde{\gamma}_3 = 1 \, ,$$

$$\tilde{\alpha}_1^2 \tilde{\beta}_1 \tilde{\beta}_2 \tilde{\alpha}_3^2 \tilde{\beta}_3 = 1 \, .$$

The completeness condition: $\tilde{\alpha}_1 = \tilde{\alpha}_3 \, .$

$$\tilde{\gamma}_1^4 - 2\tilde{\gamma}_1^3 + \tilde{\gamma}_1^2 - \tilde{\gamma}_1 - 1 = 0 \, .$$

$$\tilde{\gamma}_1 \doteq 0.293424 + 1.001441i \, .$$

$$\gamma_1 = \gamma_3 \doteq 73.669300° \, ,$$

$$\alpha_1 = \alpha_3 \doteq 54.794785° \, ,$$

$$\beta_1 = \beta_3 \fallingdotseq 51.535914° \, ,$$
$$\alpha_2 \fallingdotseq 35.920271° \, ,$$
$$\beta_2 \fallingdotseq 37.749030° \, ,$$
$$\gamma_2 \fallingdotseq 106.330699° \, ,$$
$$\text{volume} \fallingdotseq 2.78183 \, .$$
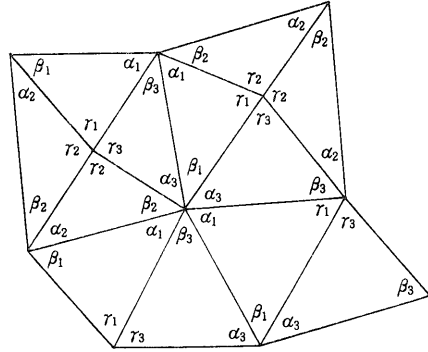

Fig. 59.

EXAMPLE 7. $\pi_1(M_7) \cong \langle a, b | a^{-4}bab^{-2}ab = 1 \rangle$.
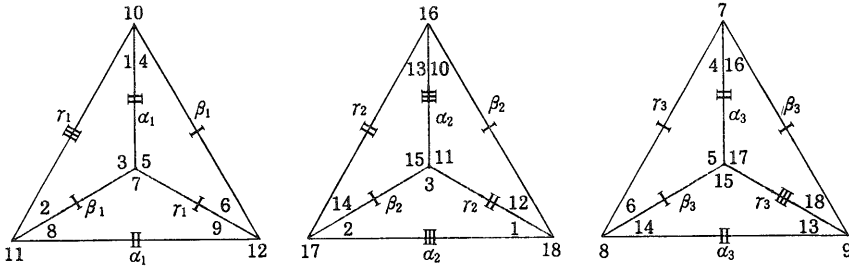
$(H_1(M_7) \cong \mathbf{Z} \times \mathbf{Z}_2.)$

Fig. 60.

$$\tilde{\beta}_1^2 \tilde{\gamma}_1 \tilde{\beta}_2^2 \tilde{\beta}_3^2 \tilde{\gamma}_3 = 1 \, ,$$
$$\tilde{\alpha}_1^2 \tilde{\gamma}_2^2 \tilde{\alpha}_3^2 = 1 \, ,$$
$$\tilde{\gamma}_1 \tilde{\alpha}_2^2 \tilde{\gamma}_3 = 1 \, .$$

The completeness condition: $\tilde{\alpha}_1 = \tilde{\alpha}_3$.

$$2\tilde{\alpha}_1^2 - \tilde{\alpha}_1 + 1 = 0 \, .$$
$$\tilde{\alpha}_1 = \frac{1 + \sqrt{-7}}{4} \, .$$
$$\alpha_1 = \gamma_1 = \alpha_3 = \gamma_3 \fallingdotseq 69.295189° \, ,$$
$$\beta_1 = \gamma_2 = \beta_3 \fallingdotseq 41.409622° \, ,$$

$$\alpha_2 \fallingdotseq 110.704911° \, ,$$
$$\beta_2 \fallingdotseq 27.885567° \, ,$$
$$\text{volume} \fallingdotseq 2.66675 \, .$$
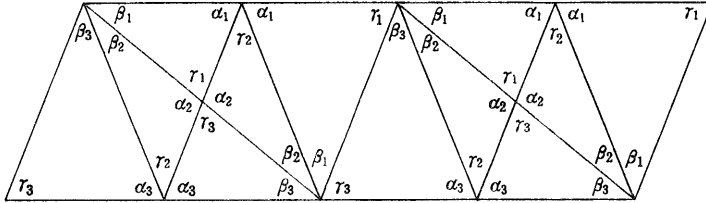
Fig. 61.

EXAMPLE 8. $\pi_1(M_8) \cong \langle a, b \,|\, a^3 b a^2 b^{-2} a^2 b = 1 \rangle \, .$
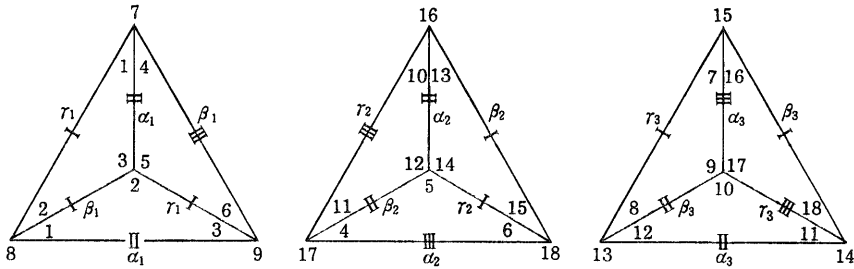
$(H_1(M_8) \cong \boldsymbol{Z} \times \boldsymbol{Z}_7.)$

Fig. 62.

$$\tilde{\beta}_1 \tilde{\gamma}_1^2 \tilde{\beta}_2 \tilde{\gamma}_2 \tilde{\beta}_3 \tilde{\gamma}_3 = 1 \, ,$$
$$\tilde{\alpha}_1^2 \tilde{\alpha}_2 \tilde{\beta}_2 \tilde{\alpha}_3 \tilde{\beta}_3 = 1 \, ,$$
$$\tilde{\beta}_1 \tilde{\alpha}_2 \tilde{\gamma}_2 \tilde{\alpha}_3 \tilde{\gamma}_3 = 1 \, .$$

The completeness condition: $\tilde{\alpha}_2 = \tilde{\alpha}_3 \, .$

$$\tilde{\beta}_2^3 - \tilde{\beta}_2 + 1 = 0 \, .$$
$$\tilde{\beta}_2 \fallingdotseq 0.662359 + 0.562280i \, .$$
$$\alpha_1 = \gamma_2 = \gamma_3 \fallingdotseq 59.015770° \, ,$$
$$\beta_1 = \alpha_2 = \alpha_3 \fallingdotseq 80.656154° \, ,$$
$$\gamma_1 = \beta_2 = \beta_3 \fallingdotseq 40.828076° \, ,$$
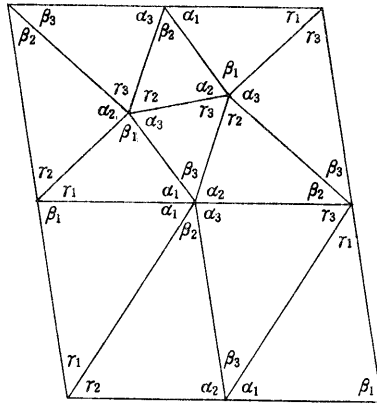$$\text{volume} \fallingdotseq 2.82812 \, .$$

Fig. 63.

EXAMPLE 9. $\pi_1(M_9) \cong \langle a, b \,|\, ab^{-2}a^2ba^2b^{-2} = 1 \rangle$.
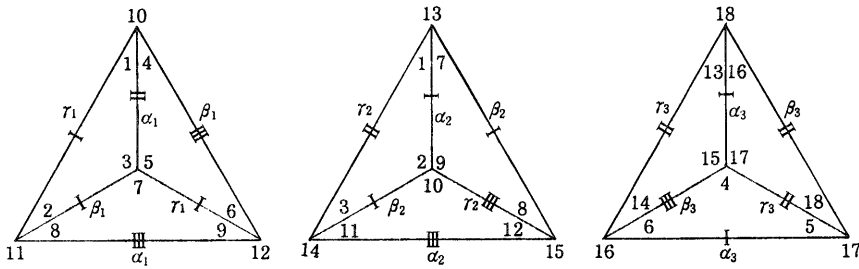
$(H_1(M_9) \cong \mathbf{Z}.)$

Fig. 64.

$$\tilde{\beta}_1 \tilde{\gamma}_1^2 \tilde{\alpha}_2 \tilde{\beta}_2^2 \tilde{\alpha}_3^2 = 1 ,$$

$$\tilde{\alpha}_1 \tilde{\gamma}_2 \tilde{\beta}_3 \tilde{\gamma}_3^2 = 1 ,$$

$$\tilde{\alpha}_1 \tilde{\beta}_1 \tilde{\alpha}_2 \tilde{\gamma}_2 \tilde{\beta}_3 = 1 .$$

The completeness condition: $\tilde{\beta}_1 \tilde{\beta}_2 \tilde{\alpha}_3 = \tilde{\gamma}_2 \tilde{\gamma}_3$.

$$\tilde{\beta}_2^3 - \tilde{\beta}_2^2 + 1 = 0 .$$

$\tilde{\beta}_2 \doteqdot 0.877439 + 0.744862 i$.

$\alpha_1 = \tilde{\gamma}_2 = \beta_3 \doteqdot 80.656154°$,

$\beta_1 = \alpha_2 = \gamma_3 \doteqdot 59.015770°$,

$\gamma_1 = \beta_2 = \alpha_3 \doteqdot 40.328076°$,
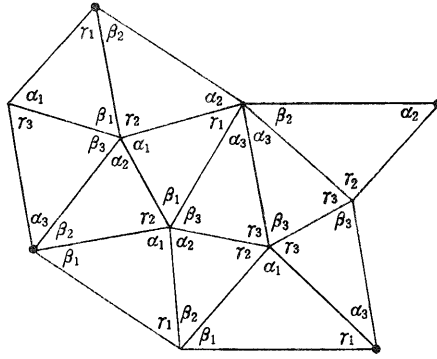
volume $\doteqdot 2.82812$.

Fig. 65.

REMARK. This manifold M_9 is not homeomorphic to the 5_2-knot complement. But they have the same homology group and the same volume.

EXAMPLE 10. $\pi_1(M_{10}) \cong \langle a, b \,|\, a^3bab^{-3}ab=1 \rangle$.

$(H_1(M_{10}) \cong \mathbf{Z}.)$
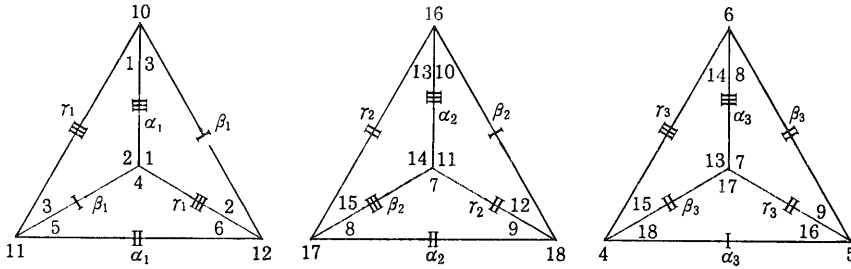

Fig. 66.

$$\tilde{\beta}_1^2 \tilde{\beta}_2 \tilde{\alpha}_3 = 1 ,$$
$$\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\gamma}_2^2 \tilde{\beta}_3^2 \tilde{\gamma}_3 = 1 ,$$
$$\tilde{\alpha}_1 \tilde{\gamma}_1^2 \tilde{\alpha}_2 \tilde{\beta}_2 \tilde{\alpha}_3 \tilde{\gamma}_3 = 1 .$$

The completeness condition: $\tilde{\beta}_2 = \tilde{\alpha}_3$.

$$\tilde{\beta}_1^4 - \tilde{\beta}_1^3 - \tilde{\beta}_1^2 - 2\tilde{\beta}_1 - 1 = 0 .$$
$$\tilde{\beta}_1 = -0.269448 + 0.919612i .$$
$$\alpha_1 \doteqdot 37.749030° ,$$
$$\beta_1 = 106.330700° ,$$
$$\gamma_1 = 35.920270° ,$$
$$\alpha_2 = \gamma_3 \doteqdot 51.535914° ,$$
$$\beta_2 = \alpha_3 \doteqdot 73.669300° ,$$

$$\gamma_2 = \beta_3 \doteqdot 54.794786°,$$
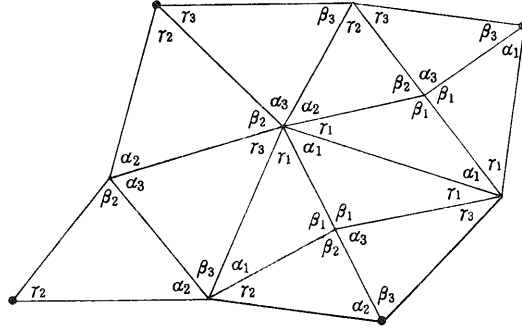$$\text{volume} \doteqdot 2.78183.$$

Fig. 67.

§ 6. Representations of $\pi_1(M)$.

In this section we shall explain how the representations to PSL$(2, \boldsymbol{C})$ of the fundamental group of a 3-manifold are constructed from its nice triangulation. We shall illustrate it for the nice triangulation of 5_2-knot complement treated in § 2. There we derived a system of equations which gives a necessary condition for the construction of hyperbolic structure:

$$(+) \quad \tilde{\alpha}_1 \tilde{\beta}_1 \tilde{\alpha}_2 \tilde{\beta}_2 \tilde{\gamma}_3^2 = 1,$$
$$(\mp) \quad \tilde{\alpha}_1 \tilde{\gamma}_1 \tilde{\alpha}_2 \tilde{\gamma}_2 \tilde{\alpha}_3 = 1,$$
$$(\equiv) \quad \tilde{\beta}_1 \tilde{\gamma}_1 \tilde{\beta}_2 \tilde{\gamma}_2 \tilde{\alpha}_3 \tilde{\beta}_3^2 = 1,$$

$$(0)_i \quad (i = 1, 2, 3)$$

$$\tilde{\beta}_i = 1/(1 - \tilde{\alpha}_i), \quad \tilde{\gamma}_i = 1/(1 - \tilde{\beta}_i), \quad \tilde{\alpha}_i = 1/(1 - \tilde{\gamma}_i),$$
$$\tilde{\alpha}_i = 1 - 1/\tilde{\beta}_i, \quad \tilde{\beta}_i = 1 - 1/\tilde{\gamma}_i, \quad \tilde{\gamma}_i = 1 - 1/\tilde{\alpha}_i,$$
$$\tilde{\alpha}_i \tilde{\beta}_i \tilde{\gamma}_i = -1.$$

We claim that, to each solution

$$r = ((\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1), (\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2), (\tilde{\alpha}_3, \tilde{\beta}_3, \tilde{\gamma}_3)) \qquad (\varDelta)$$

of the system of equations, there corresponds an equivalence class of representations of $\pi_1(M)$ to PSL$(2, \boldsymbol{C})$, where M is the complement of 5_2-knot.

Let $u \neq 0, 1, \infty$ be a complex number.

(i) If Im $(u) > 0$, then we say that u determines a "positively oriented ideal simplex". Indeed, u determines the ideal simplex in the upper half space over $\boldsymbol{C}$ (or in the projective model) illustrated in the Figure 68.
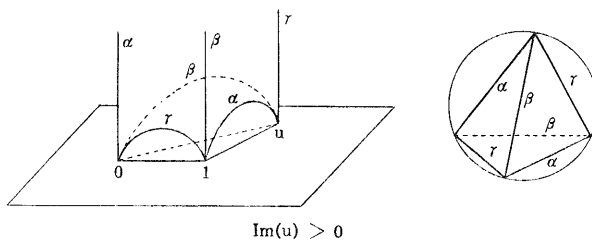
Im(u) > 0

Fig. 68.

The volume of this simplex is of course positive finite. We denote it by $v(u)$.

(ii) If Im(u)=0 (i. e., $u \in R$), then we say that u determines a "flattened ideal simplex", as illustrated in Figure 69.
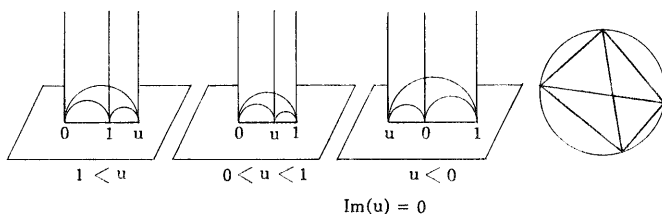

Im(u) = 0

Fig. 69.

It is natural to think that the volume of this flattened simplex is 0. So we put $v(u)$=0, in this case.

(iii) If Im(u)<0, then we say that u determines a "negatively oriented ideal simplex", as illustrated in Figure 70.
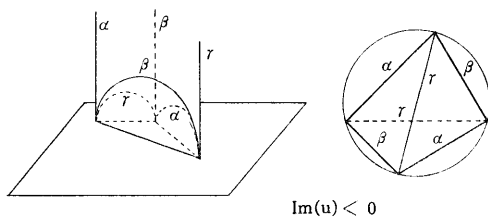

Im(u) < 0

Fig. 70.

The volume of this simplex is finite. However we think the volume to be negative, taking the minus sign to the real volume. Thus $v(u)$<0.

Putting (i), (ii), (iii) together, we say that $u \neq 0, 1, \infty$ determines a "nondegenerate ideal simplex".

REMARK. Obviously, $v(u)$ is a continuous function from $C-\{0,1\}$ to R. This

can be extended to a continuous function from C or even from $C \cup \{\infty\}$, by putting simply $u(0) = u(1) = u(\infty) = 0$. Moreover, if $u = x + yi (x, y \in \mathbf{R})$, then $f(x, y) = v(x + yi)$ is a real function of 2 variables.

Now consider the nice triangulation of 5_2-knot complement constructed in §2. If we glue the face (with the labelled vertices) 13-14-15, and then glue the face 16-17-18, we obtain a single polyhedron $\varLambda$ illustrated in the Figure 71.
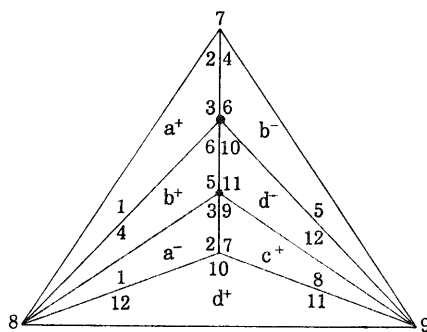
Polyhedron $\varLambda$

Fig. 71.

From this polyhedron $\varLambda$ we obtain a 3-manifold M^* homeomorphic to M, if we glue the faces a^+ and a^-, b^+ and b^-, c^+ and c^-, d^+ and d^-, respectively so that the vertices with the same label coincide and then remove the vertices. In M^* we take a base point O in the interior of $\varLambda$. Let A^+ and A^- are points in the faces a^+ and a^-, respectively, which coincide by the glueing of a^+ and a^-. Consider an arc l^+ connecting O and A^+ and an arc l^- connecting A^- and O, in the polyhedron. In M^*, $l^+ \cup l^-$ constitutes a closed loop a (with base point O). Similarly, loops b, c, d are defined. We also denote by a, b, c, d the corresponding elements of $\pi_1(M^*)$ or of $\pi_1(M)$. Then, it is not hard to see that $\pi_1(M)$ is generated by a, b, c, d and has the following relators (arround the edges $+$, $\doteqdot$, $\equiv$):

$$((+)) \quad acd^{-1}b^{-1}d = 1,$$
$$((\doteqdot)) \quad cd = 1,$$
$$((\equiv)) \quad a^{-1}b^{-1}cab = 1.$$

That is,

$$\pi_1(M) \cong \langle a, b, c, d \mid acd^{-1}b^{-1}d = cd = a^{-1}b^{-1}cab = 1 \rangle.$$

Suppose that a solution $r = ((\tilde{\alpha}_1, \tilde{\beta}_1, \tilde{\gamma}_1), (\tilde{\alpha}_2, \tilde{\beta}_2, \tilde{\gamma}_2), (\tilde{\alpha}_3, \tilde{\beta}_3, \tilde{\gamma}_3))$ of the equations under consideration is given. Then

$$\tilde{\alpha}_i, {}_i\tilde{\beta}, \tilde{\gamma}_i \neq 0, 1, \infty,$$

for if one of them is equal to 0, then the left-hand-side of $(+)$ or $(\mp)$ or $(\mp)$ would become 0, and if $\tilde{\alpha}_1=1$, for instance, then $\tilde{\gamma}_1=0$ by $(0)_1$. Since each triple $(\tilde{\alpha}_i, \beta_i, \tilde{\gamma}_i)$ satisfies the equations $(0)_i$, it determines a non-degenerated simplex in the hyperbolic 3-space as mentioned above. For $i=1,2,3$, we obtain 3 non-degenerated simplexes Δ_1, Δ_2, Δ_3. In the hyperbolic 3-space we glue the faces 13-14-15, and then the faces 16-17-18, in the following manner.

When a positively oriented simplex and a negatively oriented simplex are glued together, we glue these so that both are on the same side of the glueing surface. (See Figure 72.)

$\left(\begin{array}{l}\text{The shaded area is}\\\text{the glueing surface.}\end{array}\right)$
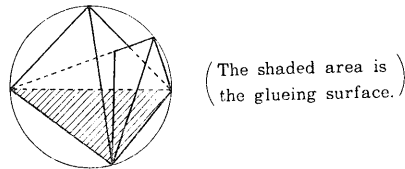
Fig. 72.

When two positively oriented simplexes (or two negatively oriented simplexes) are glued together, we glue these so that they are on the opposite side of the glueing surface, as usual. (See Figure 73.)
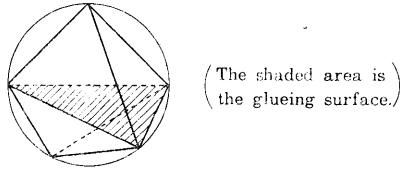

$\left(\begin{array}{l}\text{The shaded area is}\\\text{the glueing surface.}\end{array}\right)$

Fig. 73.

It would be needless to explain how to glue a flattened simplex. Thus the topological polyhedron Λ becomes a geometric polyhedron Λ^*, which may contain concave or crushed parts. Each face of Λ^* is a geometric (hyperbolic) ideal triangle. For instance a^- and a^+ with the labelled vertices 1, 2, 3 are geometric ideal triangles. Now there is a unique isometry A which maps a^- to a^- so that the vertices with the same labels coincide. We correspond this $A \in I^+(H^3) \cong \text{PSL}(2, C)$ to the generator a of $\pi_1(M)$. $(I^+(H^3)$ is the group of all orientation preserving isometries of H^3 onto itself.)

Similarly we correspond to b the isometry B which maps b^- to b^+ so that the vertices with the same labels coincide. Similarly we correspond isometries C, D to c, d respectively. This correspondence can be uniquely extended to a homomorphism of the free group generated by a, b, c, d to $\text{PSL}(2, C)$. Then the

necessary and sufficient condition for this homeomorphism to induce a homomorphism of $\pi_1(M)$ to $\mathrm{PSL}(2, C)$ is that following relations hold:

$$(((+)))\quad ACD^{-1}B^{-1}D = I, \quad (I: \text{identity mapping})$$

$$(((\mp)))\quad CD = I,$$

$$(((\mp\!\mp)))\quad A^{-1}B^{-1}CAB = I,$$

corresponding to the relators $((+))$, $((\mp))$, $((\mp\!\mp))$ of the presentation of $\pi_1(M)$. But it is not hard to see that these relations hold whenever $(+)$, $(\mp)$, $(\mp\!\mp)$ hold. Thus we could associate to each solution r a representation $\pi_1(M) \to \mathrm{PSL}(2, C)$, up to equivalence. ("Up to equivalence" is caused by the ambiguity of placing the polyhedron $\varLambda^*$ in H^3.)

§7. Miscellaneous examples of the construction of hyperbolic structures.

1. A concrete example of a hyperbolic structure of a closed 3-manifold.

Let J be a hyperbolic regular ideal dodecahedron, whose dihedral angles are the right angle. We color the faces of J with four colors (say, red, blue, yellow and green) in the manner of four color problem. Let J_i $(i=1, \cdots, 8)$ be 8 copies of J. We glue the faces of J_i $(i=1, \cdots, 8)$ pairwise as follows:

(i) for red faces, we glue the corresponding points of J_i and J_j, where the pair (i, j) is indicated as follows:
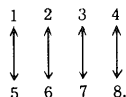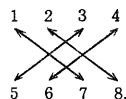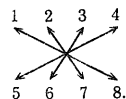

Similarly,

(ii) for blue faces, the pair (i, j) is as follows:
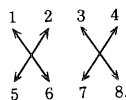
(iii) for yellow faces, the pair (i, j) is as follows:

(iv) for green faces, (i, j) is as follows:

It is easy to observe that by this glueing we obtain a closed hyperbolic 3-manifold M_1.

If we do not glue the 8 copies of one specified face of J, we obtain a hyperbolic 3-manifold M_2 with a totally geodesic boundary surface of genus 2 (It is obvious that M_2 is connected).

2. Another example of hyperbolic 3-manifold with totally geodesic boundary. Consider the Heegaard diagram of Figure 74.
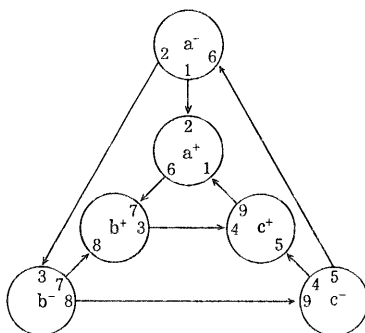

Fig. 74.

The manifold M with this diagram is obtained from genus 3 handlebody by attaching a 2-handle along the loop determined by the diagram. ∂M is a genus 2 surface. A presentation of the fundamental group of M can be read from the diagram:

$$\pi_1(M) \cong \langle a, b, c \,|\, a^2 b^{-1} c^2 a^{-1} b^2 c^{-1} = 1 \rangle .$$

As is § 4, M is homeomorphic to the manifold obtained from the polyhedron P of Figure 75 by glueing the faces as in the specified way.
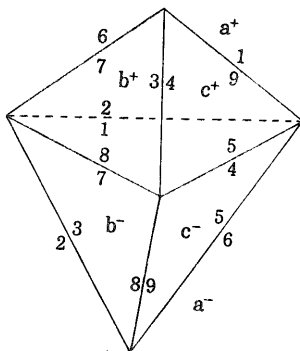

Fig. 75.

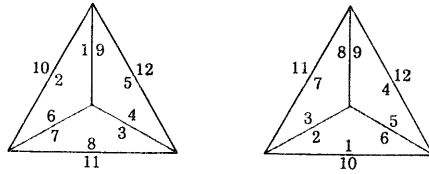Thus M has a nice triangulation as shown in the Figure 76.

Fig. 76.

12 edges are all identified after the glueing. M is homeomorphic to a manifold obtained from the two polyhedra of Figure 77 by glueing hexagon-faces in the specified way.
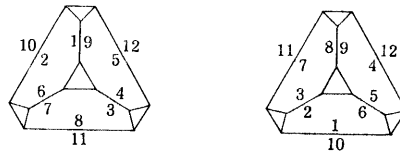

Fig. 77.

Now we invoke a theorem of hyperbolic plane geometry. First we define a relation $H(x, y)$ as follows: $H(x, y)$ $(x, y \in \mathbf{R}^+)$ if and only if the following right-angled hexagon is possible in the hyperbolic plane.
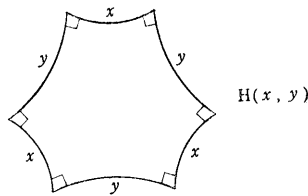

Fig. 78.

Theorem $\forall x \in \mathbf{R}^+ \exists! y \in \mathbf{R}^+ H(x, y)$.

Moreover y, as a function of x, is strictly decreasing, varying from ∞ to 0.

Now choose $a \in \mathbf{R}^+$ such that a regular triangle with sides of length a has angles of $30°$. Then we choose b such that $H(a, b)$. Then we can construct a polyhedron as illustrated in Figure 77.
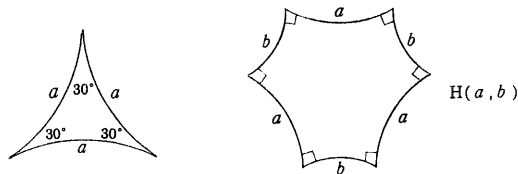

Fig. 79.

The dihedral angle along an edge of length b is 30° and the dihedral angle along an edge of length a is 90°. If we glue two such polyhedra as in the Figure 77 by isometries, obtain a desired hyperbolic structure. For the total angle along the only one edge is $30° \times 12 = 360°$, and hence no singularity occurs along the edge and also it is obvious that the boundary is totally geodesic.

§ 8. Proof of the Nice Triangulation Theorem.

In this section we shall give a rigorous proof of the Nice Triangulation Theorem stated in § 1.

Suppose that a compact 3-manifold M with boundary is given. M has a handle decomposition and hence is obtained from a handlebody by attaching some 2-handles along loops $\{l_i\}$ on the boundary of the handlebody. If we cut the handlebody by meridian disks, we obtain D^3 with a graphic picture on ∂D^3. This graphic picture is called a Heegaard diagram of M. (Of course we assume that the meridian loops and the loops $\{l_i\}$ interest transversely in a finite number of points.)

Moreover we can assume that the graph is connected and that each meridian loop intersects $\{l_i\}$ in at least 3 points. (We do not assume that the graph is normal. So it may contain the part illustrated in the Figure 80. So it is easy to obtain the graph which satisfies the above assumption.)
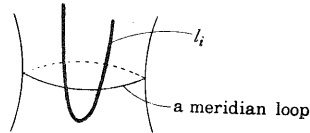

Fig. 80.

Now let m be a fixed meridian loop (named the distinguished loop) and P be a fixed intersection point of m with $\{l_i\}$. We add redundant intersection in m arround P as in the Figure 81.
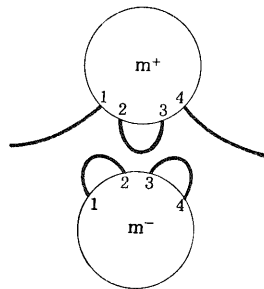

Fig. 81.

Now consider the dual graph D of this graph as in §4. The dual graph exists because the Heegaard diagram is connected. In D, the face corresponding to the distinguished loop contains the following part illustrated in the Figure 82.
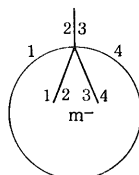
Fig. 82.

As in §2 or §4, we can add edges consistent with glueing (without adding any vertices) to obtain a triangulation of the S^2, as shown in the Figure 83. Any 1-gon or 2-gon does not occur.
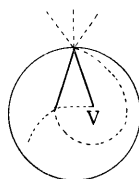
Fig. 83.

Finally consider this as a cone with the vertex V. Then we obtain a nice triangulation of M as in §2 or §4. **q.e.d.**

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