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LIMIT SYSTEMS AND CHAIN CONDITION

By

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There is an interesting theorem about the countable chain condition of complete Boolean algebras in [11]. Roughly speaking, the wellordered injective direct limit of complete Boolean algebras which satisfy the countable chain condition also satisfies the countable chain condition.

In this paper, we shall investigate relations among limit systems of topological spaces, complete Boolean algebras and complete pseudo-Boolean algebras, and state the application of the above theorem.

§1 is devoted to the basic definitions and preliminary results. Limit systems and their relations are described in §2. And in §3, we shall discuss about the chain conditions of the wellordered direct limits.

§1. Basic definitions and preliminary results.

We use the lattice theoretic symbols and the set theoretic ones. They are usual ones, but we shall give a few remarks about the symbols concerning a pseudo-Boolean algebra.

A pseudo-Boolean algebra is a lattice with the least element $\mathbf{0}$ and the operation $\Rightarrow$, where $a\Rightarrow b$ is the maximal element x such that $a\wedge x\leqq b$.

A Boolean algebra or a pseudo-Boolean algebra is complete if every subset of it has the least upper bound and the greatest lower bound.

$\bigvee^{B}$ or $\bigwedge^{B}$ means the least upper bound or the greatest lowwer bound in B respectively. If no confusion occurs we use $\bigvee$ or $\bigwedge$ instead of $\bigvee^{B}$ or $\bigwedge^{B}$ respectively.

Many informations about the relationship between a Boolean algebra and a pseudo-Boolean algebras are in [10].

Let A be a pseudo-Boolean algebra and $R(A)$ be the set $\{a\Rightarrow\mathbf{0}\,;\,a\in A\}$. An element of $R(A)$ is called a regular element.

PROPOSITION 1 [10]

$R(A)$ is a Boolean algebra the ordering of which is the restriction of the

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ordering of A. More precisely, the following hold.

For each $x, y \in R(A)$,

$$x \vee^{R(A)} y = ((x \vee^{A} y) \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0}$$

$$x \wedge^{R(A)} y = x \wedge^{A} y$$

$$x \vee^{R(A)} x \Rightarrow \mathbf{0} = \mathbf{1}$$

$$x \wedge^{R(A)} x \Rightarrow \mathbf{0} = \mathbf{0}\,.$$

If A is complete, then R(A) is complete, where

$$\bigvee{}^{R(A)} \mathfrak{A} = ((\bigvee{}^{A} \mathfrak{A}) \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0} \ \textit{and}$$

$$\bigwedge{}^{R(A)} \mathfrak{A} = \bigwedge{}^{A} \mathfrak{A}$$

PROOF. Notice that $a \leqq b$ implies $b \Rightarrow \mathbf{0} \leqq a \Rightarrow \mathbf{0}$ and $a \Rightarrow \mathbf{0} = ((a \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0}$ holds, then it is easy to prove the above proposition.

For a topological space X, $O(X)$ is the set of open subsets of X and $RO(X)$ is the set of regular open subsets of X. $O(X)$ is a complete pseudo-Boolean algebra and $RO(X) = R(O(X))$.

Let A and B be pseudo-Boolean algebras. A function $\varphi: A \to B$ is a pseudo-Boolean morphism if φ preserves the operations $\vee$, $\wedge$ and $\Rightarrow$ and $\varphi(\mathbf{0}) = \mathbf{0}$. φ is a complete pseudo-Boolean morphism if φ is a pseudo-Boolean morphism and preserves the operation $\bigvee$, i.e. if $\bigvee^{A} \mathfrak{A}$ exists in A, then $\varphi(\bigvee^{A} \mathfrak{A}) = \bigvee^{B} \varphi'' \mathfrak{A}$.

When A and B are Boolean algebras, a function $\varphi: A \to B$ is a Boolean morphism if it is a pseudo-Boolean morphism. Notice that $a \Rightarrow \mathbf{0}$ is the complement of a, when A is a Boolean algebra and consequently φ preserves the complement operation $-$.

φ is a complete Boolean morphism if it is a Boolean morphism and preserves the operation $\bigvee$.

PROPOSITION 2. *Let* $\varphi: A \to B$ *be a pseudo-Boolean morphism, where A and B are pseudo-Boolean algebras. Then,* $\varphi \upharpoonright R(A)$ *is a Bsolean morphism which maps* $R(A)$ *into* $R(B)$. *In addition, if* φ *is complete, then* $\varphi \upharpoonright R(A)$ *is also complete.*

PROOF. A routine by Prop. 1.

COROLLARY. *Let* $f: X \to Y$ *be an open continuous function. Then,* $f^{-1} \upharpoonright O(Y)$ *is a complete pseudo-Boolean morphism from* $O(Y)$ *to* $O(X)$. *And consequently* $f^{-1} \upharpoonright RO(Y)$ *is a complete Boolean morphism from* $RO(Y)$ *to* $RO(X)$. *Besides, if f is surjective,* $f^{-1} \upharpoonright O(Y)$ *is injective.*

PROOF. Let O and P are open subsets of Y.

$$\begin{aligned} f^{-1}(O \Rightarrow P) &= f^{-1}\,\mathrm{int}\,(O^c \cup P) \\ &= \mathrm{int}\,((f^{-1}O)^c \cup f^{-1}P) \\ &= f^{-1}O \Rightarrow f^{-1}P \end{aligned}$$

Similarly, the necessary property of f^{-1} for the fact that $f^{-1} \upharpoonright O(Y)$ is a complete pseudo-Boolean morphism is proved.

The rest of the corollary is clear by Prop. 2.

For $X \subseteq A$, X is dense if for any $a \in A$, $a \neq \mathbf{0}$, there is an element $x \in X$ such that $\mathbf{0} \neq x \leqq a$.

If B_0 is a Boolean algebra, every complete Boolean algebra that contains B_0 as a dense subalgebra is isomorphic to each other. (See [6].) So, such a complete Boolean algebra is called the canonical completion of B_0.

PROPOSITION 3. *Let A be a complete pseudo-Boolean algebra which contains A_0 as a dense subalgebra. Then, $R(A)$ is the canonical completion of $R(A_0)$.*

PROOF. By Prop. 1 and Prop. 2, $R(A)$ is a complete Boolean algebra which contains $R(A_0)$ as a subalgebra. It is easy to check that $R(A_0)$ is dense in $R(A)$.

COROLLARY. *Let X be a topological space and V be a base of X which is a subalgebra of $O(X)$. Then, $RO(X)$ is the canonical completion of $R(V)$.*

PROOF. Clear by Prop. 3.

§ 2. Limit systems.

Let M be a directed set, i.e. M is partially ordered by $\leqq$ and for each two elements $\alpha, \beta \in M$ there is an element γ of M such that $\alpha \leqq \gamma$ and $\beta \leqq \gamma$.

$\langle A_\alpha, \varphi_{\alpha\beta}\,;\ \alpha, \beta \in M \text{ and } \alpha \leqq \beta \rangle$ is a direct system of pseudo-Boolean algebras, if A_α is a pseudo-Boolean algebra for each $\alpha \in M$ and $\varphi_{\alpha\beta}$ is a pseudo-Boolean morphism which maps A_α into A_β for $\alpha, \beta \in M$, where $\varphi_{\alpha\alpha}$ is the identity and $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ holds for each $\alpha, \beta, \gamma \in M$ such that $\alpha \leqq \beta$ & $\beta \leqq \gamma$.

$\langle A_\alpha, \varphi_{\alpha\beta} : \alpha, \beta \in M \text{ and } \alpha \leqq \beta \rangle$ is a direct system of complete pseudo-Boolean algebras, if it is a direct system of pseudo-Boolean algebras and A_α is complete and $\varphi_{\alpha\beta}$ is a complete pseudo-Boolean morphism.

A direct system of Boolean algebras and a direct system of complete Boolean algebras are defined similarly. A direct system is injective, if its morphisms are injective.

$\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ is an inverse system of topological spaces, if X_α is a topological spaces for each $\alpha\in M$ and $f_{\alpha\beta}$ is a continuous function which maps X_β into X_α for each $\alpha, \beta\in M(\alpha\leqq\beta)$ where $f_{\alpha\alpha}$ is the identity and $f_{\alpha\beta}\circ f_{\beta\gamma}=f_{\alpha\gamma}$ for each $\alpha, \beta, \gamma\in M$ such that $\alpha\leqq\beta$ & $\beta\leqq\gamma$.

$\varinjlim_{\alpha\in M} A_\alpha$ is the ordinary direct limit of an direct system $\langle A_\alpha, \varphi_{\alpha\beta}; \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ in the category of pseudo-Boolean algebras. And, $\varphi_{\alpha\infty}$ is the morphism that maps A_α into $\varinjlim_{\alpha\in M} A_\alpha$ naturally. We shall use the same notations for the Boolean algebras.

$\varprojlim_{\alpha\in M} X_\alpha$ is the ordinary inverse limit of an inverse system $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ of topological spaces. (See [3] or [5].)

$f_{\alpha\infty}$ is the function that maps $\varprojlim_{\alpha\in M} X_\alpha$ into X_α naturally. $\{f_{\alpha\infty}^{-1}O; \alpha\in M$ & O is open in $X_\alpha\}$ is a topological base for the topological space $\varprojlim_{\alpha\in M} X_\alpha$.

An inverse system $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta\in M$ & $\alpha\leqq\beta\rangle$ of topological spaces is reflective if for any non-empty open set $O\subseteq X_\alpha$, $f_{\alpha\infty}^{-1}O$ is non-empty for each $\alpha\in M$.

It is point-reflective if for any $x\in X_\alpha$ there is an element $y\in\varprojlim_{\alpha\in M} X_\alpha$ such that $x=f_{\alpha\infty}(y)$ for each $\alpha\in M$. Of course, the point-refectiveness implies the reflectiveness.

PROPOSITION 4. *Let $\langle A_\alpha, \varphi_{\alpha\beta}; \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ be a direct system of pseudo-Boolean algebras. Then, $\varinjlim_{\alpha\in M} A_\alpha$ is a pseudo-Boolean algebra and $\varphi_{\alpha\infty}$ is a pseudo-Boolean morphism. And, $\langle R(A_\alpha), \varphi_{\alpha\beta}\restriction R(A_\alpha); \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ is a direct system of Boolean algebras. Consequently, $R(\varinjlim_{\alpha\in M} A_\alpha)=\varinjlim_{\alpha\in M} R(A_\alpha)$. If $\varphi_{\alpha\beta}$ is complete for each $\alpha, \beta\in M$ such that $\alpha\leqq\beta$, $\varphi_{\alpha\infty}$ is also complete.*

PROOF. A routine.

PROPOSITION 5. *Let $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ be an inverse system of topological spaces and $f_{\alpha\beta}$ be an open function for each $\alpha, \beta\in M$ such that $\alpha\leqq\beta$. Then, $\langle O(X_\alpha), f_{\alpha\beta}^{-1}\restriction O(X_\alpha); \alpha, \beta\in M$ and $\alpha\leqq\beta\rangle$ is a direct system of complete pseudo-Boolean algebras.*

If the inverse system is point-reflective, $\langle O(X_\alpha), f_{\alpha\beta}^{-1}\restriction O(X_\alpha); \alpha, \beta$ $M, \alpha\leqq\beta\rangle$ is injective and $\varinjlim_{\alpha\in M} O(X_\alpha)$ is a dense subalgebra of $O(\varprojlim_{\alpha\in M} X_\alpha)$.

If the inverse system is reflective, $\langle RO(X_\alpha), f_{\alpha\beta}^{-1}\restriction RO(X_\alpha); \alpha, \beta\in M, \alpha\leqq\beta\rangle$

is injective and $RO(\varprojlim_{\alpha\in M} X_\alpha)$ *is the canonical completion of* $\varinjlim_{\alpha\in M} RO(X_\alpha)$.

PROOF. By Corollary to Prop. 2, the first proposition is obvious. The point-reflectiveness implies that $f_{\alpha\beta}$ is an open continuous surjection for $\beta\in M$ or $\beta=\infty$. So, the second one holds.

The reflectiveness implies that $f''_{\alpha\beta}X_\beta$ is dense open in X_α. So, $f_{\alpha\beta}^{-1}\restriction RO(X_\alpha)$ is injective. Let $i: \varinjlim_{\alpha\in M} RO(X_\alpha)\to RO(\varprojlim_{\alpha\in M} X_\alpha)$ be a function such that $i_{\alpha\infty}(b)=\operatorname{int} f_{\alpha\infty}^{-1}b$ for such α that $b\in RO(X_\alpha)$.

Then, i is an injective Boolean morphism and the range of i is dense in $RO(\varprojlim_{\alpha\in M} X_\alpha)$. Now, the third proposition has been proved.

§ 3. The chain condition.

In this chapter, κ stands always for a regular cardinal and λ stands always for an ordinal where an ordinal is a set of ordinals less than it and a cardinal is an initial ordinal. $cf'\lambda$ is the cofinality of λ, i.e. $cf'\lambda$ is the least cardinal of X such that $X\subseteq\lambda$ & $\sup X=\lambda$.

A subset X of λ is closed unbounded if X is unbounded in λ and closed under the order-topology.

A subset S of λ is stationary if S intersects with every closed unbounded subset X of λ.

A pseudo-Boolean algebra A satisfies κ-c.c., if every subset of A each two elements of which have the meet 0 has the cardinality less than κ.

A topological space X satisfies κ-c.c., if every pair-wise disjoint family of open subsets of X has the cardinality less than κ.

The following is clear.

PROPOSITION 6. *A pseudo-Boolean algebra A satisfies κ-c.c. if and only if $R(A)$ satisfies it.*

And a topological space X satisfies κ-c.c. if and only if $RO(X)$ satisfies it.

PROPOSITION 7. *Let $\varphi: \mathcal{B}\to C$ be a complete Boolean morphism, where $\mathcal{B}$ is complete and satisfies κ-c.c.*

Then, $\varphi''\mathcal{B}$ is a complete subalgebra of C and satisfies κ-c.c.

PROOF. The fact that $\varphi''\mathcal{B}$ is a complete subalgebra of C is clear.

Let S be a pairwise disjoint family of non-zero elements of $\varphi''\mathcal{B}$ and $b_0=\bigvee\{b;\ \varphi(b)=0\}$.

Let s' be an element of $\mathcal{B}$ such that $\varphi(s')=s$ for $s\in S$.

Then, $\varphi(s'-b_0)=\varphi(s')-\varphi(b_0)=s$ for $s\in S$. And so, $s\neq t(s, t\in S)$ implies $s'-b_0\neq t'-b_0$. Now, the family $\{s'-b_0;\ s\in S\}$ is a pairwise disjoint family of non-zero elements with the same cardinality of S. Hence, the second proposition holds.

PROPOSITION 8. *Let $\langle\mathcal{B}_\alpha, \varphi_{\alpha\beta};\ \alpha, \beta\in M, \alpha\leqq\beta\rangle$ be a direct system of complete Boolean algebras and $\mathcal{B}$ be the canonical completion of its diredt limit and φ be a complete Boolean morphism that maps $\mathcal{B}$ into C.*

Then, $\varphi''\mathcal{B}$ is the canonical completion of $\bigcup_{\alpha\in M}\varphi\circ\varphi''_{\alpha\infty}\mathcal{B}_\alpha$.

PROOF. Easy to check.

We shall restate the theorem of [11], concerning the chain condition of a complete Boolean algebra, with a little modification and give an out-line of the proof. So, if one desires the complete proof, see §6 of [11].

From now on, a limit system is always a wellordered limit system, i.e. M is an ordinal. So, instead of M we use the notation λ.

THEOREM 1. *Let $\langle B_\alpha, \varphi_{\alpha\beta};\ \alpha, \beta<\lambda$ and $\alpha\leqq\beta\rangle$ be a direct system of complete Boolean algebras, where B_α satisfies κ-c.c. for each $\alpha<\lambda$. If $cf'\lambda\neq\kappa$, then the direct limit also satisfies κ-c.c.. If $cf'\lambda=\kappa$ and $\{\alpha;\ B_\alpha$ is the canonical completion of $\varinjlim_{\beta<\alpha} B_\beta\}$ is stationary in λ, then the direct limit also satisfies κ-c.c..*

Outline of the proof. The proof in the case $cf'\lambda\neq\kappa$ is a trivial one, so we omit it. In the case $cf'\lambda=\kappa$, without any loss of generality, we may assume $\lambda=\kappa$ and $\varphi_{\alpha\beta}$ is an inclusion map for each α, β by Prop. 7 and Prop. 8. Let $h_{\alpha\beta}:\ B_\beta\to B_\alpha(\alpha<\beta<\kappa$ or $\beta=\infty)$ be the basic projection, i.e. $h_{\alpha\beta}(b)=\bigwedge^{B_\alpha}\{c;\ b\leqq i_{\alpha\beta}(c)\ \&\ c\in B_\alpha\}$. $h_{\alpha\beta}(i_{\alpha\beta}(b))=b$ for each $b\in B_\alpha$ and $h_{\alpha\beta}\circ h_{\beta\gamma}=h_{\alpha\gamma}$ for each $\alpha, \beta, \gamma<\kappa(\gamma=\infty)$ such that $\alpha\leqq\beta$ and $\beta\leqq\gamma$. Let S be a subset of $\varinjlim_{\alpha<\kappa} B_\alpha$ such that the meet of any different two elements of S is equal to 0. By the assumption, we may assume $\varinjlim_{\alpha<\kappa} B_\alpha=\bigcup_{\alpha<\lambda} B_\alpha$. Let $S_\alpha=S\cap B_\alpha$ and $\sigma_\alpha=\bigvee^{A_\alpha}h_{\alpha\infty}''S$. By κ-c.c. of B_α, there exists an ordinal function $\phi:\ \kappa\to\kappa$ such that $\sigma_\alpha=\bigvee^{B_\alpha}h_{\alpha\infty}''S_{\phi(\alpha)}$. The subset $\{\gamma:\ \phi(\alpha)<\gamma$ holds for any $\alpha<\gamma\}$ is closed unbounded in κ. So, there exists an ordinal γ_0 such that $\phi(\alpha)<\gamma_0$ for any $\alpha<\gamma_0$ and B_{γ_0} is the canonical completion of $\bigcup_{\alpha<\gamma_0} B_\alpha$. We claim that $S=S_{\gamma_0}$.

Otherwise, there exists $b\in S-S_{\gamma_0}$. Then, $h_{\gamma\infty}(b)\wedge y=0$ for each $y\in S_{\gamma_0}$. There exist z and $\alpha_0<\gamma_0$ such that $0<z\leqq h_{\gamma_0\infty}(b)$ and $z\in B_{\alpha_0}$. Then, $z\wedge y=0$ for each $y\in S_{\gamma_0}$, and so $z\wedge\sigma_{\alpha_0}=z\wedge\bigvee^{B_{\alpha_0}}h_{\alpha_0\infty}''S_{\phi(\alpha_0)}\leqq z\wedge\bigvee^{B_{\alpha_0}}h_{\alpha_0\gamma_0}''S_{\gamma_0}=0$ and this contradicts to the fact $h_{\alpha_0\infty}(b)<\sigma_{\alpha_0}$.

S_{γ_0} is of the cardinality less than κ, by κ-c. c. of B_{γ_0}. So, the cardinality of S is less than κ.

COROLLARY 1. *Let $\langle A_\alpha, \varphi_{\alpha\beta}; \alpha, \beta<\lambda$ and $\alpha\leqq\beta\rangle$ be a direct system of complete pseudo-Boolean algebras. If $cf'\lambda\neq\kappa$ and A_α satisfies κ-c. c. for each $\alpha<\lambda$, then $\varinjlim_{\alpha<\lambda} A_\alpha$ satisfies κ-c. c.. If $cf'\lambda=\kappa$ and A_α satisfies κ-c. c. for each $\alpha<\lambda$ and $\{\alpha; \varinjlim_{\beta<\alpha} A_\beta$ is dense in $A_\alpha\}$ is stationary in λ, then $\varinjlim_{\alpha>\lambda} A_\alpha$ satisfies κ-c. c..*

PROOF. Clear by Prop. 2, Prop. 3, Prop. 4, Prop. 6 and Th. 1.

COROLLARY 2. *Let $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta<\lambda$ and $\alpha\leqq\beta\rangle$ be a reflective inverse system of topological spaces and $f_{\alpha\beta}$ be open. If $cf'\lambda\neq\kappa$ and X_α satisfies κ-c. c. for each $\alpha<\lambda$, then $\varprojlim_{\alpha<\lambda} X_\alpha$ also satisfies κ-c. c..*

If $cf'\lambda=\kappa$ and X_α satisfies κ-c. c. for each $\alpha<\lambda$ and $\{\alpha: X_\alpha=\varprojlim_{\beta<\alpha} X_\beta\}$ is stationary in λ, then $\varprojlim_{\alpha<\lambda} X_\alpha$ also satisfies κ-c. c.

PROOF. By Prop. 5 and Prop. 6, the κ-c. c. of $\varinjlim_{\alpha<\lambda} RO(X_\alpha)$ implies the κ-c. c. of $\varprojlim_{\alpha<\lambda} X_\alpha$. $X_\alpha=\varprojlim_{\gamma<\alpha} X_\gamma$ and the reflectiveness imply that $\langle X_\gamma, f_{\gamma\delta}; \gamma<\alpha$ and $\gamma\leqq\delta\rangle$ is reflective. So, by Prop. 5, $RO(X_\alpha)$ is the canonical completion of $\varinjlim_{\gamma<\alpha} RO(X_\gamma)$. Now, the corollary is clear.

COROLLARY 3.* *Let $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta<\lambda$ and $\alpha\leqq\beta\rangle$ be an inverse system of topological spaces and $f_{\alpha\beta}$ be an open surjection and $X_\alpha=\varprojlim_{\gamma<\alpha} X_\gamma$ for each limit α.*

If X_α satisfies κ-c. c. for each $\alpha<\lambda$, then $\varprojlim_{\alpha<\lambda} X_\alpha$ also satisfies κ-c. c.

REMARK 1. In § 2, we have investigated the relationship between the inverse limit of topological spaces and the direct limit of their open algebras. There is a more direct relationship between the direct limit of topological spaces and the inverse limit of their open algebras. $\langle A_\alpha, i_{\alpha\beta}; \alpha, \beta\in M, \alpha\leqq\beta\rangle$ is a weak inverse system of complete pseudo-Boolean algebras, if A_α is a complete pseudo-Boolean algebra and $i_{\alpha\beta}: A_\beta\to A_\alpha$ is a complete weakly pseudo-Boolean morphism, i. e. $i_{\alpha\beta}$ preserves $\bigvee$ and $\wedge$ and $i_{\alpha\beta}(\mathbf{1})=\mathbf{1}$.

Let $\varprojlim^w_{\alpha\in M} A_\alpha$ be an inverse limit of the above system. Then, $\varprojlim^w_{\alpha\in M} A_\alpha$ is a complete pseudo-Boolean algebra and the related morphism $i_{\alpha\infty}: \varprojlim^w_{\alpha\in M} A_\alpha\to A_\alpha$ is a complete weakly pseudo-Boolean morphism.

* This result is well-known and can be seen in [13].

Now, let $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta\in M, \alpha\leqq\beta\rangle$ be a direct system of topological spaces, i.e. $f_{\alpha\beta}$ is a continuous function which maps X_α into X_β. Then, $\langle O(X_\alpha), f_{\alpha\beta}^{-1}\restriction O(X_\beta); \alpha, \beta\in M, \alpha\leqq\beta\rangle$ is a weak inverse system of complete pseudo-Boolean algebras and $\varprojlim^w_{\alpha\in M} O(X_\alpha)$ is isomorphic to $O(\varinjlim_{\alpha\in M} X_\alpha)$, where $\varinjlim_{\alpha\in M} X_\alpha$ is a usual direct limit of topological spaces. See [3], for the definition of the direct limit of topological spaces.

REMARK 2. Here, we show that the conditions of the preceding theorem or corollaries are somewhat necessary. For that purpose, we give two examples. For the simplicity, we treat the case of ω_1-c. c., where ω_1 is the least uncountable cardinal. Usually, ω_1-c. c. is called the countable chain condition, abbreviated by c. c. c..

EXAMPLE 1. This example is a trivial one. Let $X_\alpha=\alpha+1$, ie. the set of ordinals less than or equal to α. The topology of X_α is the usual ordertopology. Let $f_{\alpha\beta}(x)=x$ for $x<\alpha$ and $f_{\alpha\beta}(x)=\alpha$ for $x\geqq\alpha$ for each $\alpha, \beta; \alpha\leqq\beta$. Then, $f_{\alpha\beta}$ is a continuos surjection form X_β to X_α for $\alpha, \beta; \alpha\leqq\beta$. For limit α, $X_\alpha=\varinjlim_{\gamma<\alpha} X_\gamma$ holds. And the system is point-reflective. X_α satifies c. c. c. for $\alpha<\omega_1$, but X_{ω_1}, of course, does not satisfies c. c. c..

EXAMPLE 2. For this example, we use an ω_1-Aronszajn tree. See [2] or [7], for an Aroszajn tree. Let $T=\langle T, \leqq\rangle$ be the ω_1-Aronszajn that is defined using the rational numbers, for example, one in §1 of [7]. Then, T satisfies the following:

i) T is an ω_1-tree without an ω_1-branch.

ii) Every level of T is countable.

iii) T contains an uncountable antichain.

Let $T_\alpha=\{x; x\in T$ and level of x is less than $\alpha\}$, and $B(T_\alpha)$=the set of branches through T_α, where a branch is a maximal linear ordered subset of T_α.

Now, we induce a topology into $B(T)$ and $B(T_\alpha)$ in the following way. $\{b; b\in B(T)\ \&\ x\in b\}$ is a basic open set of $B(T)$ for each $x\in T$. $\{b; b\in B(T_\alpha)\ \&\ x\in b\}$ is a basic open set of $B(T_\alpha)$ for each $x\in T_\alpha$. In addition, $\{b\}$ is a basic open set of $B(T_\alpha)$, if there exists $y\in T$ such that $x\leqq y$ holds for any $x\leqq b$.

Let $f_{\alpha\beta}: B(T_\beta)\to B(T_\alpha)$ be the function defined by the postulate: $f_{\alpha\beta}(b)=b\cap T_\alpha$, for each $\alpha, \beta; \alpha\leqq\beta$. Then, $f_{\alpha\beta}$ is an open continuous surjection and $B(T)=\varprojlim_{\alpha<\omega_1} B(T_\alpha)$. And the system is point-reflective. $B(T_\alpha)$ satisfies c. c. c. by ii), but

$B(T)$ does not satisfy c. c. c. by iii).

The countable chain codition is deeply concerned by Martin's axiom. Under Martin's axiom and the negation of the continuum hypothesis, c. c. c. is equivalent to the precaliber property. (See [12] or [4].) The precaliber property is preserved under the cartesian product of topological spaces. But, the cartesian product of a Suslin line with itself does not satisfy c. c. c. (See [9].) We can ask whether the precaliber property is preserved under the restricted limit system as in the theorem or corollaries. We do not know the answer. But, we know the following two facts. The precaliber property is preserved under the systems that satisfy the condition of Theorem 1, Corollary 1 or Corollary 2, replaced "stationary" by "closed unbounded". This is a corollary of the proof in [8]. A Suslin line is not the inverse limit of the inverse system of spaces which have the precaliber property, which satisfies the condition of Corollary 2.

For, if S is the inverse limit of the retricted inverse system $\langle X_\alpha, f_{\alpha\beta}; \alpha, \beta < \lambda\rangle$, where X_α has the precaliber property for each α, then $S \times S = \varprojlim_{\alpha<\lambda} X_\alpha \times X_\alpha$. $X_\alpha \times X_\alpha$ also has the precaliber property for each α and so satisfies c. c. c. Consequently, $S \times S$ satisfies c. c. c. by the corollary. On the other hand, as stated before, the cartesian product of a Suslin line with itself does not satisfy c. c. c.. And so, S cannot be a Suslin line.

References

[1] Birkhoff, G., Lattice Theory, Amdrican Mathematical Society Colloquim Publications Vol. 25 (1948).

[2] Devlin, k. J., The Suslin Problem, Lecture Notes in Mathematics 405 Springer-Verlag (1974).

[3] Dugundji, J., Topology, Allyn and Bacon, Inc. (1966).

[4] Eda, K., Some property of a Boolean algebra and a topological space, unpublished (1976).

[5] Engelking, R., Outline of General Topology, North-holland publishing Company (1968).

[6] Halmos, P. R., Lectures on Boolean algebras, D. Van Nostrand, (1963).

[7] Jech, T. J., Trees, The Journal of Symbolic logic Vol. 36 No. 1 (1971).

[8] Jech, T. J., Lectures in Set Theory, Lecture Notes in Mathematics 217 (1971).

[9] Kurepa, D., The cartesian multiplication and the cellularity number Publications de L institut Mathematique tome 2 (16) (1962).

[10] Rasiowa, H. and Sikorski, R., The mathematics of metamathematics, Monografie Matematyczne, Tom 41 (1966).

[11] Solovay, R. M. and Tennembaum, S., Iterated Cohen extensions and Souslin's Problem, Annals of mathematics, Vol. 94, (1971).

[12] Tall, F. D., The countable chain condition versus separability-Applications of Martin's axiom, General Topology and its applications 4, (1974).

[13] Takeuti, G. and Zaring, W.M., Axiomatic Set Theory Springer-Verlag, (1973).