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ON SOME STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS

By

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Let $A(p)$ denote the class of functions $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ which are analytic in the open disk $E=\{z: |z|<1\}$.

A function $f(z)\in A(p)$ is called p-valently starlike with respect to the origin iff

$$Re\frac{zf'(z)}{f(z)}>0 \quad \text{in } E.$$

We denote by $S^*(p)$ the subclass of $A(p)$ consisting of functions which are p-valently starlike in E.

Mocanu [3, Theorem 1] proved that if $f(z)\in A(1)$ and

$$|\arg f'(z)|<\frac{\pi}{2}\alpha_0=0.968\cdots, \qquad z\in E,$$

where $\alpha_0=0.6165\cdots$ is the unique root of the equation

$$2\tan^{-1}(1-\alpha)+\pi(1-2\alpha)=0,$$

then $f(z)\in S^*(1)$.

In [5], Nunokawa proved the following theorem.

THEOREM A. *Let* $p\geqq 2$. *If* $f(z)\in A(p)$ *satisfies*

$$|\arg f^{(p)}(z)|<\frac{3}{4}\pi \qquad in\ E,$$

then $f(z)$ *is* p*-valent in* E.

DEFINITION 1. Let $F(z)$ be analytic and univalent in E, and suppose that $F(E)=D$. If $f(z)$ is analytic in E, $f(0)=F(0)$, and $f(E)\subset D$, then we say that $f(z)$ is subordinate to $F(z)$ in E, and we write

$$f(z)\prec F(z).$$

DEFINITION 2. If the function $f(z)$ is analytic in E and if for every non-

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real z in E

$$\operatorname{sign}(Im\, f(z))=\operatorname{sign}(Im\, z),$$

then $f(z)$ is said to be typically-real in E. We owe this definition to [1, p. 184].

We shall use the following lemmas to prove our results.

LEMMA 1. *Let* $\beta^*=1.218\cdots$ *be the solution of*

$$\pi\beta=\frac{3\pi}{2}-\tan^{-1}\beta$$

and let

$$\alpha=\alpha(\beta)=\beta+\frac{2}{\pi}\tan^{-1}\beta$$

for $0<\beta\leqq\beta^*$.

If $p(z)$ *is analytic in* E, *with* $p(0)=1$, *then*

$$p(z)+zp'(z)\prec\left(\frac{1+z}{1-z}\right)^{\alpha} \Longrightarrow p(z)\prec\left(\frac{1+z}{1-z}\right)^{\beta}$$

We owe this lemma to [2, Theorem 5].

LEMMA 2. *Let* $f(z)\in A(p)$ *and suppose*

$$\operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}>0 \qquad in\ E.$$

Then we have

$$\operatorname{Re}\frac{zf^{(k)}(z)}{f^{(k-1)}(z)}>0 \qquad in\ E,$$

or

$$f^{(p-k)}(z)\in S^*(k)$$

for $k=1, 2, 3, \cdots, p$.

We owe this lemma to [4, Theorem 5].

THEOREM 1. *Let* $p\geqq 2$. *If* $f(z)\in A(p)$ *satisfies*

$$|\arg f^{(p)}(z)|<\frac{3}{4}\pi \qquad in\ E \tag{1}$$

and $f^{(p-1)}(z)/z$ *is typically-real in* E, *then* $f(z)\in S^*(p)$.

PROOF. Let us put

$$p(z)=\frac{f^{(p-1)}(z)}{p!\,z}.$$

From the assumption (1), Lemma 1 and applying the same method as the proof of [5, Main theorem], we have

$$p(z)+zp'(z)=\frac{f^{(p)}(z)}{p!}\prec\left(\frac{1+z}{1-z}\right)^{3/2} \quad \text{in } E,$$

$p(0)=1$ and therefore we have

$$\frac{f^{(p-1)}(z)}{p!z}\prec\left(\frac{1+z}{1-z}\right) \quad \text{in } E.$$

This shows that

$$\text{(2)} \qquad \operatorname{Re}\frac{f^{(p-1)}(z)}{z}>0 \quad \text{in } E.$$

By the same calculation as [6, p. 276], we have

$$\text{(3)} \qquad \frac{f^{(p-2)}(z)}{zf^{(p-1)}(z)}=\int_0^1\frac{f^{(p-1)}(tz)}{f^{(p-1)}(z)}dt$$

$$=\int_0^1\frac{z}{f^{(p-1)}(z)}\cdot\frac{tz}{z}\cdot\frac{f^{(p-1)}(tz)}{tz}dt$$

On the other hand, we easily have

$$\text{(4)} \qquad \left|\arg\left(\frac{z}{f^{(p-1)}(z)}\cdot\frac{tz}{z}\cdot\frac{f^{(p-1)}(tz)}{tz}\right)\right|$$

$$=\left|\arg\frac{f^{(p-1)}(tz)}{tz}-\arg\frac{f^{(p-1)}(z)}{z}\right|<\frac{\pi}{2}.$$

Since $f^{(p-1)}(z)/z$ is typically-real in E and satisfies the condition (2). From (3) and (4), we easily have

$$\operatorname{Re}\frac{f^{(p-2)}(z)}{zf^{(p-1)}(z)}>0 \quad \text{in } E.$$

This shows that

$$\text{(5)} \qquad \operatorname{Re}\frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}>0 \quad \text{in } E.$$

From Lemma 2 and (5), we easily have

$$\operatorname{Re}\frac{zf'(z)}{f(z)}>0 \quad \text{in } E.$$

This completes our proof.

THEOREM 2. *Let* $p\geqq 2$. *If* $f(z)\in A(p)$ *satisfies*

$$\text{(6)} \qquad |\arg f^{(p)}(z)|<\frac{\pi}{2}\cdot\alpha_1 \quad \text{in } E.$$

where $\alpha_1=1/2+(2/\pi)\tan^{-1}(1/2)=0.79516\cdots$, *then* $f(z)\in S^*(p)$.

PROOF. Let us put

$$p(z)=\frac{f^{(p-1)}(z)}{p!\,z}.$$

From the assumption (6), Lemma 1 and by the same calculation as in the proof of Theorem 1, we have

$$p(z)+zp'(z)=\frac{f^{(p)}(z)}{p!}\prec\left(\frac{1+z}{1+z}\right)^{\alpha_1} \qquad \text{in } E,$$

$p(0)=1$, $\alpha_1=\alpha(1/2)=(1/2)+(2/\pi)\tan^{-1}(1/2)=0.79516\cdots$ and therefore, we have

$$\frac{f^{(p-1)}(z)}{p!\,z}\prec\left(\frac{1+z}{1-z}\right)^{1/2} \qquad \text{in } E.$$

This shows that

$$\left|\arg\frac{f^{(p-1)}(z)}{z}\right|<\frac{\pi}{4} \qquad \text{in } E. \tag{7}$$

By the same calculation as the proof of Theorem 1, we have

$$\frac{f^{(p-2)}(z)}{zf^{(p-1)}(z)}=\int_0^1\frac{z}{f^{(p-1)}(z)}\cdot\frac{tz}{z}\cdot\frac{f^{(p-1)}(tz)}{tz}\,dt.$$

From (7), we easily have

$$\left|\arg\left(\frac{z}{f^{(p-1)}(z)}\cdot\frac{tz}{z}\cdot\frac{f^{(p-1)}(tz)}{tz}\right)\right|$$
$$\leqq\left|\arg\frac{f^{(p-1)}(z)}{z}\right|+\left|\arg\frac{f^{(p-1)}(tz)}{tz}\right|<\frac{\pi}{2} \qquad \text{in } E.$$

Therefore, we have

$$\operatorname{Re}\frac{f^{(p-2)}(z)}{zf^{(p-1)}(z)}>0 \qquad \text{in } E.$$

This shrws that

$$\operatorname{Re}\frac{zf^{(p-2)}(z)}{f^{(p-2)}(z)}>0 \qquad \text{in } E.$$

or $f(z)$ is p-valently starlike in E. This completes our proof.

From Theorem 2, we easily have the following corollary.

COROLLARY 1. *Let* $f(z)\in A(2)$ *satisfies*

$$|\arg f''(z)|<\frac{\pi}{2}\alpha_1 \qquad \text{in } E.$$

then $f(z)$ *is 2-valently starlihe in* E.

REMARK. $\alpha_0=0.6165\cdots<\alpha_1=0.79516\cdots$.

References

[1] Goodman, A. W., Univalent Functions, Vol. 1, Mariner Publishing Company, Tampa, Florida, 1983.

[2] Miller, S. S. and Mocanu, P. T., Marx-Strohhäcker differential subordination systems, Proc. Amer. Math. Soc. **99** (1987), 527-534.

[3] Mocanu, P. T., Some starlikeness conditions for analytic functions, Rev. Roumaine Math. Pures. Appl. **33** (1988), 1-2, 117-124.

[4] Nunokawa, M., On the theory of multivalent functions, Tsukuba J. Math. **11** (1987), 273-236.

[5] ———, A note on multivalent functions, Tsukuba J. Math. **13** (1989), 453-455.

[6] ———, On certain multivalently starlike functions, Tsukuba J. Math. **14** (1990), 275-277.

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