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# ARCWISE CONNECTEDNESS OF THE COMPLEMENT IN A HYPERSPACE

By

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**Abstract.** The hyperspace $C(X)$ of a continuum $X$ is always arcwise connected. In [6], S. B. Nadler Jr. and J. Quinn show that if $C(X)-\{A_i\}$ is arcwise connected for each $i = 1, 2$, then $C(X)-\{A_1,A_2\}$ is also arcwise connected. Nadler raised questions in his book [5]: *Is it still true with the two sets $A_1$ and $A_2$ replaced by n sets, n finite? What about countably many? What about a collection $\{A_\lambda : \lambda \in \Lambda\}$ which is a compact zero-dimensional subset of the hyperspace?* In this paper we prove that if $\mathcal{A} \subset C(X)$ is a closed countable subset, $\mathcal{U}$ is an arc component of an open set of $C(X)$ and $C(X)-\{A\}$ is arcwise connected for each $A \in \mathcal{A}$, then $\mathcal{U}-\mathcal{A}$ is arcwise connected.

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## 1. Notation and Preliminary Lemmas

A continuum is a nonempty compact connected metric space. The letter $X$ will always denotes a nondegenerate continuum with a metric function $d$. Let $Y$ be a subcontinuum of $X$ and $\varepsilon$ a positive number. The set $N(Y;\varepsilon)$ denotes the $\varepsilon$-neighborhood of $Y$ in $X$, *i.e.*, $N(Y;\varepsilon)=\{x \in X : d(x,y)<\varepsilon$ for some $y \in Y\}$ and $Y_\varepsilon$ denotes the component of the closure of $N(Y;\varepsilon)$ containing $Y$. The hyperspace $C(X)$ of $X$ is the space of all subcontinuum of $X$ with the Hausdorff metric $H_d$ defined by

$$H_d(A,B)=\inf\{\varepsilon>0 : A \subset N(B;\varepsilon) \text{ and } B \subset N(A;\varepsilon)\}.$$

With this metric, $C(X)$ becomes a continuum. If $Y$ is a subcontinuum of $X$, then

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we consider $C(Y)$ as a subspace of $C(X)$. For two subsets $\mathcal{A}$ and $\mathcal{B}$ of $C(X)$, let $H_d(\mathcal{A},\mathcal{B}) = \inf\{H_d(A,B): A \in \mathcal{A}$ and $B \in \mathcal{B}\}$. A map is a continuous function. Any map $\mu: C(X) \to [0,1]$ satisfying

(1) if $A \subset B$ and $A \neq B$, then $\mu(A) < \mu(B)$,

(2) $\mu(\{x\}) = 0$ for each $x \in X$ and $\mu(X) = 1$

is called a *Whitney map* for $C(X)$. Such a map always exists (see [7]). An *order arc* is a map $\sigma: [a,b] \to C(X)$ such that if $a \leq t_0 < t_1 \leq b$, then $\sigma(t_0) \subset \sigma(t_1)$ and $\sigma(t_0) \neq \sigma(t_1)$. It is also called an order arc from $\sigma(a)$ to $\sigma(b)$.

If $A$, $B$ are distinct elements of $C(X)$, then there is an order arc from $A$ to $B$ if and only if $A \subset B$ (see [1]).

We often use the following lemmas which are easy to prove hence we omit their proofs.

LEMMA 1. *Let $Y$ be a proper subcontinuum of $X$. If there is a subcontinuum $M$ of $X$ such that $M \cap Y \neq \phi \neq M - Y$, then for any $\varepsilon > 0$ and $y \in M \cap Y$, there is a subcontinuum $N$ of $M \cap Y_\varepsilon$ such that $N \cap Y \neq \phi \neq N - Y$ and $y \in N$.*

The diameter of a subset $A$ of $X$ is denoted by $\delta(A)$, *i.e.*, $\delta(A) = \sup\{d(x,y): x, y \in A\}$.

REMARK. If $\mathcal{A}$ is a connected subset of $C(X)$ such that $Y \in \mathcal{A}$ and $\delta(\mathcal{A}) \leq \varepsilon$, then $\mathcal{A} \subset C(Y_\varepsilon)$.

LEMMA 2. *If a subset $\{A,B,C,D\} \subset C(X)$ satisfies $A \subset B \cap C \subset B \cup C \subset D$, then $H_d(B,C) \leq H_d(A,D)$. In particular, if $\sigma$ is an order arc, then $\delta(\sigma([a,b])) = H_d(\sigma(a),\sigma(b))$.*

Furthermore we need the following Krasinkiewiz-Nadler's Theorem (Theorem 3.1 of [2]).

PROPOSITION 3. *Let $\mu: C(X) \to [0,1]$ be a Whitney map and $A_1, A_2 \in \mu^{-1}(t_0)$, where $t_0 \in [0,1]$. Let $K$ be a subcontinuum of $A_1 \cap A_2$. Then there is a map $\alpha: [0,1] \to \mu^{-1}(t_0) \cap C(A_1 \cup A_2)$ such that $\alpha(0) = A_1$, $\alpha(1) = A_2$ and $K \subset \alpha(t)$ for all $t \in [0,1]$. If $A_1 \neq A_2$, then $\alpha$ can be taken to be an emmbedding.*

In fact Theorem 3.1 of [2] is much more general, and from its proof we obtain the following lemma.

LEMMA 4. *Let $\mu: C(X) \to [0,1]$ be a Whitney map and let $A$, $B$, $C$ be* *subcontinua of X such that* $A \cap B \supset C$. *Then there is a map* $\alpha:[0,1] \to \mu^{-1}(\mu(A)) \cap C(A \cup B)$ *such that* $\alpha(0)=A, \alpha(t) \supset C$ *for each* $t \in [0,1]$, *and*

*if* $\mu(A) \leq \mu(B)$ *then* $\alpha(1) \subset B$,

*if* $\mu(A) > \mu(B)$ *then* $\alpha(1) \supset B$.

In the same paper they proved (Theorem 3.5 in [2]) that:

PROPOSITION 5. *Let X be decomposable and* $\mu: C(X) \to [0,1]$ *a Whitney map. Then there is* $s_0 \in [0,1)$ *such that if* $s \in [s_0,1]$, *then* $\mu^{-1}(s)$ *is arcwise connected.*

The following proposition is Theorem 4.6 of [4].

PROPOSITION 6. *If Y is a non-degenerate proper subcontinuum of X, then the following two statements are equivalent:*

(1) $C(X)-\{Y\}$ *is not arcwise connected.*

(2) *There is a dense subset D of Y such that if M is a subcontinuum of X satisfying* $M \cap D \neq \phi \neq M-Y$, *then* $M \supset Y$.

**2. Bypass Lemma**

Let $K, L \in C(X)$ and $\mathscr{A} \subset C(X)$. An arc from $K$ to $L$ in $\mathscr{A}$ is a map $\alpha:[a,b] \to \mathscr{A}$ such that $\alpha(a)=K$ and $\alpha(b)=L$. If $\alpha$ is an embedding, then we call it an embedding arc. Following is a key lemma.

LEMMA 7. *Let Y be a nondegenerate proper subcontinuum of a continuum X such that* $C(X)-\{Y\}$ *is arcwise connected. Let* $\alpha:[0,1] \to C(X)$ *be a map such that* $\alpha(1)=Y$ *and* $\alpha(t) \in C(Y)-\{Y\}$ *for each* $t \in [0,1)$. *Then for a given* $\varepsilon > 0$, *there is a map* $\beta:[0,1] \to C(X)-\{Y\}$ *such that* $\alpha(0)=\beta(0)$, $H_d(\alpha(t),\beta(t)) < \varepsilon$ *for each* $t \in [0,1]$ *and* $\beta(1)-Y \neq \phi$.

PROOF. First suppose that $Y$ is indecomposable. Put $\varepsilon_1 = \varepsilon/3$. Since $\alpha$ is continuous, there is $t_0 \in [0,1)$ such that $\delta(\alpha([t_0,1])) < \varepsilon_1$. Let $\lambda$ be the composant of $Y$ such that $\alpha(t_0) \subset \lambda$. By Lemma 1 and Proposition 6, there is a subcontinuum $M$ of $Y_{\varepsilon_1}$ such that $M-Y \neq \phi \neq Y-M$ and $M \cap \lambda \neq \phi$. We may assume that $M \cap \alpha(t_0) \neq \phi$. (Because let $\lambda'$ be a composant of $Y$ different from $\lambda$. Since $M$ is compact and $Y-M \neq \phi, \lambda'-M \neq \phi$. Thus we can replace $M$ by $M \cup N$, where $N$ is a continuum contained in $\lambda$ such that $M \cap N \neq \phi \neq N \cap \alpha(t_0)$.). Let $\sigma:[t_0,1] \to C(X)$ be an order arc from $\alpha(t_0)$ to $M \cup \alpha(t_0)$. Then

$$\delta(\sigma([t_0,1])) = H_d(\alpha(t_0), M \cup \alpha(t_0)) \le H_d(\alpha(t_0), Y_{\varepsilon_1})$$
$$\le H_d(\alpha(t_0), Y) + H_d(Y, Y_{\varepsilon_1}) < 2\varepsilon_1.$$

Define an arc $\beta$ in $C(X)$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0], \\ \sigma(t) & \text{if } t \in (t_0, 1]. \end{cases}$$

Clearly $\beta$ is continuous and its image does not contain $Y$. If $t \in [0, t_0]$, then $H_d(\alpha(t), \beta(t)) = 0$. Suppose that $t \in (t_0, 1]$. Then since $\alpha(t_0) = \beta(t_0)$,

$$H_d(\alpha(t), \beta(t)) \le H_d(\alpha(t), \alpha(t_0)) + H_d(\beta(t_0), \beta(t))$$
$$\le \delta(\alpha([t_0,1])) + \delta(\sigma([t_0,1])) < 3\varepsilon_1 = \varepsilon.$$

For the second case, suppose that $Y$ is decomposable. Put $\varepsilon_1 = \varepsilon/5$ and let $\mu$ be a Whitney map for $C(X)$. By Proposition 5, there is $s_0 < \mu(Y)$ such that if $s \in [s_0, \mu(Y)]$, then $\mu^{-1}(s) \cap C(Y)$ is arcwise connected. Moreover $s_0$ can be taken so that $\delta(\mu^{-1}([s_0,1]) \cap C(Y)) < \varepsilon_1$. Since $\alpha$ is continuous, there is $t_0 \in [0,1)$ such that $\mu(\sigma([t_0,1])) \subset [s_0, 1]$. For simplicity, put $s_1 = \mu(\alpha(t_0))$. By Proposition 6 and Lemma 1, there is a subcontinuum $M$ of $Y_{\varepsilon_1}$, such that $M - Y \ne \phi \ne Y - M$ and $M \cap Y \ne \phi$. There are two cases.

(i) Suppose there is $A \in \mu^{-1}(s_1) \cap C(Y)$ such that $A \cap M \ne \phi \ne Y - (A \cup M)$. Put $t_1 = (t_0 + 1)/2$ and let $\sigma_1 : [t_0, t_1] \to \mu^{-1}(s_1) \cap C(Y)$ be an arc from $\alpha(t_0)$ to $A$ (such an arc exists since $s_0 \le s_1 < \mu(Y)$) and $\sigma_2 : [t_1, 1] \to C(X)$ an order arc from $A$ to $A \cup M$. Note that $\delta(\sigma_2([t_1,1])) < 2\varepsilon_1$. Define an arc $\beta$ in $C(X)$ by

$$\beta(t) = \begin{cases} \alpha(t) & \text{if } t \in [0, t_0], \\ \sigma_1(t) & \text{if } t \in (t_0, t_1], \\ \sigma_2(t) & \text{if } t \in (t_1, 1]. \end{cases}$$

Clearly $\beta$ is continuous and $\beta(t) \ne Y$ for each $t \in [0,1]$. If $t \in [0, t_0]$, then $H_d(\alpha(t), \beta(t)) = 0$. Suppose $t \in [t_0, t_1]$. Then since $\beta([t_0, t]) \subset \mu^{-1}(s_1) \cap C(Y)$,

$$H_d(\alpha(t), \beta(t)) \le H_d(\alpha(t), \alpha(t_0)) + H_d(\beta(t_0), \beta(t))$$
$$\le \delta(\alpha[t_0,1])) + \delta(\mu^{-1}(s_1) \cap C(Y)) < 2\varepsilon_1 < \varepsilon.$$

Finally suppose $t \in [t_1, 1]$. Then

$$H_d(\alpha(t), \beta(t)) \le H_d(\alpha(t), \alpha(t_1)) + H_d(\alpha(t_1), \beta(t_1)) + H_d(\beta(t_1), \beta(t))$$
$$< \delta(\alpha([t_1,1])) + 2\varepsilon_1 + \delta(\sigma_2([t_1,1])) < \varepsilon_1 + 2\varepsilon_1 + 2\varepsilon_1 = 5\varepsilon_1 = \varepsilon.$$

(ii) Suppose that for each $A \in \mu^{-1}(s_1) \cap C(Y)$, $A \cap M \ne \phi$ implies $Y \subset A \cup M$. In this case, each element of $\mu^{-1}(s_1) \cap C(Y)$ intersects $M$. In particular,

$\alpha(t_0)\cap M\neq\phi$. Considering an order arc from $M$ to $M\cup Y$, we can enlarge $M$ and hence we can assume $\mu(M)>s_1$. By Lemma 4, there is a map $\sigma_1:[t_0,t_1]\to\mu^{-1}(s_1)\cap C(Y\cup M)$ from $\alpha(t_0)$ to $\sigma_1(t_1)\subset M$, where $t_1=(t_0+1)/2$. Let $\sigma_2:[t_1,1]\to C(X)$ be an order arc from $\sigma_1(t_1)$ to $M$. Define an arc $\beta$ in $C(X)$ by

$$\beta(t)=\begin{cases}\alpha(t) \text{ if } t\in[0,t_0],\\ \sigma_1(t) \text{ if } t\in(t_0,t_1],\\ \sigma_2(t) \text{ if } t\in(t_1,1].\end{cases}$$

As in case (i), $\beta$ satisfies all the required conditions.

Now we prove the main lemma.

BYPASS LEMMA 8. *Let $Y$ be a subcontinuum of $X$ such that $C(X)-\{Y\}$ is arcwise connected and let $\alpha:[0,1]\to C(X)$ be an arc such that $\alpha(t)=Y$ if and only if $t$ = 1/2. Then for each $\varepsilon>0$ and each a, b, where $0\le a<1/2<b\le 1$, there is a map $\beta:[0,1]\to C(X)-\{Y\}$ such that $\alpha(t)=\beta(t)$ for all $t\in[0,a]\cup[b,1]$ and $H_d(\alpha(t),\beta(t))<\varepsilon$ for all $t\in[0,1]$.*

PROOF. If $Y$ = $X$, then $X$ is decomposable (by Theorem 11.4 and Corollary 11.8 of [5]). Let $\mu$ be a Whitney map for $C(X)$. By Proposition 5, there is $s_0\in[0,1)$ such that $\mu^{-1}(s)$ is arcwise connected for each $s\in[s_0,1]$. Moreover $s_0$ can be chosen so that $\delta(\mu^{-1}[s_0,1])<\varepsilon/2$. Since $\alpha$ is continuous, there exist two numbers $t_0,t_1$ such that $a\le t_0<1/2<t_1\le b, \mu(\alpha(t_0))=\mu(\alpha(t_1))\in[s_0,1]$ and $\delta(\alpha([t_0,\ t_1]))<\varepsilon/2$. Put $\mu(\alpha(t_0))=s_1$. Then since $s_1\in[s_0,1]$, there is a map $\sigma:[t_0,t_1]\to\mu^{-1}(s_1)$ from $\alpha(t_0)$ to $\alpha(t_1)$. Define an arc $\beta$ in $C(X)-\{Y\}$by

$$\beta(t)=\begin{cases}\alpha(t) \text{ if } t\in[0,t_0]\cup[t_1,1],\\ \sigma(t) \text{ if } t\in(t_0,t_1).\end{cases}$$

If $t\in(t_0,t_1)$, then

$$\begin{aligned}H_d(\alpha(t),\beta(t))&\le H_d(\alpha(t),\alpha(t_0))+H_d(\sigma(t_0),\sigma(t))\\ &\le\delta(\alpha([t_0,t_1]))+\delta(\mu^{-1}(s_1))<\varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Therefore $\beta$ satisfies the required conditions.

Next suppose that $Y$ is a proper subcontinuum of $X$. Put $\varepsilon_1=\varepsilon/4$. There exist two numbers $t_0,t_1$ such that $a\le t_0<1/2<t_1\le b$ and $\delta(\alpha([t_0,t_1]))<\varepsilon_1$. Note that $\alpha([t_0,t_1])\subset C(Y_{\varepsilon_1})$. If $\alpha([t_0,1/2])\subset C(Y)$, then by Lemma 7, there is a map $\sigma:[t_0,1/2]\to C(Y_{\varepsilon_1})-\{Y\}$ such that $\sigma(t_0)=\alpha(t_0),\sigma(1/2)-Y\neq\phi$ and $H_d(\alpha(t),\sigma(t))<\varepsilon_1$ for each $t\in[t_0,1/2]$. If $\alpha([t_0,1/2])-C(Y)\neq\phi$, then put $\sigma=\alpha|[t_0,1/2]$. There is $r\in(t_0,1/2)$ such that $\sigma(r)-Y\neq\phi\neq Y_{\varepsilon_1}-\sigma(r)$. Let $\tau:[r,1/2]\to C(X)$ be an order arc from $\sigma(r)$ to $Y_{\varepsilon_1}$. Define $\beta_0:[t_0,1/2]\to C(X)$ by

$$\beta_0(t)=\begin{cases}\sigma(t) & \text{if } t\in[t_0,r],\\ \tau(t) & \text{if } t\in(r,1/2].\end{cases}$$

It is easy to see that $H_d(\alpha(t),\beta_0(t))<4\varepsilon_1$ for each $t\in[t_0,1/2]$.

As in the same way, we can find a map $\beta_1:[1/2,t_1]\to C(X)$ such that $\beta_1(1/2)=Y_{\varepsilon_1}, \beta_1(t_1)=\alpha(t_1)$ and $H_d(\alpha(t),\beta_1(t))<4\varepsilon_1$ for each $t\in[1/2,t_1]$. Then the arc $\beta$ defined by

$$\beta(t)=\begin{cases}\alpha(t) & \text{if } t\in[0,t_0]\cup[t_1,1],\\ \beta_0(t) & \text{if } t\in[t_0,1/2],\\ \beta_1(t) & \text{if } t\in[1/2,t_1]\end{cases}$$

satisfies the required conditions.

## 3. Arcwise Connectedness of the Complement

Let $\mathcal{Y}$ be a closed subset of $C(X)$ such that $C(X)-\{Y\}$ is arcwise connected for each $Y\in\mathcal{Y}$. We will show that if $\mathcal{Y}$ is a finite set, then its complement is also arcwise connected. Using this, we show that the same is fold if $\mathcal{Y}$ is a closed countable set. If $\mathcal{A}\subset\mathscr{C}(X)$ and $\varepsilon>0$, then we wright the $\varepsilon$-neighborhood of $\mathcal{A}$ in $C(X)$ by $N(\mathcal{A};\varepsilon)$.

THEOREM 9. *Let $\mathcal{Y}$ be a finite subset of $C(X)$ such that $C(X)-\{Y\}$ is arcwise connected for each $Y\in\mathcal{Y}$ and let $\alpha:[0,1]\to C(X)$ be an arc from $K$ to $L$, where $K,L\in C(X)-\mathcal{Y}$. Then for each $\varepsilon>0$, there is a map $\beta:[0,1]\to C(X)-\mathcal{Y}$ from $K$ to $L$ such that $\beta([0,1])\subset N(\alpha([0,1]);\varepsilon)$.*

PROOF. If $K=L$, then we can take $\beta$ to be a constant map. Hence let us suppose $K\neq L$. There is an embedding $\alpha':[0,1]\to\alpha([0,1])$ such that $\alpha(t)=\alpha'(t)$ for $t=0,1$. Therefore we can assume that $\alpha$ is an embedding arc and hence $\alpha^{-1}(\mathcal{Y})$ is a finite set. Let $\alpha^{-1}(\mathcal{Y})=\{t_1,t_2,\cdots,t_n\}$, where $0<t_i<t_{i+1}<1$ for $i=1,2,\cdots,n-1$.

(i) Suppose $n=1$ and without loss of generality, assume $t_1=1/2$. Put $\alpha(1/2)=Y$. Then $\alpha([0,1])$ and $\mathcal{Y}_1=\mathcal{Y}-\{Y\}$ are closed and disjoint. Put $\delta=H_d(\alpha([0,1]),\mathcal{Y}_1)$ and $\varepsilon_1=\min\{\varepsilon,\delta\}$. Then $\varepsilon_1>0$. Applying Bypass Lemma, there is a map $\beta:[0,1]\to C(X)-\{Y\}$ from $K$ to $L$ such that $H_d(\alpha(t),\beta(t))<\varepsilon_1$. By the choice of $\varepsilon_1,\beta$ satisfies the required conditions.

(ii) Suppose $k\geq 2$and the Theorem holds for $n=k-1$. Let $\alpha^{-1}(\mathcal{Y})=\{t_1,t_2,\cdots,t_k\}$ where $0<t_i<t_{i+1}<1$ for $i-1,2,\cdots,k-1$. Put $\delta=H_d(\alpha[t_0,1]),\mathcal{Y}-\{\alpha(t_k)\}$ and

$\varepsilon_1 = \min\{\varepsilon/2, \delta\}$, where $t_0 = (t_{k-1} + t_k)/2$. Then partially applying Bypass Lemma, there is a map $\beta_1 : [0,1] \to C(X)$ such that $\alpha | [0, t_0] = \beta_1 | [0, t_0], \alpha(1) = \beta_1(1)$, $H_d(\alpha(t), \beta_1(t)) < \varepsilon_1$ and $\beta_1([0,1])$ does not contain $\alpha(t_k)$. Let $\alpha_1$ be an embedding arc from $K$ to $L$ such that $\alpha_1([0,1]) \subset \beta_1([0,1])$. Then it is easy to see that the image of $\alpha_1$ intersects at most $n-1$ elements of $\mathcal{Y}$. Therefore by the inductive hypothesis, there is an arc $\beta$ from $K$ to $L$ in $C(X) - \mathcal{Y}$ such that $\beta([0,1]) \subset N(\alpha_1([0,1]); \varepsilon/2)$. Hence $\beta$ is a required arc.

COROLLARY 10. *Let $\mathcal{F}$ be a closed subset of $C(X)$ and let $\mathcal{A}$ be an arc component of $C(X) - \mathcal{F}$. If $\mathcal{Y}$ is a finite subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$, then $\mathcal{A} - \mathcal{Y}$ is arcwise connected.*

PROOF. Let $K$, $L$ be arbitrary elements of $\mathcal{A} - \mathcal{Y}$. There is a map $\alpha : [0,1] \to \mathcal{A}$ from $K$ to $L$. Put $\varepsilon = (1/2) H_d(\alpha([0,1]), \mathcal{F})$. Then $\varepsilon > 0$ and hence by Theorem 9, there is a map $\beta : [0,1] \to C(X) - \mathcal{Y}$ from $K$ to $L$ such that $\beta([0,1]) \subset N(\alpha([0,1]); \varepsilon)$. By the definition of $\varepsilon, N(\alpha([0,1]); \varepsilon) \cap \mathcal{F} = \phi$. Therefore $\beta$ is an arc in $\mathcal{A} - \mathcal{Y}$ from $K$ to $L$.

Let $A'$ denote the derived set of the space $A$. The derived set of $A$ of order $\lambda$ is defined by

$$A^{(1)} = A', \quad A^{(n+1)} = (A^{(n)})' \quad \text{and} \quad A^{(\lambda)} = \cap_{n<\lambda} A^{(n)}$$

if $\lambda$ is a limit ordinal (see [3]).

We say that a triple $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ is *admissible* if $\mathcal{F}$ is a closed subset of $C(X)$, $\mathcal{A}$ is an arc component of $C(X) - \mathcal{F}$, $\mathcal{Y}$ is a closed countable subset of $C(X)$ such that $C(X) - \{Y\}$ is arcwise connected for each $Y \in \mathcal{Y}$.

THEOREM 11. *If $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ is admissible, then $\mathcal{A} - \mathcal{Y}$ is arcwise connected.*

PROOF. First observe that the least ordinal $\nu$ such that $\mathcal{Y}^{(\nu)} = \phi$ (such an ordinal $\nu$ exists since $\mathcal{Y}$ does not contain perfect sets) is not a limit ordinal. Therefore there is the least ordinal $\lambda$ such that $\mathcal{Y}^{(\lambda)} = \phi$. Denote such the ordinal $\lambda$ by $\Lambda(\mathcal{Y})$. To prove the Theorem, we shall proceed by transfinite induction on $\Lambda(\mathcal{Y})$.

If $\Lambda(\mathcal{Y}) = 0$, then $\mathcal{Y}$ is a finite set. Hence Theorem follows from Corollary 10.

Suppose that the Theorem holds for any admissible triple $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ such that $\Lambda(\mathcal{Y}) < \lambda$. Let $\{\mathcal{F}, \mathcal{A}, \mathcal{Y}\}$ be an admissible triple such that $\Lambda(\mathcal{Y}) = \lambda$ and let $K$, $L$ be arbitrary elements of $\mathcal{A} - \mathcal{Y}$. It is sufficient to show that there is an arc from $K$ to $L$ in $\mathcal{A} - \mathcal{Y}$. Since $\mathcal{Y}^{(\lambda+1)} = \phi, \mathcal{Y}^{(\lambda)}$ is a finite set. Therefore by Corollary 10, there is a map $\alpha:[0,1]\to\mathscr{A}-\mathscr{Y}^{(\lambda)}$ from $K$ to $L$. Put $\varepsilon=(1/2)H_d(\alpha([0,1]),\mathscr{Y}^{(\lambda)})$, $\mathscr{Y}_1=\mathscr{Y}-N(\mathscr{Y}^{(\lambda)};\varepsilon)$, $\mathscr{F}_1=\mathscr{F}\cup\overline{N(\mathscr{Y}^{(\lambda)};\varepsilon)}$, where $\overline{N(\mathscr{Y}^{(\lambda)};\varepsilon)}$ is the closure of $N(\mathscr{Y}^{(\lambda)};\varepsilon)$ in $C(X)$, and let $\mathscr{A}_1$ be the arc component of $C(X)-\mathscr{F}_1$ containing $K$ (and hence $L$). Note that $\mathscr{A}_1\subset\mathscr{A}$. The triple $\{\mathscr{F}_1,\mathscr{A}_1,\mathscr{Y}_1\}$ is admissible and $\Lambda(\mathscr{Y}_1)<\lambda$. Hence by inductive hypothesis, there is an arc from $K$ to $L$ in $\mathscr{A}_1-\mathscr{Y}_1$. Since $\mathscr{A}_1-\mathscr{Y}_1\subset\mathscr{A}-\mathscr{Y}$ and $K$, $L$ are arbitrary elements of $\mathscr{A}-\mathscr{Y}$, $\mathscr{A}-\mathscr{Y}$ is arcwise connected.

COROLLARY 12. *If $\mathscr{Y}$ is a countable closed subset of $C(X)$ such that $C(X)-\{Y\}$ is arcwise connected for each $Y\in\mathscr{Y}$, then $C(X)-\mathscr{Y}$ is arcwise connected.*

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