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ON ARONSZAJN TREES WITH A NON-SOUSLIN BASE

By

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§1. Introduction.

A tree is a partially ordered set $(T, <_T)$ with the property that for every element $x \in T$, $\hat{x}=\{y \in T: y <_T x\}$ is well-ordered by $<_T$. The order type of $\hat{x}$ is then an ordinal, which is called the height of x, $ht(x)$. When a subset of a tree is totally ordered by $<_T$, it is called a chain. When a subset of a tree has no comparable elements, it is called an antichain. We deal with only ω_1-trees which have cardinality ω_1, whose α-th level $T_\alpha=\{x \in T: ht(x)=\alpha\}$ is countable for every countable ordinal α, and which have additionally certain minor properties. An ω_1-tree T is said to be non-Souslin if every uncountable subset of T contains an uncountable antichain. A non-Souslin tree has clearly no uncountable chain and nevertheless for every countable ordinal α, the α-th level T_α is non-empty. This notion was introduced in Baumgartner [1]. The first example of a non-Souslin tree is the special Aronszajn tree which was given by Aronszajn (see Kurepa [5]). A special Aronszajn tree is characterized by Q-embeddability that means the existence of an order preserving function $f: T \to Q$. An R-(a fortiori, Q-) embeddable tree is always non-Souslin. Other examples of non-Souslin trees are found in Baumgartner [1], Hanazawa [2], [3] and Shelah [6]. Except for only one, the properties characterizing them are given as modifications of R-embeddability. The exception is the one given in [3], which has a non-Souslin base of cardinality ω_1. A non-Souslin base is a family F of uncountable antichains satisfying that whenever S is an uncountable subset of the tree T, there is an element A of F such that for every $x \in A$, there is $y \in S$ satisfying $x \leqq_T y$. Notice that a non-Souslin tree has always a non-Souslin base of cardinality 2^{ω_1}. We call a tree with a non-Souslin base of cardinality less than 2^{ω_1} an NSB-tree.

In this paper we discuss about NSB-trees, mainly to show that the property NSB is independent of R-embeddability. We first observe (in theorem 1) that under the standard set theory ZFC alone, even the existence of NSB-trees can

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not be proved. We use the axiom of constructibility $V=L$. It is shown in [3] that if $V=L$, there is an NSB-tree which is even not R-embeddable. On the other hand, if $V=L$, there is a Q- (a fortiori, R-) embeddable tree which is nevertheless not NSB (Theorem 2). The existence of such a tree may be one of rare examples which can be proved from $\diamondsuit^+$ but can not be proved from $\diamondsuit^*$, where $\diamondsuit^+$ and $\diamondsuit^*$ are Jensen's combinatorial principles, which are consequences of $V=L$. Finally we remark that if $V=L$, there is also a Q-embeddable NSB-tree. Hence property NSB is independent of and compatible with the property of being special Aronszajn under $V=L$.

§ 2. Definitions and results.

We write T instead of $(T, <_T)$ and $<$ instead of $<_T$. We refer the reader to [3] for the concepts undefined here.

DEFINITION 1. Let F be a family of uncountable antichains of an ω_1-tree T. F is an NS-base if and only if for every uncountable subset S of T, there exists an element A of F such that

$$\forall x \in A \exists y \in S (x \leqq y).$$

DEFINITION 2. T is called a κ-NSB tree if it has an NS-base of cardinality κ.

REMARK 1. A non-Souslin tree is trivially a 2^{ω_1}-NSB tree and vice versa. Note that there always exists a non-Souslin tree because a special Aronszajn tree is non-Souslin.

DEFINITION 3. T is called an NSB tree if it has an NS-base of cardinality less than 2^{ω_1}.

REMARK 2. There is no ω-NSB tree. (Suppose $\{A_n : n \in \omega\}$ were an NS-base. Take $\alpha < \omega_1$ so that for every $n \in \omega$, $|A_n \cap T \upharpoonright \alpha| \geqq 2$. Take $x \in T_\alpha$ arbitrarily. Then the set $S = \{y \in T : x \leqq y\}$ gives a contradiction.)

Let MA stand for Martin's axiom as usual (see Kunen [4, p. 54]).

THEOREM 1. (*MA*) *If* $\kappa < 2^\omega$, *there is no* κ-*NSB tree.*

COROLLARY 1.1. ($MA + \neg CH$) *There is no NSB tree. Because* $MA + \neg CH$ *implies* $2^\omega = 2^{\omega_1}$.

COROLLARY 1.2. *The existence of an NSB-tree can not be proved in ZFC alone.* (*cf. Remark* 1)

REMARK 3 ([3]). ($\diamondsuit$) There is an NSB tree which is not R-embeddable.

THEOREM 2. ($\diamondsuit^+$) *There is a special Aronszajn tree which is not NSB.*

COROLLARY 2.1. *Q-embeddability (a fortiori, R-embeddability) does not imply property NSB even under $V=L$.*

QUESTION 2.2. Can Theorem 2 be proved under ZFC alone (or even under ZFC$+\diamondsuit^*$)?

THEOREM 3. ($\diamondsuit$) *There is a special Aronzajn tree which is also NSB.*

Similarly an R-embeddable, not Q-embeddable, NSB tree can be obtained under $\diamondsuit$. On the other hand, by combining the trees given by Theorem 2 and Baumgartner [1], we can obtain (1) an R-embeddable, not Q-embeddable, not NSB tree, and (2) a not R-embeddable, not NSB, non-Souslin tree, under $\diamondsuit^+$.

§ 3. Proofs.

3.1. Proof of Theorem 1. Assume MA and $\kappa<2^\omega$. To the contrary, suppose T is a κ-NSB tree. As described in Remark 2, κ is not ω, and so $\neg$CH is the case. Since MA$+\neg$CH implies that every Aronszajn tree is special (Baumgartner, see Kunen [4, p. 91]), T must be special. Take a function $f: T\to Q$ satisfying that for any $x, y\in T$ with $x<y$, $f(x)<f(y)$. Let $\{A_\alpha: \alpha<\kappa\}$ be a κ-NS base of T. Define a poset P by the following:

$P=\{\langle X, Y\rangle$: (1) X and Y are disjoint finite subsets of T, (2) if $y\in Y$ then $ht(y)>\omega$, and (3) for every $w\in T$, if there are $x\in X$ and $y\in Y$ satisfying $w<x$ and $f(y)=f(w)$, then $w\in X\}$,

$$\langle X_1, Y_1\rangle\leqq\langle X_2, Y_2\rangle \quad \text{iff} \quad X_1\supseteqq X_2 \text{ and } Y_1\supseteqq Y_2\,.$$

Note that if $x\in X$ and $y\in Y$ where $\langle X, Y\rangle\in P$, then $y\nleqq x$. First we show that P satisfies c.c.c. Suppose S is an uncountable subset of P. By the Δ-system lemma (see Kunen [4, p. 49]), there is an uncountable subset $S'=\{\langle X_\xi, Y_\xi\rangle: \xi<\omega_1\}$ of S such that there is a finite set X^* satisfying $X_\xi\cap X_\eta=X^*$ for all $\xi, \eta<\omega_1$ with $\xi\neq\eta$, and further such that there is Y^* satisfying $Y_\xi\cap Y_\eta=Y^*$ for all ξ, η with $\xi\neq\eta$. Then take an uncountable subset $\{\langle X_\xi, Y_\xi\rangle: \xi\in I\}$ of S' such that for all $\xi, \eta\in I$, $f[X_\xi]=f[X_\eta]$ and $f[Y_\xi]=f[Y_\eta]$. We can easily take two pairs $\langle X_\xi, Y_\xi\rangle$ and $\langle X_\eta, Y_\eta\rangle$, $\xi, \eta\in I$, such that $X_\xi\cap Y_\eta=\emptyset$ and $X_\eta\cap Y_\xi=\emptyset$. Then clearly $\langle X_\xi\cup X_\eta, Y_\xi\cup Y_\eta\rangle$ is in P. This shows that P satisfies c.c.c. Now put

$$D_\alpha=\{\langle X, Y\rangle\in P: \exists x\in X(ht(x)>\alpha)\}\,.$$

Then D_α is dense in P for each $\alpha<\omega_1$. For, suppose that $\langle X, Y\rangle\in P$ and $\alpha<\omega_1$. As Y is finite and T_ω is infinite, there is $z\in T_\omega$ such that $(\forall w\in T)(w>z\Rightarrow w\notin Y)$. Take x so that $x>z$ and $ht(x)>\alpha$ and put $X'=X\cup\{x\}\cup\{w\in T: w<x \ \& \ f(w)\in f[Y]\}$. Then $\langle X', Y\rangle\in P$ and $\langle X', Y\rangle\leqq\langle X, Y\rangle$. Thus D_α is dense. Next put

$$E_\beta=\{\langle X, Y\rangle\in P: Y\cap A_\beta\neq\emptyset\}.$$

E_β is also dense in P for each $\beta<\kappa$. For, suppose $\langle X, Y\rangle\in P$ and $\beta<\kappa$. Take $a\in A_\beta\backslash(X\cup\hat{X}\cup T\restriction(\omega+1))$, where $\hat{X}=\{z\in T: z<x$ for some $x\in X\}$. Put $X'=X\cup\{z\in\hat{X}: f(z)=f(a)\}$. Then $\langle X', Y\cup\{a\}\rangle$ is in P. (It suffices to show $X'\cap(Y\cup\{a\})=\emptyset$. Suppose $z\in X'\backslash X$. Then $z\in\hat{X}$. Hence $z\neq a$ and $z\notin Y$.) E_β is thus dense. Therefore, by MA$+\neg$CH, there exists a $\{D_\alpha: \alpha<\omega_1\}\cup\{E_\beta: \beta<\kappa\}$-generic subset G of P. Now put $S=\cup\{X: \exists Y\langle X, Y\rangle\in G\}$. Clearly S is an uncountable subset of T and for each $\beta<\kappa$ there is an element $y\in A_\beta$ such that for any $x\in S$, $y\nleqq x$. This contradicts that $\{A_\alpha: \alpha<\kappa\}$ is an NS-base, q. e. d.

3.2. Proof of Theorem 2. The principle $\diamondsuit^+$ asserts the existence of a $\diamondsuit^+$-sequence $\langle S_\alpha: \alpha<\omega_1\rangle$ which satisfies:

(1) S_α is a countable family of subsets of α,

(2) for each $A\subset\omega_1$, there is a cub (closed unbounded) $C\subset\omega_1$, such that for every $\alpha\in C$, $A\cap\alpha\in S_\alpha$ and $C\cap\alpha\in S_\alpha$.

LEMMA 2.1. *Let $\langle S_\alpha: \alpha<\omega_1\rangle$ be a $\diamondsuit^+$-sequence. Put*

$$S_\alpha^+=S_\alpha\cup\{U\cap V: U, V\in S_\alpha\}.$$

Then for each subset $A\subset\omega_1$ and for each cub $C\subset\omega_1$, there is a cub $C'\subseteqq C$ such that $\forall\alpha\in C'$ ($A\cap\alpha\in S_\alpha^+$ and $C'\cap\alpha\in S_\alpha^+$).

PROOF. By the property of $\diamondsuit^+$-sequence, there is cub $C_0\subset\omega_1$ such that $\forall\alpha\in C_0$ ($A\cap\alpha\in S_\alpha$ and $C_0\cap\alpha\in S_\alpha$). By the same reason, for some cub $C_1\subset\omega_1$, $\forall\alpha\in C_1$ ($C\cap C_0\cap\alpha\in S_\alpha$ & $C_1\cap\alpha\in S_\alpha$). Then $\forall\alpha\in C\cap C_0\cap C_1$ ($A\cap\alpha\in S_\alpha^+$ & $C\cap C_0\cap C_1\cap\alpha\in S_\alpha^+$), q. e. d.

LEMMA 2.2 *Let $\langle S_\alpha: \alpha<\omega_1\rangle$ be a $\diamondsuit^+$-sequence and $\{P_\xi: \xi<\omega_1\}$ be a partition of ω_1. Then the following holds:*

(*) *for each subset $A\subset\omega_1$ satisfying $\forall\xi\in\omega_1$ ($|A\cap P_\xi|\leqq\omega$) and for each cub $C\subset\omega_1$, there is a cub $C'\subseteqq C$ such that*

$$\forall\alpha\in C' \ (A\cap\bigcup_{\xi<\alpha}P_\xi\in S_\alpha^+ \text{ and } C'\cap\alpha\in S_\alpha^+).$$

PROOF. By the assumption, $A\cap\bigcup_{\xi<\alpha}P_\xi$ is (at most) countable for every $\alpha<\omega_1$.

Hence $C_0=\{\alpha<\omega_1: A\cap\bigcup_{\xi<\alpha}P_\xi=A\cap\alpha\}$ is cub (the proof is routine, cf. Kunen [4, p, 78 or p. 79]). By the previous lemma, for some cub $C_1\subset C\cap C_0$, $\forall\alpha\in C_1$ $(A\cap\alpha\in S^+_\alpha$ & $C_1\cap\alpha\in S^+_\alpha)$. The desired conclusion follows immediately from this.

COROLLARY 2.2.1. *Let* $|Z|=\omega_1$ *and* $\langle Z_\xi:\xi<\omega_1\rangle$ *a partion of* Z. *Then there is a sequence* $\langle U_\alpha:\alpha<\omega_1\rangle$ *such that*

(1) U_α *is a countable set of pairs* $\langle s, c\rangle$ *of a countable subset* $s\subseteq\bigcup_{\xi<\alpha}Z_\xi$ *and a set* c *closed in* α, *and*

(2) *whenever a set* $A\subset Z$ *satisfies* $\forall\xi<\omega_1|A\cap Z_\xi|\leqq\omega$, *then for each cub* $C\subset\omega_1$, *there is a cub* $C'\subseteq C$ *such that*

$$\forall\alpha\in C'\ (\langle A\cap\bigcup_{\xi<\alpha}Z_\xi,\ C'\cap\alpha\rangle\in U_\alpha).$$

PROOF. Fix a one-to-one onto function $\pi: Z\to\omega_1$. Let $\langle S_\alpha:\alpha<\omega_1\rangle$ be a $\diamondsuit^+$-sequence. Put $U_\alpha=\{\langle\pi^{-1}[s]\cap\bigcup_{\xi<\alpha}Z_\xi, c\rangle: s, c\in S^+_\alpha, c$ is closed in $\alpha\}$. By the lemma, this satisfies the required conditions, q. e. d.

REMARK. We may assume without loss of generality the sequence $\langle U_\alpha:\alpha<\omega_1\rangle$ satisfies the following:

(3) every $\langle s, c\rangle\in U_\alpha$ satisfies that for every $\beta\in c$, $\langle s\cap\bigcup_{\xi<\beta}Z_\xi, c\cap\beta\rangle\in U_\beta$. Because, if the element $\langle s, c\rangle\in U_\alpha$ does not have this property, we may remove it from U_α.

CONVENTION. Put $\boldsymbol{T}=\bigcup_{\alpha<\omega_1}{}^\alpha\omega$, where ${}^\alpha\omega=\{f: f:\alpha\to\omega\}$. $\boldsymbol{T}$ is a tree (not an ω_1-tree) by defining $x<y$ by $x\subset y$ for $x, y\in\boldsymbol{T}$. In the rest of this paper, an ω_1-tree means always a subtree T of $\boldsymbol{T}$ such that T is ω_1-tree in the usual sense and an initial segment of $\boldsymbol{T}$. When f is a function: $\alpha\to\mathfrak{P}(\boldsymbol{T}\restriction\alpha)$, where $\alpha\leqq\omega_1$, then for each $\beta\leqq\alpha$, $f\restriction\restriction\beta$ stands for $\{\langle\xi, f(\xi)\cap\boldsymbol{T}\restriction\beta\rangle:\xi<\beta\}$, a function from β to $\mathfrak{P}(\boldsymbol{T}\restriction\beta)$. Hence if T is an ω_1-tree and $f:\alpha\to\mathfrak{P}(T\restriction\alpha)$ then for each $\beta<\alpha$, $f\restriction\restriction\beta=\{\langle\xi, f(\xi)\cap T\restriction\beta\rangle:\xi<\beta\}$.

LEMMA 2.3. *There is a sequence* $\langle\diamondsuit^+_\alpha:\alpha<\omega_1\rangle$ *such that*

(1) $\diamondsuit^+_\alpha$ *is a countable set of pairs* $\langle f, c\rangle$ *of a function* $f:\alpha\to\mathfrak{P}(\boldsymbol{T}\restriction\alpha)$ *and a set* c *closed in* α,

(2) *if* $\langle f, c\rangle\in\diamondsuit^+_\alpha$, *then for every* $\beta\in c$, $\langle f\restriction\restriction\beta, c\cap\beta\rangle\in\diamondsuit^+_\beta$,

(3) *if a function* $F:\omega_1\to\mathfrak{P}(\boldsymbol{T})$ *satisfies the condition that* $\forall\xi<\omega_1\forall\alpha<\omega_1|f(\xi)\cap\boldsymbol{T}\restriction\alpha|\leqq\omega$, *then for each cub set* C,

there is a cub set $C'\subseteq C$ *such that*

$$\forall\alpha\in C'\ (\langle F\restriction\restriction\alpha,\ C'\cap\alpha\rangle\in\diamondsuit^+_\alpha).$$

PROOF. A function $F: \omega_1 \to \mathfrak{P}(\boldsymbol{T})$ can be identified by one-to-one manner with $F^* = \{\langle \alpha, x \rangle : \alpha \in \omega_1, x \in F(\alpha)\} \subseteqq \omega_1 \times \boldsymbol{T}$. $\{((\alpha+1) \times \boldsymbol{T} \restriction (\alpha+1)) \setminus (\alpha \times \boldsymbol{T} \restriction \alpha) : \alpha < \omega_1\}$ is a partition of $\omega_1 \times \boldsymbol{T}$. $|\omega_1 \times \boldsymbol{T}| = \omega_1$ since $\diamondsuit^+$ implies CH. So the assertion follows directly from Corollary 2.2.1 and the remark after it, q. e. d.

We fix this sequence $\langle \diamondsuit_\alpha^+ : \alpha < \omega_1 \rangle$ in this section. For a technical reason, we assume without loss of generality that $\langle \varnothing, \varnothing \rangle \in \diamondsuit_0^+$ and $\diamondsuit_\alpha^+ = \varnothing$ if α is a successor ordinal.

To show the theorem, we construct T and $e: T \to Q$ such that

(1) T is an ω_1-tree, and

(2) if $x < y$ in T then $e(x) < e(y)$ in Q.

Besides, for each $\langle f, c \rangle \in \diamondsuit_\alpha^+$, we give $X(f, c) \subseteqq T_\alpha$ (not $T \restriction \alpha$) such that

(3) $\beta \in c \ \& \ x \in X(f, c) \Rightarrow \exists y < x \ (y \in X(f \restriction \restriction \beta, c \cap \beta))$

(in other words, every element of $X(f, c)$ is an extension of some elements of $X(f \restriction \restriction \beta, c \cap \beta)$ if $\beta \in c$),

(4) $\forall \xi < \alpha \ \exists y \in f(\xi) \ \forall x > y \ (x \notin X(f, c))$

(i. e., every ξ-th subset $f(\xi) \subset T \restriction \alpha$ has an element which has no extensions in $X(f, c)$),

(5) $X(f, c) \neq \varnothing$, if $f \subseteqq \alpha \times \mathfrak{P}(T \restriction \alpha)$ and $\forall \alpha' \in c \cup \{\alpha\} \ \forall \xi < \alpha' \ \forall \beta < \alpha' \ \exists y \in f(\xi) \cap T \restriction \alpha' \ (ht(y) > \beta)$.

CLAIM. Such a tree T is Q-embeddable and not NSB.

PROOF. T is clearly Q-embeddable by e. To show $T \notin$NSB, let $\{A_\xi : \xi < \omega_1\}$ be any family of uncountable antichains of T. Put

$$A = \{\langle \xi, A_\xi \rangle : \xi < \omega_1\},$$

and

$$C = \{\alpha : \forall \xi < \alpha \forall \beta < \alpha \exists y \in T \restriction \alpha (y \in A_\xi \text{ and } ht(y) > \beta)\}.$$

Then C is cub in ω_1. By Lemma 2.3, there is a cub $C' \subseteqq C$ such that

$$\forall \alpha \in C' \ \langle A \restriction \restriction \alpha, C' \cap \alpha \rangle \in \diamondsuit_\alpha^+.$$

Put

$$X = \cup \{X(A \restriction \restriction \alpha, C' \cap \alpha) : \alpha \in C'\}.$$

Then by (5) X is uncountable and $\forall \xi < \omega_1 \ \exists y \in A_\xi \ \forall x \in X (y \nleqq x)$. (For, let $\xi < \omega_1$. Let α be the least ordinal satisfying $\xi < \alpha \in C'$. Then by (4) there is $y \in A_\xi \cap T \restriction \alpha$ such that for no x, $y < x \in X(A \restriction \restriction \alpha, C' \cap \alpha)$. Such y satisfies $\forall x \in X (y \nleqq x)$ by (3).) This means $\{A_\xi : \xi < \omega_1\}$ is not an NS-base, q. e. d.

Now we define T_α, $e \restriction T_\alpha$ and $X(f, c) \subseteqq T_\alpha$ by induction on α. At each stage α, we make the following hold together with the above conditions (1)-(5):

(6) if $x \in T \restriction \alpha$ and $e(x) < q \in Q$, then there is $y \in T_\alpha$ such that $x < y$ and

$e(y)=q$,

(7) if $X(f, c)\neq\varnothing$, $\beta\in c\cup\{0\}$, $y\in X(f\upharpoonright\upharpoonright\beta, c\cup\beta)$, and $e(y)<q\in Q$, then there is $x\in X(f, c)$ such that $x>y$ and $e(x)=q$.

(I) If $\alpha=0$, put $T_0=\{\emptyset\}$, $e(\emptyset)=0$, and $X(\emptyset, \emptyset)=\{\varnothing\}$.

(II) If $\alpha=\beta+1$, put $T_{\beta+1}=\{x^\frown\langle n\rangle : x\in T_\beta \ \&\ n\in\omega\}$ and $e(x^\frown\langle n\rangle)=e(x)+q_n$, where $x^\frown\langle n\rangle$ stands for $x\cup\{\langle\beta, n\rangle\}$ and $\{q_n : n\in\omega\}$ is a list of Q^+.

(III) Suppose $\mathrm{Lim}(\alpha)$,

(III.1) For each $x\in T\upharpoonright\alpha$ and for each $q\in Q$ with $e(x)<q$, we define $t_\alpha(x, q)\in {}^\alpha\omega(=\boldsymbol{T}_\alpha)$ as follows:

Take a sequence $q_0=e(x)<q_1<q_2<\cdots\rightarrow q$ with $q_n\in Q$, $n\in\omega$, and a sequence $\alpha_0=ht(x)<\alpha_1<\alpha_2<\cdots\rightarrow\alpha$. Construct a sequence $x_0=x<x_1<x_2<\cdots$ with $x_n\in T\upharpoonright\alpha$, by induction on $n\in\omega$ so that $e(x_n)=q_n$ and $ht(x_n)=\alpha_n$. This is possible by induction hypothesis (6). Put $t_\alpha(x, q)=\bigcup_{n\in\omega}x_n$.

(III.2) For each pair $\langle f, c\rangle\in\diamondsuit_\alpha^+$, we define $X(f, c)\subseteq\boldsymbol{T}_\alpha$, as follows:

There are three cases to consider.

CASE 1. $f\subset\alpha\times\mathfrak{P}(T\upharpoonright\alpha)$, $\forall\alpha'\in c\cup\{\alpha\}\,\forall\xi<\alpha'\ \forall\beta<\alpha'\ \exists y\in f(\xi)\cap T\upharpoonright\alpha'\ (ht(y)>\beta)$, and c is bounded in α. In this case, put $\gamma=$the maximum element of $c\cup\{0\}$. Let $\langle\xi_i : i\in\omega\rangle$ be an ω-type enumeration of the elements of $\alpha\backslash\gamma$. Fix arbitrarily a sequence $\alpha_0=\gamma<\alpha_1<\alpha_2<\cdots\rightarrow\alpha$. Take $y_0\in T\upharpoonright\alpha$ so that $ht(y_0)>\gamma$ and $y_0\in f(\xi_0)$, and take $y_{n+1}\in f(\xi_n)$ so that $ht(y_{n+1})>ht(y_n)\cup\alpha_n$. This is possible by the assumption. Now, by the assumption and the induction hypothesis (5), $X(f\upharpoonright\upharpoonright\gamma, c\cap\gamma)$ is not empty. For each $x\in X(f\upharpoonright\upharpoonright\gamma, c\cap\gamma)$ and for each $q\in Q$ with $q>e(x)$, define $u_\alpha(x, q, f, c)\in\boldsymbol{T}_\alpha$ as follows:

Take a sequence $q_0=e(x)<q_1<q_2<\cdots\rightarrow q$ from Q. Put $x_0=x$. For $n>0$, take x_n so that $x_n>x_{n-1}$, $ht(x_n)=ht(y_n)$, $x_n\neq y_n$, and $e(x_n)=q_{2n}$ or q_{2n+1}. This is possible by induction hypthesis (6). Put $u_\alpha(x, q, f, c)=\bigcup_{n\in\omega}x_n$, and $X(f, c)=\{u_\alpha(x, q, f, c) : x\in X(f\upharpoonright\upharpoonright\gamma, c\cap\gamma), e(x)<q\in Q\}$.

CASE 2. The same as Case 1 but c is unbounded in α. In this case we first fix a sequence $\alpha_0<\alpha_1<\cdots\rightarrow\alpha$ such that $\alpha_n\in c$, $n\in\omega$. Note that $X(f\upharpoonright\upharpoonright\alpha_n, c\cap\alpha_n)\neq\emptyset$ for each $n\in\omega$. For each x and q such that $x\in X(f\upharpoonright\upharpoonright\alpha_n, c\cap\alpha_n)$ and $e(x)<q\in Q$, take a sequence $q_0=e(x)<q_1<q_2<\cdots\rightarrow q$. Put $x_0=x$, and for $k>0$, take $x_k\in X(f\upharpoonright\upharpoonright\alpha_{k+n}, c\cap\alpha_{k+n})$ so that $e(x_k)=q_{k+n}$ and $x_k>x_{k-1}$. This is possible by induction hypothesis (7). Put $u_\alpha(x, q, f, c)=\bigcup_{n\in\omega}x_n$ and $X(f, c)=\{u_\alpha(x, q, f, c) : x\in\bigcup_{n\in\omega}X(f\upharpoonright\upharpoonright\alpha_n, c\cap\alpha_n), e(x)<q\in Q\}$.

CASE 3. Otherwise. Put $X(f, c)=\varnothing$.

(III.3) Now, we set

$$T_\alpha=\{t_\alpha(x, q): x\in T\restriction\alpha,\ e(x)<q\in Q\}\cup\bigcup\{X(f, c): \langle f, c\rangle\in\diamondsuit_\alpha^+\},$$

$$e(t_\alpha(x, q))=q, \quad \text{and} \quad e(u_\alpha(x, q, f, c))=q.$$

Thus T_α, $e\restriction T_\alpha$, and $X(f, c)$ for $\langle f, c\rangle\in\diamondsuit_\alpha^+$ are defined. We must check that they have the required properties. But it needs only calculation. We only show (4) and leave the rest to the reader. Let $\langle f, c\rangle\in\diamondsuit_\alpha^+$. To show (4), suppose $\xi<\alpha$. Suppose that $X(f, c)$ has been defined in Case 1 and recall the terminologies used there. If $\xi\geqq\gamma$, then $\xi=\xi_n$ for some n. Then $y_{n+1}\in f(\xi_n)=f(\xi)$. But every element $u_\alpha(x, q, f, c)=\bigcup_{n\in\omega}x_n$ of $X(f, c)$ is not an extension of y_{n+1}, because $y_{n+1}\neq x_{n+1}<u_\alpha(x, q, f, c)$ and $ht(y_{n+1})=ht(x_{n+1})$ by the definition. If $\xi<\gamma$, note that $\langle f\restriction\restriction\gamma, c\cap\gamma\rangle\in\diamondsuit_\gamma^+$. By induction hypothesis, we can find $y\in(f\restriction\restriction\gamma)(\xi)$ which has no extension in $X(f\restriction\restriction\gamma, c\cap\gamma)$. Since every element of $X(f, c)$ is an extension of some element of $X(f\restriction\restriction\gamma, c\cap\gamma)$ by the definition, such y has no extension in $X(f, c)$. Next, suppose that $X(f, c)$ has been defined in Case 2. Then $\xi<\alpha_n$ for some n. Note that $\alpha_n\in c$ and $X(f\restriction\restriction\alpha_n, c\cap\alpha_n)\neq\emptyset$. The rest is similar to the one in the case $\xi<\gamma$ of the above. If $X(f, c)$ has been defined in Case 3, it is trivial, q. e. d.

3.3. Proof of Theorem 3. We refer the reader to Convention in the previous section for the definition of $\boldsymbol{T}$ and for the meaning of the concept of ω_1-tree. Assume $\diamondsuit$. Then there is a sequence $\langle\diamondsuit_\alpha: \alpha<\omega_1\rangle$ such that

(1) $\diamondsuit_\alpha$ is a countable subset of $\boldsymbol{T}\restriction\alpha$,

(2) if $A\subset\boldsymbol{T}$ satisfies $|A\cap\boldsymbol{T}\restriction\alpha|\leqq\omega$, then the set $\{\alpha: A\cap\boldsymbol{T}\restriction\alpha=\diamondsuit_\alpha\}$ is stationary in ω_1.

The purpose is to define an ω_1-tree T and a Q-embedding $e: T\to Q$ so that $\{A(x, q): x\in T, q\in Q^+\}$ forms an NS-base, where $A(x, q)$ stands for $\{y\in T: x<y, e(y)=q\}$. We define T_α and $e\restriction T_\alpha$ by induction on α. At each stage α, we ensure the following:

(*) $x\in T\restriction\alpha\ \&\ e(x)<q\Rightarrow\exists y\in T_\alpha(x<y\ \&\ e(y)=q)$.

(I) $T_0=\{\emptyset\}$ and $e(\emptyset)=0$.

(II) $T_{\beta+1}=\{x^\frown\langle n\rangle: x\in T_\beta, n\in\omega\}$ and $e(x^\frown\langle n\rangle)=e(x)+q_n$, where $\langle q_n: n\in\omega\rangle$ is a list of Q^+.

(III) Suppose $\mathrm{Lim}(\alpha)$. For every pair of $x\in T\restriction\alpha$ and $q\in Q$ with $e(x)<q$, we define $t_\alpha(x, q)$. First define x_0 as follows: If $\diamondsuit_\alpha$ is an initial segment of $T\restriction\alpha$ and there is $y\in T\restriction\alpha$ such that $x<y$, $e(y)<q$, and $y\notin\diamondsuit_\alpha$, then put $x_0=$such y. Otherwise put $x_0=x$. Fix a sequence $q_0=e(x_0)<q_1<q_2<\cdots\to q$ and a sequence $\alpha_0=ht(x_0)<\alpha_1<\alpha_2<\cdots\to\alpha$. Take inductively

x_k so that $x_k > x_{k-1}$, $ht(x_k) = \alpha_k$, and $e(x_k) = q_k$. Put $t_\alpha(x, q) = \bigcup_{k \in \omega} x_k$ and $T_\alpha = \{t_\alpha(x, q) : x \in T \upharpoonright \alpha, e(x) < q\}$ and $e(t_\alpha(x, q)) = q$.

Finally we put $T = \bigcup_{\alpha < \omega_1} T_\alpha$, which is clearly Q-embedded by e. To show that T is NSB, we prove that $\{A(x, q) : x \in T, e(x) < q\}$ is an NS-base. Let S be an uncountable subset of T. Put $I = \{y \in T : \exists x \in S(y \leqq x)\}$. Put $C = \{\alpha : \mathrm{Lim}(\alpha), \forall q \in Q \, \forall x \in T \upharpoonright \alpha (\exists y(x < y \; \& \; e(y) = q \; \& \; y \notin I) \Rightarrow \exists$ such y in $T \upharpoonright \alpha)\}$, which is cub in ω_1. Take $\alpha \in C$ such that $I \cap T \upharpoonright \alpha = \diamondsuit_\alpha$. Since S is uncountable, $T_\alpha \cap I \neq \emptyset$. Take x, q so that $t_\alpha(x, q) \in T_\alpha \cap I$. Recall x_0 used in the definition of $t_\alpha(x, q)$. Since $x_0 < t_\alpha(x, q)$, x_0 is also in I, and so $x_0 \in I \cap T \upharpoonright \alpha = \diamondsuit_\alpha$. By the choice of x_0, it must hold that $\forall y \in T \upharpoonright \alpha (e(y) < q \; \& \; x < y \Rightarrow y \in I \cap T \upharpoonright \alpha)$. Hence every $y \in T$ satisfying $x < y$ and $e(y) < q$ belongs to I, because $\alpha \in C$. Therefore $A(x, (e(x) + q)/2) \subseteq I$, q. e. d.

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