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MINIMAL IMMERSION OF SURFACES IN QUATERNIONIC PROJECTIVE SPACES

By

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Abstract. For a minimal immersion of a surface in a quaternionic Kähler manifold a concept of non-degeneracy is defined. Then using a theorem on elliptic differential systems we show a non-degenerate surface is in a sense generic, and around each point with possible exception of an isolated set of degenerate points we can define a smooth Darboux frame. The frame is continuous at a degenerate point.

Next, by reducing the structure group we define a symmetric sextic form of type $(6, 0)$ and we show in the case that ambient space is $\boldsymbol{HP}^n$ this form is a holomorphic (abelian) differential. The last section is a brief note on the relation of our work to Glazebrook's twistor spaces for $\boldsymbol{HP}^n$.

Introduction.

In recent years the study of minimal immersions of oriented surfaces into compact manifolds especially S^n, $\boldsymbol{CP}^n$ and $\boldsymbol{HP}^n$ has attracted a lot of attention. One basic idea originally due to H. Hopf is to define on the strface symmetric differentials of type $(1, 0)$ by reducing the structure group. The minimality condition is used next to prove they are holomorphic. The surfaces for which these forms vanish form a class that in many cases can be constructed using holomorphic or algebraic maps. For the case of S^n cf. Calabi [2], Chern [3], Chern-Wolfson [4] and Bryant [1], also for $\boldsymbol{CP}^n$ cf. Chern-Wolfson [4] and Wolfson [9]. For a different approach cf. Eells-Wood [6] in case of $\boldsymbol{CP}^n$ and Glazebrook [7], [8] for $\boldsymbol{HP}^n$.

In this paper we use the method of moving frames and the structure equations of quaternionic Kähler manifolds (cf. [10]) to study minimal immersions of oriented surfaces in $\boldsymbol{HP}^n$. First we define a concept of non-degeneracy for immersions of surfaces in quaternionic Kähler manifolds. Then we prove that non-degenerate minimal immersions are in a sense generic. Finally as an in-

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variant of a minimal immersion in $\boldsymbol{HP}^n$, $n>1$, we define a sextic differential form on the surface and show it is in fact a holomorphic (abelian) differential. The last section is a brief note on the relation of our work to Glazebrook's and his twistor spaces.

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§ 0. Notations and Conventions.

For basic definitions and notations about the skew field of quaternions, cf. Chevalley [5].

We recall that if $q \in \boldsymbol{H}$, where $\boldsymbol{H}$ is the skew field of quaternions, then

$$q=q'+jq'', \qquad q', q'' \in \boldsymbol{C}.$$

In the same way for any $A \in W \otimes_{\boldsymbol{R}} \boldsymbol{H}$, where W is a real vector space.

$$A=A'+jA'', \qquad A', A'' \in W \otimes_{\boldsymbol{R}} \boldsymbol{C}.$$

We use the above notation of prime and double prime throughout the paper, which are called complex and quaternionic imaginary parts, respectively.

We also notice that if $\theta, \omega \in \Lambda(T^*M) \otimes_{\boldsymbol{R}} \boldsymbol{H}$ are of degree p, q respectively, then

$$\overline{\theta \wedge \omega}=(-1)^{pq} \bar{\omega} \wedge \bar{\theta}.$$

Also the extension of the symmetric product is no longer symmetric but it satisfies

$$\overline{\theta \cdot \omega}=\bar{\omega} \cdot \bar{\theta},$$

where θ, ω are as above.

We agree, unless otherwise stated, on the following range of indices:

$$\begin{aligned}
&1 \leqq i, j, \cdots \leqq 2 \\
&1 \leqq A, B, \cdots \leqq N \text{ or } 4n \\
&3 \leqq r, s, \cdots \leqq N \text{ or } 4n \\
&1 \leqq \alpha, \beta, \cdots \leqq n \\
&3 \leqq \lambda, \mu, \cdots \leqq n .
\end{aligned}$$

We also assume M is an oriented surface, X a Riemannian manifold and $f: M \rightarrow X$ is an immersion.

§ 1. Quaternionic Kähler Manifolds.

For more detail regarding this section cf. Zandi [10].

A quaternionic metric manifold X is a Riemannian manifold of dimension

$4n$ where the structure group can be reduced to $Sp(n)\cdot Sp(1)=Sp(n)\times Sp(1)/\text{Center}\subset SO(4n)$. In other words around each point $p\in X$ there is a coframe $\{\theta_1, \cdots, \theta_{4n}\}$ called admissible coframe such that the forms

$$\omega_\alpha=1\otimes\theta_\alpha+i\otimes\theta_{n+\alpha}+j\otimes\theta_{2n+\alpha}+k\otimes\theta_{3n+\alpha} \tag{1.1}$$

in $T^*X\otimes_{\boldsymbol{R}}\boldsymbol{H}$ are defined up to $q_{\alpha\beta}\omega_\beta\bar{q}$ where $(q_{\alpha\beta})\in Sp(n)$ and $q\in Sp(1)$.

Using the $Sp(1)$ part of the structure group we can locally define three automorphisms of TX, F_1, F_2, F_3 which are called the standard automorphisms. In more detail let $\{e_1, \cdots, e_{4n}\}$ be a frame dual to an admissible coframe and let

$$\left.\begin{aligned} &F_1(e_\alpha)=-e_{n+\alpha}\\ &F_1(e_{n+\alpha})=e_\alpha\\ &F_1(e_{2n+\alpha})=e_{3n+\alpha}\\ &F_1(e_{3n+\alpha})=-e_{2n+\alpha}.\end{aligned}\right\} \tag{1.2}$$

Similarly using multiplication by j and k on the right hand side of ω_α, F_2 and F_3 can be defined. Notice that $\{F_1, F_2, F_3\}$ is defined up to an element of $SO(3)$.

DEFINITION 1.1. Let X be a quaternionic metric manifold. Let $\{\omega_\alpha\}$ and $\{F_1, F_2, F_3\}$ be as above. X is called Kähler if there exist quaternionic forms $\{\omega_{\alpha\beta}\}$ and ω such that

$$\left.\begin{aligned} &d\omega_\alpha=\sum_\beta\omega_{\alpha\beta}\wedge\omega_\beta+\omega_\alpha\wedge\omega\\ &\omega_{\alpha\beta}+\bar{\omega}_{\beta\alpha}=0\\ &\omega+\bar{\omega}=0.\end{aligned}\right\} \tag{1.3}$$

Equivalently X is quaternionic Kähler if there exist real forms $\{\varphi_{ij}\}$ such that

$$\left.\begin{aligned} &DF_i=\varphi_{ij}F_j\\ &\varphi_{ij}+\varphi_{ji}=0,\end{aligned}\right\} \tag{1.4}$$

where D is the Riemannian covariant derivative.

We call $\omega_{\alpha\beta}$, ω the quaternionic connection forms of X and define the quaternionic curvature forms as follows,

$$\left.\begin{aligned} &\Omega_{\alpha\beta}=d\omega_{\alpha\beta}-\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}\\ &\Omega=d\omega-\omega\wedge\omega.\end{aligned}\right\} \tag{1.5}$$

REMARK 1.2. When $n\geqq 2$, X has constant scaler curvature R and

$$\Omega=-\frac{R}{16n(n+2)}\sum_\alpha\bar{\omega}_\alpha\wedge\omega_\alpha. \tag{1.6}$$

REMAK 1.3. Under change of quaternionic coframe

$$\tilde{\omega}_\alpha = q_{\alpha\beta}\omega_\beta\bar{q}\,, \tag{1.7}$$

the connection and curvature forms change according to

$$\left.\begin{aligned}\tilde{\omega}_{\alpha\beta} &= \sum dq_{\alpha\gamma}\cdot\bar{q}_{\beta\gamma} + \sum q_{\alpha\gamma}\omega_{\gamma\lambda}\bar{q}_{\beta\lambda}\\ \tilde{\omega} &= q\omega\bar{q} - qd\bar{q}\,.\end{aligned}\right\} \tag{1.8}$$

$$\left.\begin{aligned}\tilde{\Omega}_{\alpha\beta} &= q_{\alpha\gamma}\Omega_{\gamma\lambda}\bar{q}_{\beta\lambda}\\ \tilde{\Omega} &= q\Omega\bar{q}\,.\end{aligned}\right\} \tag{1.9}$$

REMARK 1.4. For $X = \boldsymbol{HP}^n$ which is of constant quaternionic sectional curvature we have

$$\left.\begin{aligned}\Omega_{\alpha\beta} &= -\omega_\alpha\wedge\bar{\omega}_\beta\\ \Omega &= -\sum_\alpha \bar{\omega}_\alpha\wedge\omega_\alpha\,.\end{aligned}\right\} \tag{1.10}$$

§ 2. General Theory of Minimal Immersions in Quaternionic Kähler Manifolds.

Let $f: M \to X$ be an immersion of an oriented surface in a Riemannian manifold X of dimension N. Let $\{e_1, \cdots, e_N\}$ be an orthonormal frame along M on X, so that $\{e_1, e_2\}$ is an oriented orthonormal frame for M (tangent to M). The dual coframe $\{\theta_1, \cdots, \theta_N\}$ satisfies

$$f^*\theta_r = 0 \text{ on } M \quad \text{for } 3 \leqq r \leqq N,$$

and $\{f^*\theta_1, f^*\theta_2\}$ forms an oriented orthonormal coframe on M.

A frame (coframe) as above is called a Darboux frame (coframe) for $f: M \to X$.

Let $\{\theta_{AB}\}$ be the Levi-Civita connection forms of X with respect to a Darboux coframe. Then the second fundamental forms II_r, $3 \leqq r \leqq N$ are defined as follows:

$$\mathrm{II}_r = \sum_{i=1}^{2}\theta_i\cdot\theta_{ir} = -\sum\theta_i\theta_{ri} \qquad \text{(symmetric product)}$$

where by θ_i, θ_{ir} we mean $f^*\theta_i$, $f^*\theta_{ir}$ but from now one we drop f^* when there is no danger of confusion. We also put

$$\begin{cases}\theta_{ri} = \sum h_{rij}\theta_j\\ h_{rij} = h_{rji}\,.\end{cases}$$

Hence

$$\mathrm{II}_r = \sum_{i,j} h_{rij}\theta_i\theta_j\,.$$

DEFINITION 2.1. Let $f: M \to X$ be an immersion. f is minimal if the traces of all the second fundamental forms vanish. In otheer words

$$\sum_r h_{rii}=0, \qquad 3 \leqq r \leqq N. \tag{2.1}$$

It is well known that $\varphi=\theta_1+i\theta_2$ defines an integrable almost complex (hence complex) structure on M by choosing φ to be a (1.0) form. Now we have the following whose proof is clear.

PROPOSITION 2.2. *Let $f: M \to X$ be a minimal immersion. Let $\{\theta_1, \cdots, \theta_N\}$ be a Darboux coframe. Then*

$$\left.\begin{aligned} \theta_{r1}+i\theta_{r2} &\equiv 0 \mod \bar{\varphi} \\ \theta_{r1}-i\theta_{r2} &\equiv 0 \mod \varphi . \end{aligned}\right\} \tag{2.2}$$

Let $f: M \to X$ be a minimal immersion where M is an oriented surface and X a quaternionic Kähler manifold of real dimension $4n>4$. Let $x \in X$ and let $\{e_1, e_2\}$ be an orthonormal frame on M around x. It is clear that whether $\{e_1, e_2\}$ are quaternionic linearly independent as vectors of TX is independent of the choice of the frame $\{e_1, e_2\}$. Hence we can have the following definition:

DEFINITION 2.3. $x \in M$ is a non-degenerate point or f is non-degenerate at x if $\{e_1, e_2\}$ as above are quaternionic linearly independent as vectors of T_xX.

We also need the following:

DEFINITION 2.4. Let $f: M \to X$ be an immersion. Let $x \in M$. A quaternionic coframe $\{\omega_\alpha\}$ around $f(x) \in X$ is called a quaternionic Darboux coframe if

$$\omega_3=\cdots=\omega_n=0 \quad \text{on } M. \tag{2.4}$$

PROPOSITION 2.5. *Let $f: M \to X$ be an immersion. Then around each non-degenerate $x \in M$ there exists a smooth quatermionic Darboux coframe.*

PROOF. Let $x \in M$ be a non-degenerate point and $\{e_1, e_2\}$ an orthonormal frame around x. Since $\{e_1, e_2\}$ is quaternionic linearly independent at x it is linearly independent in a neighborhood of x. We complete $\{e_1, e_2\}$ to a quaternionic basis for TX and dualize to get a quaternionic Darboux coframe around $f(x) \in X$.

When $x \in M$ is degenerate we have the following:

THEOREM 2.6. *Let $f: M \to X$ be a minimal immersion. Then f is degenerate in a neighborhood of any non-isolated degenerate point. Moreover around every* *isolated degenerate point there exists a continuously defined quaternionic Darboux coframe.*

PROOF. Let $\{e_1, e_2\}$ be an oriented orthonormal frame around $x\in M$. Let $\{F_1, F_2, F_3\}$ be a set of standard transformation of TX as in (1.2). It is clear that x is a non-degenerate (degenerate) point if the dimension of the space generated by $\{e_1, e_2, F_\rho(e_i);\ 1\leqq i\leqq 2,\ 1\leqq\rho\leqq 3\}$ is eight (four). Let $x\in M$ be a degenerate point and let

$$\vec{v}=(\langle F_1(e_1), e_2\rangle, \langle F_2(e_1), e_2\rangle, \langle F_3(e_1), e_2\rangle)\in \boldsymbol{R}^3,$$

where $\langle\ ,\ \rangle$ is the Riemannian inner product. Since $\vec{v}\neq 0$, by applying an element of $SO(3)$ if necessary we can change F_1, F_2, F_3 so that $\vec{v}=(0, \alpha, 0)$. Hence we can assume $\langle F_1(e_1), e_2\rangle=0$ and complete $\{e_1, e_2\}$ to a Riemannian orthonormal frame $\{e_A\}$, $1\leqq A\leqq 4n$, for X along M such that $e_{2n+a}=F_1(e_a)$, $1\leqq a\leqq 2n$. Let $F_2=u_{AB}e_A^*\otimes e_B$. From $\langle F_3(e_1), e_2\rangle=0$ we get $u_{i,2n+j}=0$, $1\leqq i, j\leqq 2$.

Consider now the following set of functions defined in a neighborhood U of X:

$$\{u_{1\lambda}, u_{1,2n+\lambda}, u_{2\lambda}, u_{2,2n+\lambda}\}, \qquad 3\leqq\lambda\leqq 2n.$$

It is clear that these functions vanish simultaneously at a point of U if and only if the point is degenerate. To keep things under control we list the formulas we need.

$$\left.\begin{array}{ll}
\{e_A\},\ 1\leqq A\leqq 4n, & \text{is a Darboux frame for } f: M\to X \text{ such that} \\
 & e_{2n+a}=F_1(e_a),\ 1\leqq a\leqq 2n. \\
\{\theta_A\},\ 1\leqq A\leqq 4n, & \text{is the Darboux coframe dual to } \{e_A\}. \\
\varphi=\varphi_1+i\theta_2 & \text{is a basis for the space of (1, 0)-form on } M. \\
\langle F_3(e_1), F_2\rangle=0 & \\
F_2=u_{AB}\theta_A\otimes e_B, & u_{AB}\in 0(4n). \\
u_{AB}+u_{BA}=0. & \\
u_{i,2n+j}=0, & 1\leqq i, j\leqq 2.
\end{array}\right\} \tag{2.4}$$

Let $\{\theta_{AB}\}$ be the Levi-Civita connection forms. Then

$$\left.\begin{array}{l}
De_A=\sum\theta_{AB}\otimes e_B \\
\theta_{AB}+\theta_{BA}=0,
\end{array}\right\} \tag{2.5}$$

where D is the covariant derivative.

Since $f: M\to X$ is minimal, from Prop. 2.2 we have

$$\theta_{ri}-i\theta_{r2}\equiv 0 \bmod \varphi, \qquad 3\leqq r\leqq 4n. \tag{2.6}$$

Since X is quaternionic Kähler from (1.4) we have

$$DF_2=\varphi_{21}F_1+\varphi_{23}F_3. \tag{2.7}$$

We differentiate $F_2=u_{AB}\theta_A\otimes e_B$ to get

$$DF_2=\sum(du_{AB}+u_{CB}\theta_{CA}+u_{AC}\theta_{CB})\theta_A\otimes e_B=\varphi_{21}F_1+\varphi_{23}F_3. \tag{2.8}$$

Let

$$F_1=P_{AB}\theta_A\otimes e_B \tag{2.9}$$

Then

$$\left.\begin{aligned}&P_{ab}=P_{2n+a,2n+b}=0\\&P_{a,2n+b}=\delta_{ab}\\&P_{2n+a,b}=-\delta_{ab}.\end{aligned}\right\} \tag{2.10}$$

Since $F_3=F_2F_1$, we have

$$F_3=u_{AC}P_{CB}\theta_A\otimes e_B. \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.8) we obtain

$$du_{AB}+u_{CB}\theta_{CA}+u_{AC}\theta_{CB}=\varphi_{21}P_{AB}+\varphi_{23}u_{AC}P_{CB}. \tag{2.12}$$

Using (2.12) and (2.6) we get for $3\leqq\lambda,\ \mu\leqq n$,

$$\left.\begin{aligned}&du_\lambda\equiv[\sum(i\theta_{12}\delta_{\lambda\mu}+\theta_{\lambda\mu})u_\mu+\sum(\varphi_{23}\delta_{\lambda\mu}+\theta_{\lambda,2n+\mu})u_{2n+\mu}]\bmod\varphi\\&du_{2n+\lambda}\equiv[\sum(-\varphi_{23}\delta_{\lambda\mu}+\theta_{2n+\lambda,\mu}u_\mu)u_\mu+\sum(i\theta_{12}\delta_{\lambda\mu}+\theta_{\lambda\mu})u_{2n+\mu}]\bmod\varphi,\end{aligned}\right\}$$

where $u_\lambda=u_{1\lambda}-iu_{2\lambda}$ and $u_{2n+\lambda}=u_{1,2n+\lambda}-iu_{2,2n+\lambda}$.

Since the above system satisfies the conditions of Thm. in section 4 in Chern [3] we obtain that $\{u_\lambda, u_{2n+\lambda}\}$, $3\leqq\lambda\leqq 2n$, either identically vanish in a neighborhood of x or they have an isolated zero at x. In other words f is either degenerate in a neighborhood of x or x is an isolated degenerate point. This completes the first part of the theorem.

To complete the proof we consider a smooth map on a neighborhood U of an isolated degenerate point $x\in M$,

$$F: U-\{x\}\longrightarrow G_r(8;TX),$$

where $G_r(8;TX)$ is the Grassman bundle of 8-planes in TX. From (2) in section 4 of Chern [3] F can be extended continuously to U, so that $F(x)$ would be contained in T_xM. Therefore in a neighborhood of x we can continuously choose a quaternionic basis $\{e_\alpha\}$, $1\leqq\alpha\leqq n$, for TX so that $e_1, e_2\in F(y)$, $y\in U$. The dual quaternionic coframe gives a desired Darboux coframe.

DEFINITION 2.7. A minimal immersion $f: M\to X$ is degenerate if it is degenerate at every $x\in M$.

REMARK 2.8. From Thm. 2.6. we observe that if f is not degenerate then the set of degenerate points are isolated. From now on we assume f is non-degenerate.

§ 3. Reduucton of the Structure Group.

We recall from § 1, our convention on the range of indices. Let M be an oriented connected surface, X a quaternionic Kähler manifold and $f: M \to X$ a non-degenerate [Remark (2.8)] minimal immersion. Let φ be a $(1,0)$-form on M such that $ds_M^2 = \varphi\bar{\varphi}$.

Let $x \in M$ be a non-degenerate point. From Prop. (2.5), there exists a quaternionic Darboux coframe $\{\omega_\alpha\}$ around x. Hence,

$$\left.\begin{aligned} & f^*\omega_1 = s_1\varphi + t_1\bar{\varphi} && \\ & && s_i,\ t_i \in \boldsymbol{H} \\ & f^*\omega_2 = s_2\varphi + t_2\bar{\varphi} && \\ & f^*\omega_\lambda = 0 && 3 \leqq \lambda \leqq n\,. \end{aligned}\right\} \tag{3.1}$$

From now on we drop f^* unless there is danger of confusion.

Notice that $\{\omega_\alpha\}$ is defined up to

$$\left.\begin{aligned} & \tilde{\omega}_\alpha = q_{\alpha\beta}\omega_\beta\bar{q} \\ & (q_{\alpha\beta}) \in Sp(n),\ q_{\lambda i} = q_{i\lambda} = 0,\ q \in Sp(1)\,. \end{aligned}\right\} \tag{3.2}$$

We proceed to reduce the structure group even further.

The Riemannian metric on X is defined by

$$ds^2 = \bar{\omega}_1 \cdot \omega_1 + \cdots + \bar{\omega}_n \cdot \omega_n\,.$$

Since $\omega_\lambda = 0$ on M, the induced metric on M, $ds_M^2 = \varphi\bar{\varphi}$ satisfies the following relation:

$$\bar{\omega}_1 \cdot \omega_1 + \bar{\omega}_2 \cdot \omega_2 = \varphi\bar{\varphi}\,. \tag{3.3}$$

Substituting (3.1) in (3.3) we obtain

$$\left.\begin{aligned} & |s_1|^2 + |s_2|^2 + |t_1|^2 + |t_2|^2 = 1 \\ & (\bar{s}_1 t_1 + \bar{s}_2 t_2)' = 0\,. \end{aligned}\right\} \tag{3.4}$$

From the first equation in (3.4) we obseve that (s_1, s_2), $(t_1, t_2) \in \boldsymbol{H}$ cannot both be the $\vec{0}$ vector. If necessary we can multiply ω_α's by j on the right to make $(s_1, s_2) \neq (0,0)$. Therefore there exists a smooth family of matrices $Q(x) \in Sp(2)$ such that

$$Q\,{}^t[s_1, s_2] = {}^t[\tilde{s}, 0] \quad \text{where } \tilde{s}_1 \in \boldsymbol{C}\,.$$

Let $Q\,{}^t[t_1, t_2] = {}^t[\tilde{t}_1, \tilde{t}_2]$. Put $Q = (q_{ij})$. We change ω_1, ω_2 by Q to get

$$\left.\begin{aligned}&\tilde{\omega}_1=\tilde{s}_1\varphi+\tilde{t}_1\bar{\varphi}, \quad \tilde{s}_1\in \boldsymbol{C}\\&\tilde{\omega}_2=\tilde{t}_2\bar{\varphi}.\end{aligned}\right\}\tag{3.5}$$

From the second equation in (3.4) we see that $(\tilde{t}_1)'=0$. Hence $\tilde{t}_1=j\tilde{\tau}$, where $\tilde{\tau}_1\in \boldsymbol{C}$. Now once more we transform $\tilde{\omega}_1$, $\tilde{\omega}_2$ by the matrix

$$\tilde{Q}=\begin{bmatrix}1 & 0\\ 0 & q\end{bmatrix}\in Sp(2)$$

where $q(x)$ is chosen smoothly so that $q\tilde{t}_2\in \boldsymbol{C}$. This can be done since $x\in M$ is non-degenerate and thus $\tilde{t}_2\neq 0$. Therefore we now have a quaternionic Darboux coframe as follows:

$$\left.\begin{aligned}&\omega_1=s\varphi+j\tau\bar{\varphi}\\&\omega_2=j\sigma\varphi t\bar{\varphi} \quad s,\ t,\ \sigma,\ \tau\in \boldsymbol{C}.\\&\omega_\lambda=0\end{aligned}\right\}\tag{3.6}$$

In the above normalization we actually made $\sigma=0$, but we write it as above to preserve symmetry. Notice that now the non-degeneracy condition at $x\in M$ is simply

$$s\bar{t}+\sigma\bar{\tau}\neq 0.\tag{3.7}$$

At this point we check how much the structure group has been reduced. Let $Q=(q_{ij})\in Sp(2)$, $q\in Sp(1)$ be such that $\tilde{\omega}_i=\sum q_{ij}\omega_j\bar{q}$ for $i=1, 2$ are in the normal form (3.6). Since multiplication on the right by $\bar{q}$ preserves the normal form (3.6) including the non-generacy condition (3.7), we only have to consider the change induced by Q. Writing the equations out and using the non-degeneracy condition (3.7) we easily get

$$\left.\begin{aligned}&Q=(q_{ij})=\begin{bmatrix}a_{11} & ja_{12}\\ ja_{21} & a_{22}\end{bmatrix}, \quad a_{11},\ a_{12},\ a_{21},\ a_{22}\in \boldsymbol{C} \quad \text{and}\\&U=\begin{bmatrix}\bar{a}_{11} & -a_{12}\\ \bar{a}_{21} & a_{22}\end{bmatrix}\in U(2).\end{aligned}\right\}\tag{3.8}$$

To see how the complex valued functions s, τ, σ, t in (3.6) are changed when ω_i's are changed by the reduced structure group (3.8), we calculate the effect of change on the matrix

$$S=\begin{bmatrix}\bar{s} & \tau\\ \bar{\sigma} & -t\end{bmatrix}.\tag{3.9}$$

Let $q\in \boldsymbol{H}$, $|q|=1$. Write $q=q'+jq''$ and define

$$V=\begin{bmatrix} q' & -q'' \\ \bar{q}'' & \bar{q}' \end{bmatrix} \in SU(2). \tag{3.10}$$

Then S changes according to

$$\tilde{S}=USV, \tag{3.11}$$

where U is defined in (3.8).

Now since S is non-degenerate [(3.7)] its first column is a non-zero vector. Hence by choosing U properly and smoothly, we can make it to be ${}^t[s, 0]$, $s\in \boldsymbol{C}$. Hence, S can be recuced to

$$\begin{bmatrix} \bar{s} & \tau \\ 0 & -t \end{bmatrix}.$$

Moreover by simple calculation we can make $\tau(x)=0$ (at one point). Hence (3.6) and (3.7) are reduced to

$$\left.\begin{aligned} &\omega_1=s\varphi+j\tau\bar{\varphi}=\varphi p \\ &\omega_2=t\bar{\varphi} \\ &\omega_\lambda=0 \\ &s,\ \tau,\ t\in \boldsymbol{C},\ \bar{s}t\neq 0,\ \tau(x)=0,\ p=s+j\tau,\ |p|^2+|t|^2=1. \end{aligned}\right\} \tag{3.12}$$

These last reductions are not geometrically significant, but they simplify the calculations.

§ 4. Minimality Condition.

Following the notations and normalizations of the previous section we let

$$\left.\begin{aligned} &\phi_1=\omega_1\bar{p}+t\bar{\omega}_2 \\ &\phi_2=\bar{t}\omega_1-\bar{\omega}_2 p \\ &\phi_\lambda=\omega, \quad 3\leqq\lambda\leqq n. \end{aligned}\right\} \tag{4.1}$$

Notice that

$$\sum_{\alpha=1}^{n} \bar{\phi}_\alpha\cdot\phi_\alpha=\sum_{\alpha=1}^{n} \bar{\omega}_\alpha\cdot\omega_\alpha. \tag{4.2}$$

Thus if we put

$$\phi_\alpha=\theta_{4\alpha-3}+i\theta_{4\alpha-2}+j\theta_{4\alpha-1}+k\theta_{4\alpha},$$

Then $\{\theta_A\}$, $1\leqq A\leqq 4n$, form an oriented orthonormal Riemannian coframe. Moreover since restricted to M we have

$$\left.\begin{aligned}&\phi^{1}=\varphi=\theta_1+i\theta_2\\&\phi_2=0\\&\phi_\lambda=0,\end{aligned}\right\}\qquad(4.3)$$

$\{\theta_A\}$ is a Darboux coframe for $f: M\rightarrow X$.

Now we differentiate ϕ_α's, evaluate them at x and use the fact that $\phi_\alpha=0$, $\alpha>1$, $\phi_1''=0$. Throughout we use the structure equation (1.3).

First we differentiate ϕ_1 to get

$$\left.\begin{aligned}d\phi_1&=d\omega_1\cdot\bar{s}-\omega_1\wedge d\bar{p}+dt\wedge\bar{\omega}_2+t\bar{\omega}_2\\&=(\omega_{11}\wedge\omega_1+\omega_{12}\wedge\omega_2+\omega_1\wedge\omega)\bar{s}-\omega_1\wedge d\bar{p}+dt\wedge\bar{\omega}_2\\&\quad+t\overline{(\omega_{21}\wedge\omega_1+\omega_{22}\wedge\omega_2+\omega_2\wedge\omega)}\\&=(\omega_{11}\wedge s\varphi+\omega_{12}\wedge t\bar{\varphi}+s\varphi\wedge\omega)\bar{s}-s\varphi\wedge d\bar{p}+dt\wedge\bar{t}\varphi\\&\quad+t(\bar{s}\bar{\varphi}\wedge\omega_{12}+\bar{t}\varphi\wedge\omega_{22}+\omega\wedge\bar{t}\varphi).\end{aligned}\right\}$$

Let

$$\left.\begin{aligned}S_1&=|s|^2\omega_{11}+t\omega\bar{t}\\S_2&=|t|^2\omega_{22}-sd\bar{p}+s\omega\bar{s}\end{aligned}\right\}\qquad(4.4)$$

Since $\phi_1''=0$ on M, we put $d\phi_1''=0$ in the above calculations to get

$$(S_1''-s\bar{t}\omega_{12}'')\wedge\varphi-((S_2''-\bar{s}t)\omega_{12}'')\wedge\bar{\varphi}=0.$$

Hence by Cartan's lemma we obtain

$$\left.\begin{aligned}S_1''-s\bar{t}\omega_{12}''&=\alpha\varphi+\beta\bar{\varphi}\\S_2''-\bar{s}t\omega_{12}''&=-\beta\varphi+\gamma\bar{\varphi}\end{aligned}\right\}\qquad(4.5)$$

Next we differentiate ϕ_2. Similar calculations as above and separating the complex parts of quaternionic forms we obtain

$$\left.\begin{aligned}sR_1'-\bar{t}R_2'&=\alpha'\varphi+\beta'\bar{\varphi}\\\omega_{12}'&=\beta'\varphi+\gamma'\bar{\varphi}\end{aligned}\right\}\qquad(4.6)$$

and

$$\left.\begin{aligned}sR_1''+s^2\omega_{12}''&=\alpha''\varphi+\beta''\bar{\varphi}\\-tR_2''+t^2\omega_{12}''&=\beta''\varphi+\gamma''\bar{\varphi},\end{aligned}\right\}\qquad(4.7)$$

where R_1 and R_2 are defined by

$$\left.\begin{aligned}R_1&=d\bar{t}+\bar{t}\omega_{11}-\omega\bar{t}\\R_2&=dp-\omega_{22}s+s\omega.\end{aligned}\right\}\qquad(4.8)$$

Finally differentiating $\phi_\lambda=0$ gives

$$\left.\begin{aligned}&\omega_{\lambda 1}s=a\varphi+b\bar{\varphi}\\&\omega_{\lambda 2}t=b\varphi+c\bar{\varphi}\,,\quad a,\,b,\,c\in\boldsymbol{H}.\end{aligned}\right\}\tag{4.9}$$

Using the minimality of the immersion, from (2.2) and uniqueness of the Riemannian connection forms, we conclude that β, β', β'' and b in (4.5), (4.6), (4.7) and (4.9) all vanish. Therefore after substituting for S_1, S_2, R_1 and R_2 from (4.4) and (4.8) into (4.5), (4.6) and (4.7) we obtain

$$\left.\begin{aligned}&|s|^2\omega''_{11}+\bar{t}^2\omega''-s\bar{t}\omega''_{12}\equiv 0 \bmod \varphi\\&|t|^2\omega''_{22}+\bar{s}\,d\tau+\bar{s}^2\omega''-\bar{s}t\omega''_{12}\equiv 0 \bmod \bar{\varphi}\\&s(d\bar{t}+\bar{t}\omega'_{11}-\bar{t}\omega')-\bar{t}(ds-s\omega'_{22}+s\omega')\equiv 0 \bmod \varphi\\&\omega'_{12}\equiv - \bmod \bar{\varphi}\\&s(t\omega''_{11}-\bar{t}\omega'')+s^2\omega''_{12}\equiv 0 \bmod \varphi\\&-t(d\tau-s\omega''_{22}+\bar{s}\omega'')+t^2\omega''_{12}\equiv 0 \bmod \bar{\varphi}\\&\omega_{\lambda 1}=a_{\lambda}\varphi\,,\quad a_{\lambda}\in\boldsymbol{H}\\&\omega_{\lambda 2}=c_{\lambda}\bar{\varphi}\,,\quad c_{\lambda}\in\boldsymbol{H}.\end{aligned}\right\}\tag{4.10}$$

From the fifth and sixth equations of (4.10) we get respectively

$$\begin{aligned}&\bar{t}\omega''-s\omega''_{12}\equiv t\omega''_{11} \bmod \varphi\\&\bar{s}\omega-t\omega''_{12}+d\tau\equiv s\omega''_{22} \bmod \bar{\varphi}\,.\end{aligned}$$

Substituting these into the first and second equations of (4.10) respectively, we obtain

$$\begin{aligned}&|s|^2\omega''_{11}+|t|^2\omega''_{11}=\omega''_{11}\equiv 0 \bmod \varphi\\&|t|^2\omega''_{22}+|s|^2\omega''_{22}=\omega''_{22}\equiv 0 \bmod \bar{\varphi}\,.\end{aligned}$$

Hence these and the fourth equation of (4.10) give

$$\left.\begin{aligned}&\omega''_{11}\equiv 0 \bmod \varphi\\&\bar{\omega}''_{22}\equiv 0 \bmod \varphi\\&\omega'_{21}=-\bar{\omega}'_{12}\equiv 0 \bmod \bar{\varphi}\,.\end{aligned}\right\}\tag{4.11}$$

§ 5. Construction of the Sextic Form on M.

We recall from (3.1) and (3.8) that the structure group $Sp(n)\cdot Sp(1)$ has been reduced to the subgroup consisting of $A\cdot q$, where $A=(q_{\alpha\beta})$, $1\leqq\alpha$, $\beta\leqq n$, satisfies $q_{\lambda i}=q_{i\lambda}=0$ and $Q=q_{ij}$, $1\leqq j$, $j\leqq 2$ is as in (3.8).

Now under this reduced group we calculate the effect of change of quaternionic coframe ω_α to $\tilde{\omega}_\alpha=q_{\alpha\beta}\omega_\beta\bar{q}$, on the forms given in (4.11). From the equation for the change of connection forms (1.8) we have

$$\tilde{\omega}_{\alpha\beta}=dq_{\alpha\gamma}\cdot\bar{q}_{\beta\gamma}+q_{\alpha\gamma}\omega_{\gamma\lambda}\bar{q}_{\beta\lambda}.$$

Thus from the above we obtain

$$\tilde{\omega}_{11}=dq_{11}\cdot\bar{q}_{11}+dq_{12}\cdot\bar{q}_{12}+q_{11}\omega_{11}\bar{q}_{11}+q_{12}\omega_{21}\bar{q}_{11}+q_{11}\omega_{12}\bar{q}_{12}+q_{12}\omega_{22}\bar{q}_{22}. \tag{5.1}$$

$$\tilde{\omega}_{22}=dq_{21}\cdot\bar{q}_{21}+dq_{22}\cdot\bar{q}_{22}+q_{21}\omega_{11}\bar{q}_{21}+q_{22}\omega_{21}\bar{q}_{21}+q_{21}\omega_{12}\bar{q}_{22}+q_{22}\omega_{22}\bar{q}_{22}. \tag{5.2}$$

$$\tilde{\omega}_{21}=dq_{21}\cdot\bar{q}_{11}+dq_{22}\cdot\bar{q}_{12}+q_{21}\omega_{11}\bar{q}_{11}+q_{22}\omega_{21}\bar{q}_{11}+q_{21}\omega_{12}\bar{q}_{12}+q_{22}\omega_{22}\bar{q}_{12}. \tag{5.3}$$

From (3.8) we have $q_{11}=a_{11}$, $q_{22}=a_{22}$, $q_{12}=ja_{12}$, $q_{21}=ja_{21}$. Therefore we observe that the first two terms in (5.1) and (5.2) are complex and the first two terms in (5.3) are imaginary quaternionic. Hence these terms do not cortribute to the quaternionic imaginary parts of $\tilde{\omega}_{11}$, $\tilde{\omega}_{22}$ and the complex part of $\tilde{\omega}_{21}$. Thus we obtain

$$\left.\begin{aligned}
\tilde{\omega}''_{11}&=\bar{a}^2_{11}\omega''_{11}+2a_{12}\bar{a}_{11}\omega'_{21}+a^2_{12}\bar{\omega}''_{22}\\
\bar{\tilde{\omega}}''_{22}&=\bar{a}^2_{21}\omega''_{11}-2\bar{a}_{21}a_{22}\omega'_{21}+a^2_{22}\bar{\omega}''_{22}\\
\tilde{\omega}'_{21}&=-\bar{a}_{11}\bar{a}_{21}\omega''_{11}+(\bar{a}_{11}a_{22}-a_{12}a_{21})\omega'_{21}+a_{22}a_{12}\bar{\omega}''_{22}.
\end{aligned}\right\} \tag{5.4}$$

The relations in (5.4) can be summarized in the following matrix form:

$$\begin{bmatrix}\tilde{\omega}''_{11} & -\tilde{\omega}'_{21}\\ -\tilde{\omega}'_{21} & \bar{\tilde{\omega}}''_{22}\end{bmatrix}=U\begin{bmatrix}\omega''_{11} & -\omega'_{21}\\ -\omega'_{21} & \bar{\omega}''_{22}\end{bmatrix}{}^tU \tag{5.5}$$

where $U\in U(2)$ is defined in (3.8).

Using $\tilde{\omega}_\alpha=q_{\alpha\beta}\omega_\beta\bar{q}$ [(1.7)], we also calculate the effect of change of a coframe on $(\omega_1\cdot\omega_2)'$. First

$$\begin{aligned}\tilde{\omega}_1\cdot\bar{\tilde{\omega}}_2&=\sum q_{ij}\omega_j\bar{q}\cdot q\bar{\omega}_j\bar{q}_{\alpha j}=\sum q_{1i}\omega_i\bar{\omega}_j\bar{q}_{2j}\\
&=q_{11}\omega_1\bar{\omega}_1\bar{q}_{21}+q_{11}\omega_1\bar{\omega}_2\bar{q}_{22}+q_{12}\omega_2\bar{\omega}_1\bar{q}_{21}+q_{12}\omega_2\bar{\omega}_2\bar{q}^2{}_2.\end{aligned}$$

Hence taking the complex parts from both sides and using (3.8) we get

$$(\tilde{\omega}_1\cdot\bar{\tilde{\omega}}_2)'=(a_{11}\bar{a}_{22}+\bar{a}_{12}a_{21})(\omega_1\cdot\bar{\omega}_2)'=\overline{\det(U)}(\omega_1\cdot\bar{\omega}_2)'. \tag{5.6}$$

We also observe from (3.12) that

$$\omega_1\cdot\bar{\omega}_2=(s\varphi+a\tau\bar{\varphi})\cdot\bar{t}\varphi=s\bar{t}\varphi^2+j\tau\bar{t}\bar{\varphi}^2.$$

Hence

$$(\omega_1\cdot\bar{\omega}_2)'=s\bar{t}\varphi^2.$$

Therefore $(\omega_1\cdot\bar{\omega}_2)'$ is a symmetric complex form of type (2, 0). Also from (4.11), the elements of the matrix

$$\begin{bmatrix} \omega''_{11} & -\omega'_{21} \\ -\omega'_{21} & \bar{\omega}''_{22} \end{bmatrix}$$

are 1-forms of type $(1,0)$. Thus the determinant of the above matrix $(\omega''_{11}\bar{\omega}''_{22}-(\bar{\omega}'_{21})^2)$ is a symmetric form of type $(2,0)$. Let

$$\Lambda=[(\omega_1\cdot\bar{\omega}_2)']^2\cdot\det\begin{bmatrix} \omega''_{11} & -\omega'_{21} \\ -\omega'_{21} & \bar{\omega}''_{22} \end{bmatrix}. \tag{5.9}$$

Then from (5.7) and (5.8), Λ is a sextic form of type $(6,0)$. From (5.5) and (5.6) and the fact that U is unitary hence $|\det(U)|=1$, we observe that Λ is globally defined.

§ 6. The Case of $X=\boldsymbol{HP}^n$.

When the ambient space X is $\boldsymbol{HP}'\cong S^4$, the problem has been extensively studied (cf. Calabi [2], Chern [3], Chern-Wolfson [4], Bryant [1]). In fact that case falls under the class of degenerate immersions by our definition (2.3). For $n>1$ we have the following:

THEOREM 6.1. *Let $f: M\to\boldsymbol{HP}^n$ be a non-degenerate minimal immersion, where M is an oriented surface. Then the* 6*-form Λ in* (5.9) *is a holomorphic* (*abelian*) *differential of order* 6 *on M.*

PROOF. Let $\varphi=\lambda dz$ where z is a local complex coordinate on M. From (3.12) and (4.11) we can write

$$\left.\begin{aligned} &\omega'_1=\alpha dz,\ \omega'_2=\beta dz,\ \omega''_1=\gamma d\bar{z},\ \omega''_2=0 \\ &\omega''_{11}=a dz,\ \omega''_{22}=b d\bar{z},\ \omega'_{21}=c dz. \end{aligned}\right\} \tag{6.1}$$

From structure equation (1.3) we have

$$d\omega_1=\omega_{11}\wedge\omega_1+\omega_{12}\wedge\omega_2+\omega_1\wedge\omega.$$

Hence

$$d\omega'_1=\omega'_{11}\wedge\omega'_1-\bar{\omega}''_{11}\wedge\omega''_1+\omega'_{12}\wedge\omega'_2-\bar{\omega}''_{12}\wedge\omega''_2+\omega'_1\wedge\omega'-\bar{\omega}''_1\wedge\omega''.$$

Therefore after substitution of (6.1) into the above, we obtain

$$d\omega'_1=d(\alpha dz)=d\alpha\wedge dz=(\alpha\omega'_{11}-\alpha\omega'+\bar{\gamma}\omega'')\wedge dz$$

Thus,

$$d\alpha\equiv(\alpha\omega'_{11}-\alpha\omega'+\bar{\gamma}\omega'')\bmod dz. \tag{6.2}$$

Similarly differentiating ω_2 gives

$$d\bar{\beta}\equiv(-\bar{\beta}\omega'_{22}+\bar{\beta}\omega'-\bar{\gamma}\omega''_{12})\bmod dz. \tag{6.3}$$

To find the derivatives of a, b, c we pɹoced as follows. From (1.5) and (1.10) we have

$$d\omega_{\alpha\beta}=\sum_{\gamma}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}-\omega_{\alpha}\wedge\bar{\omega}_{\beta} \tag{6.4}$$

Using (6.4) we calculate $d\omega''_{11}$, $d\omega''_{22}$ and $d\omega'_{21}$. From (6.4) for $\alpha=\beta=1$ we have

$$d\omega_{11}=\omega_{11}\wedge\omega_{11}+\omega_{12}\wedge\omega_{21}+\sum_{\lambda}\omega_{1\lambda}\wedge\omega_{\lambda 1}-\omega_{1}\wedge\bar{\omega}_{1}. \tag{6.5}$$

From (4.10) and $\omega_{\alpha\beta}+\bar{\omega}_{\beta\alpha}=0$ we have

$$\sum\omega_{1\lambda}\wedge\omega_{\lambda 1}=\sum-\bar{\omega}_{\lambda 1}\wedge\omega_{\lambda 1}=-\sum\bar{\varphi}\bar{a}_{\lambda}\wedge a_{\lambda}\varphi=(\sum_{\lambda}|a_{\lambda}|^{2})\varphi\wedge\bar{\varphi},$$

which is complex and hence does not contribute to $d\omega''_{11}$. Also from (3.12) we have

$$\omega_{1}\wedge\bar{\omega}_{1}=\varphi p\wedge\bar{p}\bar{\varphi}=|p|^{2}\varphi\wedge\bar{\varphi},$$

which is again complex. Therefore from (6.5) we obtain

$$\begin{aligned}d\omega''_{11}&=[(\omega'_{11}+j\omega''_{11})\wedge(\omega'_{11}+j\omega''_{11})+(\omega'_{12}+j\omega''_{12})\wedge(\omega'_{21}+j\omega''_{21})]''\\&=-2\omega'_{11}\wedge\omega''_{11}+\bar{\omega}'_{12}\wedge\omega''_{12}+\omega''_{12}\wedge\omega'_{21}.\end{aligned}$$

Hence substituting (6.1) into the above gives

$$d\omega''_{11}=d(adz)=da\wedge dz=(-2a\omega'_{11}+2c\omega''_{12})\wedge dz.$$

Thus

$$da\equiv 2(-a\omega'_{11}+c\omega''_{12}) \bmod dz. \tag{6.6}$$

Similarly differentiating ω_{22} and using (6.4) we get

$$d\bar{b}\equiv 2(\bar{b}\omega'_{22}-c\omega''_{12}) \bmod dz. \tag{6.7}$$

To calculate dc, we differentiate ω_{21} and use (6.4) as before to get,

$$d\omega_{21}=\omega_{21}\wedge\omega_{11}+\omega_{22}\wedge\omega_{21}+\sum_{\lambda}\omega_{2\lambda}\wedge\omega_{\lambda 1}-\omega_{2}\wedge\bar{\omega}_{1}.$$

From (4.10) we have

$$\sum_{\lambda}\omega_{2\lambda}\wedge\omega_{\lambda 1}=-\sum_{\lambda}\bar{\omega}_{\lambda 2}\wedge\omega_{\lambda 1}=-\sum\varphi\bar{c}_{\lambda}\wedge a_{\lambda}\varphi.$$

Let $\sum_{\lambda}\bar{c}_{\lambda}a_{\lambda}=A$. Then

$$\sum\omega_{2\lambda}\wedge\omega_{\lambda 1}=-\varphi A\wedge\varphi=-\varphi(A'+jA'')\wedge\varphi=jA''\varphi\wedge\bar{\varphi},$$

which is quaternionic imaginary and thus does not contribute to $d\omega'_{21}$. Also from (3.12) we obtain

$$\omega_{2}\wedge\bar{\omega}_{1}=t\bar{\varphi}\wedge(\overline{s\varphi}-j\tau\bar{\varphi})=(-j\bar{t}\tau)\varphi\wedge\bar{X},$$

which is again quaternionic imaginary. Therefore

$$d\omega'_{21}=(\omega_{21}\wedge\omega_{11}+\omega_{22}\wedge\omega_{21})',$$

and similar substitutions as before lead to

$$d\omega'_{21}=d(c\,dz)=dc\wedge dz=(-c\omega'_{11}+c\omega'_{22}-a\bar{\omega}''_{12}+\bar{b}\omega''_{12})\wedge dz\,.$$

Hence

$$dc\equiv(-c\omega'_{11}+c\omega'_{22}-\bar{a}\omega'_{12}+\bar{b}\omega''_{12})\bmod dz\,. \tag{6.8}$$

Now from (6.2) and (6.3) we get

$$d(\alpha\bar{\beta})^2\equiv 2\alpha\bar{\beta}[\alpha\bar{\beta}(\omega'_{11}-\omega'_{22})+\bar{\gamma}(\bar{\beta}\omega''-\alpha\omega''_{12})]\bmod dz\,. \tag{6.9}$$

From (3.12) and (6.1) we have

$$\omega'_1=s\varphi=\alpha\,dz$$
$$\omega'_2=t\bar{\varphi}=\beta\,d\bar{z}\,.$$

Using $\varphi=\lambda dz$, we get $\alpha=s\lambda$, $\beta=t\bar{\lambda}$. Hence from the sixth equation of (4.10) we obtain

$$\begin{aligned}\bar{\beta}\omega''-\alpha\omega''_{12}&=\bar{t}\lambda\omega''-s\lambda\omega''_{12}=\lambda(\bar{t}\omega''-s\omega''_{12})\\&\equiv 0\bmod\varphi\equiv 0\bmod dz\,.\end{aligned}$$

Substituting above in (6.9) we obtain

$$d(\alpha\bar{\beta})^2=2(\alpha\bar{\beta})^2(\omega'_{11}-\omega'_{22})\bmod dz\,. \tag{6.10}$$

Also using (6.6), (6.7) and (6.8) we obtain

$$d(a\bar{b}-c^2)=-2(a\bar{b}-c^2)(\omega'_{11}-\omega'_{22})\bmod dz\,. \tag{6.11}$$

Now for the proof of the theorem we recall from (5.9)

$$\Lambda=(\omega'_1\cdot\bar{\omega}'_2)^2[\omega''_{11}\bar{\omega}''_{22}-(\bar{\omega}'_{21})^2]=(\alpha\bar{\beta})^2(a\bar{b}-c^2)dz^6\,.$$

Hence from (6.10) and (6.11) we finally get

$$d[(\alpha\bar{\beta})^2(a\bar{b}-c^2)]\equiv 0\bmod dz\,.$$

Therefore Λ is holomorphic.

§ 7. Isotropic Minimal Surfaces in $\boldsymbol{HP}^n$.

In this section we try to explain the relation between our work and Glazebrook's isotropic minimal surfaces [7]. The section is brief and mainly restricted to $\boldsymbol{HP}^2$.

We recall that $\boldsymbol{HP}^n\cong Sp(n+1)/Sp(n)\times Sp(1)$. When the subgroup $Sp(n)\times Sp(1)$ is reduced to $Sp(n)\times U(1)$ we get $Sp(n+1)/Sp(n)\times U(1)\cong\boldsymbol{CP}^{2n+1}$ which is the standard twistor space. If $Sp(n)\times Sp(1)$ is reduced to $U(n)\times Sp(1)$ we get Glazebrook's twistor space (Glazebrook [7, 8]) which is again a complex manifold. We give the following definition of an isotropic minimal immersion in $\boldsymbol{HP}^n$.

For details cf. Glazebrook [7].

DEFINITION 7.1. A minimal immersion of a surface in $\boldsymbol{HP}^n$ is isotropic provided that it can be lifted in a "natural" way as a holomorphic or antiholomorphic map to $Z=Sp(n+1)/U(n)\times Sp(1)$.

Now in the case of $\boldsymbol{HP}^2$, for example, from (3.8) we see the $Sp(2)\times Sp(1)$ is reduced naturally to $U(2)\times Sp(1)$. Therefore the surface can be lifted not to the standard twistor space $Sp(3)/Sp(2)\times Sp(1)\cong \boldsymbol{CP}^5$ but to Glazebrook's twistor space $Z=Sp(3)/U(2)\times Sp(1)$.

The standard method of reduction of G-structures gives a set of quaternionic 1-forms $\{\phi_1, \phi_2, \phi_{ij}, 1\leqq i, j\leqq 2\}$ and 2-forms $\{\Psi_{ij}\}$ on Z. These forms satisfy the same structure equations as ω's [(1.3), (1.5), (1.6) and (1.10)]. The difference is that on Z, $\{\phi_1', \bar\phi_1'', \bar\phi_2', \phi_2'', \phi_{12}', \bar\phi_{11}'', \phi_{22}''\}$ form a basis for forms of type $(1,0)$. (The reason for this basis being non-symmetrical is that the embedding of $U(2)$ in $Sp(2)$ in (3.8) is not the natural one.) These $(1,0)$-forms satisfy the integrability conditions of Newlander-Nirenberg and define a complex structure on Z. It is easy to see that for the lift of M to Z we also havə the same equations for ϕ's as we had for ω's. Hence (3.12) and (4.11) become

$$\left.\begin{array}{l}\phi_1'\equiv\bar\phi_1''\equiv\bar\phi_2'\equiv\phi_2''\equiv 0 \bmod \varphi \\ \phi_{11}''\equiv\bar\phi_{22}''\equiv\bar\phi_{12}'\equiv 0 \bmod \varphi .\end{array}\right\} \tag{7.1}$$

Therefore to get a holomorphic curve in Z we must have $\phi_{11}''=\bar\phi_{22}''=\bar\phi_{12}'=0$, on M. Notice that the sextic form Λ in (5.9) is, up to a non-zero factor, equal to $\phi_{11}''\bar\phi_{22}''-(\bar\phi_{12}')^2$. Hence to get a holomorphic curve in Z it is necessary but not evidently sufficient to have $\Lambda\equiv 0$. This could expiain why not all minimal immersions of S^2 in (for instance) $\boldsymbol{HP}^2$ are isotropic, even though $\Lambda\equiv 0$ for S^2 (Riemann-Roch).

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