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ON MINIMAL SURFACES WITH CONSTANT KÄHLER ANGLE IN CP^3 AND CP^4

By

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0. Introduction.

Let M be a 2-dimensional Riemannian manifold isometrically immersed in a Kähler manifold N with the complex structure J, and let $\{e_1, e_2\}$ be a local orthonormal frame on M. The Kähler angle α of M is defined to be the angle between Je_1 and e_2. The surface M is holomorphic, anti-holomorphic or totally real if and only if $\alpha = 0$, π or $\pi/2$, respectively. In [6] Chern and Wolfson pointed out that the Kähler angle plays an important role in the study of minimal surfaces in Kähler manifolds.

Here we consider the problem to classify minimal surfaces with constant Kähler angle in CP^n, where CP^n denotes the complex projective space of constant holomorphic sectional curvature 4. Concerning this problem, several results are known (see [1], [10], [8], [4] and [9]). In particular, we recall the following three facts. (I) A minimal surface with constant Kähler angle in CP^2 is either holomorphic, anti-holomorphic or totally real (see [6, (2.32)] and [8, Lemma 2.1]). (II) A pseudo-holomorphic minimal surface with constant Kähler angle in CP^3 is either holomorphic, anti-holomorphic, totally real or of constant curvature (see the proof of Theorem 9.1 of [1]). (III) A minimal 2-sphere with constant Kähler angle in CP^4 is either holomorphic, anti-holomorphic, totally real or of constant curvature (see [1, Theorem 9.1], cf. [8, p. 372]).

REMARK 1. (i) A minimal surface in CP^n is called pseudo-holomorphic if its harmonic sequence terminates at each end (see [3] and [5]).

(ii) Minimal surfaces with constant curvature and Kähler angle in CP^n are classified in [10].

In this paper we prove the following:

THEOREM 1. *Let M be a superconformal minimal surface with constant Kähler angle in CP^3. Then M is totally real.*

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THEOREM 2. *Let M be a pseudo-holomorphic minimal surface with constant Kähler angle in CP^4. Then M is either holomorphic, anti-holomorphic, totally real or of constant curvature.*

REMARK 2. (i) A minimal surface in CP^n is called superconformal if its harmonic sequence is orthogornally periodic (see [5] and [2]).

(ii) Every minimal 2-sphere in CP^n is pseudo-holomorphic (see [3] and [5]). So Theorem 2 is a generalization of the fact (III). We note that the global assumption is used in the proof of the fact (III) in [1] (cf. [8]).

(iii) In Section 5, we discuss pseudo-holomorphic totally real minimal surfaces in CP^n and superconformal totally real minimal surfaces in CP^3.

In Section 1 we follow [11], [3], [5] and recall the theory of harmonic sequences. In Section 2, using the method of moving frames developed in [6] and [8], we introduce some local functions and formulas for minimal surfaces in CP^n. In Section 3 we describe the k-orthogonality and the pseudo-holomorphicity of minimal surfaces in CP^n in terms of the local functions introduced in Section 2. In Section 4 we prove Theorems 1 and 2. In Section 5 we deal with the above Remark 2 (iii).

1. Harmonic sequences.

Let $\psi : S \to CP^n$ be a map from a metric Riemann surface S to CP^n, and set

$$L = \{(x, v) \in CP^n \times C^{n+1}; v \in x\}.$$

There is a bijective correspondence between maps $\psi : S \to CP^n$ and complex line subbundles of $S \times C^{n+1}$ given by $\psi \leftrightarrow \psi^*L$. We have the identification $TCP^n = \mathrm{Hom}(L, L^\perp)$, where the orthogonal complement is taken with respect to the standard Hermitian inner product on C^{n+1}. If X is a tangent vector field on S, then $d\psi(X) : \psi^*L \to \psi^*L^\perp$ is given by

$$d\psi(X)s = \pi_{L^\perp}(Xs),$$

where $\pi_{L^\perp}$ denotes the orthogonal projection to $L^\perp$ and the section s of ψ^*L is considered as a C^{n+1}-valued map on S.

Let $\partial : T^{(1,0)}S \otimes \psi^*L \to \psi^*L^\perp$ and $\bar{\partial} : T^{(0,1)}S \otimes \psi^*L \to \psi^*L^\perp$ be the bundle maps obtained by the restriction of $d\psi$ to $T^{(1,0)}S$ and $T^{(0,1)}S$, respectively. Then, in terms of a local complex coordinate z on S, we have

$$\partial(\partial/\partial z \otimes s) = \pi_{L^\perp}(\partial s/\partial z) = d\psi(\partial/\partial z)s$$

and

$$\bar{\partial}(\partial/\partial\bar{z}\otimes s)=\pi_{L^{\perp}}(\partial s/\partial\bar{z})=d\psi(\partial/\partial\bar{z})s.$$

Every complex vector subbundle V of $S\times C^{n+1}$ inherits a holomorphic structure for which s is a local holomorphic section if and only if $\partial s/\partial\bar{z}$ is orthogonal to V. Then ψ is harmonic if and only if ∂ is a holomorphic bundle map, which is also equivalent to that $\bar{\partial}$ is an anti-holomorphic bundle map (see [11, Theorem 2.1]).

Write $\psi=\psi_0, \psi^*L=L_0$ and assume that ψ is harmonic. If ∂ is not the zero-map, that is, ψ is not anti-holomorphic, then the zeros of ∂ are isolated and we obtain a unique complex line subbundle L_1 of $L_0^{\perp}$ containing the image of ∂. The map ∂ defines a holomorphic bundle map

$$\partial_0: T^{(1,0)}S\otimes L_0\to L_1.$$

Similarly, if $\bar{\partial}$ is not the zero-map, that is, ψ is not holomorphic, then we have a complex line subbundle L_{-1} of $L_0^{\perp}$ and an anti-holomorphic bundle map

$$\bar{\partial}_0: T^{(0,1)}S\otimes L_0\to L_{-1}.$$

The maps ψ_1 corresponding to L_1 and ψ_{-1} corresponding to L_{-1} are harmonic (see [11, Theorem 2.2]). In this way, one inductively builds up a sequence of harmonic maps $\{\psi_p\}$ together with corresponding complex line subbundles $\{L_p\}$, holomorphic bundle maps

$$\partial_p: T^{(1,0)}S\otimes L_p\to L_{p+1},$$

and anti-holomorphic bundle maps

$$\bar{\partial}_p: T^{(0,1)}S\otimes L_p\to L_{p-1}$$

with Hermitian adjoint $\partial_p^*=-\bar{\partial}_{p+1}$ (see [11] and [3]).

If, for some $q\in Z, \psi_q$ is holomorphic (resp. anti-holomorphic), then $\bar{\partial}_q$ (resp. ∂_q) is identically zero and ψ_{q-1} (resp. ψ_{q+1}) cannot be defined. In this case, the sequence $\{\psi_p\}$ terminates at the left- (resp. right-) hand end. If $I=\{p\in Z:\psi_p \text{is defined}\}$, then $\{\psi_p\}_{p\in I}$ is the harmonic sequence determined by $\psi=\psi_0$ and $\{L_p\}_{p\in I}$ is the corresponding bundle sequence of complex line subbundles of $S\times C^{n+1}$.

If I is a finite set, then ψ is called pseudo-holomorphic. We shall say that ψ is k-orthogonal if k consecutive bundles of $\{L_p\}$ are mutually orthogonal. In particular, ψ is always 2-orthogonal, ψ is 3-orthogonal if and only if ψ is conformal, and ψ is at most $(n+1)$-orthogonal. If ψ is pseudo-holomorphic and the image of ψ lies fully in CP^n, then ψ is $(n+1)$-orthogonal. Here a subset in CP^n is said to lie fully in CP^n if it does not lie in a totally geodesic CP^{n-1}. If ψ

is $(n+1)$-orthogonal but not pseudo-holomorphic, then $\{L_p\}$is orthogonally periodic with period n+1. In this case we say that ψ is superconformal. We note that every conformal harmonic map $\psi: S \to CP^2$ is either pseudo-holomorphic or superconformal (see [3] and [5]).

A minimal surface M in CP^n is the image of a conformal harmonic map $\psi: S \to CP^n$ with induced metric. We say that M is pseudo-holomorphic if ψ is pseudo-holomorphic, M is k-orthogonal if ψ is k-orthogonal, and M is superconformal if ψ is superconformal.

REMARK 3. As noted above, if ∂_p is not the zero-map, then the zeros of ∂_p are isolated. As our argument in this paper is local in nature, we may assume that ∂_p has no zeros if ∂_p is not the zero-map. Similarly we may assume that $\bar{\partial}_p$ has no zeros if $\bar{\partial}_p$ is not the zero-map.

2. Moving frames.

Throughout this paper we will adopt the following ranges of indices: $1 \le \alpha, \beta, \gamma \le n, 3 \le \lambda, \mu, \nu \le n, 1 \le j, k \le 2$.

Let $\{\omega_\alpha\}$ be a local field of unitary coframes on CP^n so that the metric is represented by $ds^2 = \sum_\alpha \omega_\alpha \overline{\omega}_\alpha$. We denote by $\{\omega_{\alpha\beta}\}$ the unitary connection forms with respect to $\{\omega_\alpha\}$. Then we have

$$d\omega_\alpha = \textstyle\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta, \ \omega_{\alpha\beta} + \overline{\omega}_{\beta\alpha} = 0, \tag{2.1}$$

$$d\omega_{\alpha\beta} = \textstyle\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Omega_{\alpha\beta}, \tag{2.2}$$

$$\Omega_{\alpha\beta} = -\omega_\alpha \wedge \overline{\omega}_\beta - \delta_{\alpha\beta} \textstyle\sum_\gamma \omega_\gamma \wedge \overline{\omega}_\gamma. \tag{2.3}$$

Let M be a minimal surface in CP^n. By using isothermal parameters, we may write the induced metric on M as $ds_M^2 = \phi\overline{\phi}$, where ϕ is a complex valued 1-form and it is defined up to a complex factor of norm one. Let $\{e_1, e_2\}$ be a local orthonormal frame on M, and let J denote the complex structure of CP^n. The Kähler angle $\alpha \in [0, \pi]$ of M is defined to be the angle between Je_1 and e_2.

We assume that $0 < \alpha < \pi$ on M. It is proved in [6] that there exist fields of unitary coframes such that

$$\omega_1 = \cos(\alpha/2)\phi, \ \omega_2 = \sin(\alpha/2)\overline{\phi}, \ \omega_\lambda = 0 \tag{2.4}$$

along M, and they satisfy

$$\frac{1}{2}\{d\alpha + \sin(\alpha)(\omega_{11} + \omega_{22})\} = a\phi, \tag{2.5}$$

(2.6) $$\omega_{12} = c\bar{\phi},$$

(2.7) $$\cos(\alpha/2)\omega_{\lambda 1} = a_\lambda \phi,$$

(2.8) $$\sin(\alpha/2)\omega_{\lambda 2} = c_\lambda \bar{\phi}$$

for some complex valued functions a, c, a_λ and c_λ defined locally on M. We note that $|a|^2, |c|^2, \Sigma_\lambda |a_\lambda|^2$ and $\Sigma_\lambda |c_\lambda|^2$ are scalar invariants of M (see [8]).

The metric $ds_M^2 = \phi\bar{\phi}$ has a connection form ρ, which is a real 1-form satisfying the equation $d\phi = -i\rho \wedge \phi$. Its exterior derivative gives the Gaussian curvature K as follows:

(2.9) $$d\rho = -\frac{i}{2} K \phi \wedge \bar{\phi}$$

The Gauss equation of M is written as

(2.10) $$K = 1 + 3\cos^2(\alpha) - 2(|a|^2 + |c|^2 + \Sigma_\lambda |a_\lambda|^2 + \Sigma_\lambda |c_\lambda|^2)$$

(see [6, (2.31)] and [8, (2.3)]).

The functions a_λ and c_λ satisfy

(2.11) $$da_\lambda - 2ia_\lambda \rho - \Sigma_\mu a_\mu \omega_{\lambda\mu} = a_{\lambda,1}\phi + a_{\lambda,2}\bar{\phi}, \ a_{\lambda,2} = -\bar{c}c_\lambda \cot(\alpha/2),$$

(2.12) $$dc_\lambda + 2ic_\lambda \rho - \Sigma_\mu c_\mu \omega_{\lambda\mu} = c_{\lambda,1}\phi + c_{\lambda,2}\bar{\phi}, \ c_{\lambda,1} = ca_\lambda \tan(\alpha/2)$$

for some complex valued functions $a_{\lambda,j}$ and $c_{\lambda,j}$ defined locally on M (see [8, (2.4)], where the equality for $a_{\lambda,2}$ should be corrected as above in (2.11)).

Let Δ denote the Laplacian of M with respect to ds_M^2. Then

(2.13) $$\Delta\alpha = 4\cot(\alpha)|a|^2 - 4\tan(\alpha/2)\Sigma_\lambda |a_\lambda|^2 + 4\cot(\alpha/2)\Sigma_\lambda |c_\lambda|^2 + 3\sin(2\alpha)$$

(see [6, (2.32)] and [8, Lemma 2.1], where the coefficient 3/2 of the last term of (2.32) of [6] should be corrected as 3/4).

3. k-orthogonality and pseudo-holomorphicity.

We begin by giving a description of the geometry of CP^n. For $W = (w_0, \cdots, w_n), Z = (z_0, \cdots, z_n) \in C^{n+1}$, the usual Hermitian inner product is defined by $< W, Z >= \Sigma_a w_a \bar{z}_a$, where we use the index range $0 \le a, b, c \le n$. The complex projective space CP^n is the orbit space of C^{n+1} - {0} under the action of the group $\{Z \to \alpha Z; \alpha \in C - \{0\}\}$. Let $\pi : C^{n+1} - \{0\} \to CP^n$ denote the projection. For a point $x \in CP^n$, we take a vector $Z \in \pi^{-1}(x) =: [x]$, which is called a homogeneous coordinate vector of x. We have the identification $T_x CP^n = \{W \in C^{n+1}; < Z, W >= 0\}$.

The complex projective space CP^n is diffeomorphic to the coset space of the unitary group U(n+1):

$$\text{(3.1)} \qquad U(n+1) \xrightarrow{\lambda} U(n+1)/U(n) \xrightarrow{h} U(n+1)/U(1) \times U(n) = CP^n.$$

We identify U(n+1) with the space of all unitary frames $\{Z_a\}$ in $C^{n+1}, Z_a \in C^{n+1} - \{0\}$ satisfying $< Z_a, Z_b >= \delta_{ab}$. Under this identification, the first projection λ in (3.1) is defined by assigning to the frame $\{Z_a\}$ its first vector Z_0. The second projection h in (3.1) is the Hopf fibering.

The Maurer-Cartan forms θ_{ab} of U(n+1) are defined by

$$\text{(3.2)} \qquad dZ_a = \Sigma_b \theta_{ab} Z_b, \quad \theta_{ab} + \bar{\theta}_{ba} = 0$$

They satisfy the Maurer-Cartan equations: $d\theta_{ab} = \Sigma_c \theta_{ac} \wedge \theta_{cb}$The Fubini-Study metric on CP^n is given by $ds^2 = \Sigma_\alpha \theta_{0\alpha} \bar{\theta}_{0\alpha}$.

If we set

$$\text{(3.3)} \qquad \omega_\alpha = \theta_{0\alpha}, \quad \omega_{\alpha\beta} = -(\theta_{\beta\alpha} - \delta_{\alpha\beta}\theta_{00}),$$

then they satisfy the condition (2.1). It follows that they are the connection forms of the Fubini-Study metric. Its curvature forms are

$$\Omega_{\alpha\beta} = -\omega_\alpha \wedge \bar{\omega}_\beta - \delta_{\alpha\beta} \Sigma_\gamma \omega_\gamma \wedge \bar{\omega}_\gamma.$$

Thus the Fubini-Study metric has constant holomorphic sectional curvature 4.

Let M be a minimal surface in CP^n, which is the image of a conformal harmonic map $\psi : S \to CP^n$ from a Riemann surface S to CP^n with induced metric. We wish to define a unitary frame firld $\{Z_a\}$ in C^{n+1} over a neighborhood $U \subset S$ along ψ as maps $Z_a : U \to C^{n+1} - \{0\}$ such that: (i) $\pi \circ Z_0 : U \to CP^n$ is the restriction of ψ; and (ii) $\{Z_0, \cdots, Z_n\}$ is a unitary frame in C^{n+1} for each point $p \in U$.

Let p be any point on S. Choose $Z : U \to C^{n+1} - \{0\}$ to be a homogeneous coordinate vector for $\psi(p)$ and put $Z_0 = Z/\langle Z, Z\rangle^{1/2}$. Assume that the Kähler angle α of M satisfies $0 < \alpha < \pi$ on M. Then, from the fact in Section 2, there is a unitary frame $\{e_\alpha\}$ along $M = \psi(S)$ whose dual coframe $\{\omega_\alpha\}$ satisfies (2.4)–(2.8). The vector e_α corresponds to a vector $Z_\alpha \in C^{n+1} - \{0\}$ by the identification $T_{\psi(p)}(CP^n) = \{W \in C^{n+1}; \langle Z, W\rangle = 0\}$. Then $\{Z_0, \cdots, Z_n\}$ is a unitary frame field in C^{n+1} over U along ψ. We have by (3.2), (3.3), (2.4), (2.6), (2.7), (2.8),

$$dZ_0 = \theta_{00}Z_0 + \cos(\alpha/2)\phi Z_1 + \sin(\alpha/2)\bar{\phi}Z_2,$$

(3.4) $$dZ_1 = -\cos(\alpha/2)\bar{\phi}Z_0 + \theta_{11}Z_1 + \bar{c}\phi Z_2 - \sec(\alpha/2)\phi\sum_\lambda a_\lambda Z_\lambda,$$

$$dZ_2 = -\sin(\alpha/2)\phi Z_0 - c\bar{\phi}Z_1 + \theta_{22}Z_2 - \operatorname{cosec}(\alpha/2)\bar{\phi}\sum_\lambda c_\lambda Z_\lambda,$$

where we use the notation in Section 2. Let $\{L_p\}$ be the bundle sequence corresponding to the harmonic sequence generated by ψ, together with the corresponding bundle maps $\{\partial_p\}$ and $\{\bar{\partial}_p\}$. Then by (3.4), $L_0 = [Z_0], L_1 = [Z_1], L_{-1} = [Z_2], L_2 = [\bar{c}Z_2 - \sec(\alpha/2)\sum_\lambda a_\lambda Z_\lambda]$ if ∂_1 is not the zero-map, and $L_{-2} = [-cZ_1 - \operatorname{cosec}(\alpha/2)\sum_\lambda c_\lambda Z_\lambda]$ if $\bar{\partial}_{-1}$ is not the zero-map (cf. Remark 3).

Assume that M lies fully in CP^n where $n \geq 3$. Then either ∂_1 or $\bar{\partial}_{-1}$ is not the zero-map (see [3, Lemma 1.2]). So either L_2 or L_{-2} can be defined as above, and we can see the following:

LEMMA 1. *Let M be a minimal surface lying fully in CP^n where $n \geq 3$, and assume that the Kähler angle α of M satisfies $0 < \alpha < \pi$ on M. Then M is 4-orthogonal if and only if $c = 0$ under the notation above.*

Assume that M lies fully in CP^n where $n \geq 4$, and neither ∂_1 nor $\bar{\partial}_{-1}$ is the zero-map. Then both L_2 and L_{-2} can be defined as above, and we can see the following:

LEMMA 2. *Let M be a minimal surface lying fully in CP^n where $n \geq 4$. Assume that the Kähler angle α of M satisfies $0 < \alpha < \pi$ on M, and neither ∂_1 nor $\bar{\partial}_{-1}$ is the zero-map. Then M is 5-orthogonal if and only if $c = 0$ and $\sum_\lambda a_\lambda \bar{c}_\lambda = 0$ under the notation above.*

Assume that M lies fully in CP^4, neither ∂_1 nor $\bar{\partial}_{-1}$ is the zero-map, and M is 5-orthogonal. Then by Lemma 2, $c = 0$, and $L_0, L_1, L_{-1}, L_2 = [\sum_\lambda a_\lambda Z_\lambda], L_{-2} = [\sum_\lambda c_\lambda Z_\lambda]$ are mutually orthogonal. So we may replace $\sum_\lambda a_\lambda Z_\lambda / \|\sum_\lambda a_\lambda Z_\lambda\|$ and $\sum_\lambda c_\lambda Z_\lambda / \|\sum_\lambda c_\lambda Z_\lambda\|$ by Z_3 and Z_4, respectively (cf. Remark 3). With respect to this new frame, $a_4 = c_3 = 0$. So by (2.11), (2.12) and that c = 0,

(3.5) $$-a_3\omega_{43} = a_{4,1}\phi,$$

(3.6) $$-c_4\omega_{34} = c_{3,2}\bar{\phi}$$

We have by (3.2), (3.3), (3.4), (3.5), (3.6) and that $a_4 = c_3 = 0$,

(3.7) $$dZ_3 = \bar{a}_3\sec(\alpha/2)\bar{\phi}Z_1 + \theta_{33}Z_3 + (a_{4,1}/a_3)\phi Z_4,$$
$$dZ_4 = \bar{c}_4\operatorname{cosec}(\alpha/2)\phi Z_2 + (c_{3,2}/c_4)\bar{\phi}Z_3 + \theta_{44}Z_4 :$$

As $L_2=[Z_3]$ and $L_{-2}=[Z_4]$, by (3.7), we have:

LEMMA 3. *Let M be a minimal surface lying fully in* CP^4. *Assume that the Kähler angle* α *of M satisfies* $0<\alpha<\pi$ *on M, and neither* ∂_l *nor* $\bar{\partial}_{-l}$ *is the zero-map. Then M is pseudo-holomorphic if and only if M is 5-orthogonal and* $a_{4,1}=c_{3,2}=0$ *under the notation above.*

4. Proof of Theormes 1 and 2.

First we show the following lemma (cf. [8, Lemma 3.2]).

LEMMA 4. *Let M be a minimal surface in* CP^n. *Assume that the Kähler angle* α *of M satisfies* $0<\alpha<\pi$ *on M, and M is 4-orthogonal. Then, under the notation in Section 2,*

$$|d(\Sigma_\lambda |a_\lambda|^2)|^2=4|\Sigma_\lambda a_\lambda \bar{a}_{\lambda,1}|^2, \tag{4.1}$$

$$\begin{aligned}\Delta(\Sigma_\lambda |a_\lambda|^2)=4\{&\Sigma_\lambda |a_{\lambda,1}|^2+K\Sigma_\lambda |a_\lambda|^2-\sec^2(\alpha/2)(\Sigma_\lambda |a_\lambda|^2)^2\\&+\operatorname{cosec}^2(\alpha/2)|\Sigma_\lambda a_\lambda \bar{c}_\lambda|^2-\cos(\alpha)\Sigma_\lambda |a_\lambda|^2\},\end{aligned} \tag{4.2}$$

$$|d(\Sigma_\lambda |c_\lambda|^2)|^2=4|\Sigma_\lambda c_\lambda \bar{c}_{\lambda,2}|^2, \tag{4.3}$$

$$\begin{aligned}\Delta(\Sigma_\lambda |c_\lambda|^2)=4\{&\Sigma_\lambda |c_{\lambda,2}|^2+K\Sigma_\lambda |c_\lambda|^2+\sec^2(\alpha/2)|\Sigma_\lambda a_\lambda \bar{c}_\lambda|^2\\&-\operatorname{cosec}^2(\alpha/2)(\Sigma_\lambda |c_\lambda|^2)^2+\cos(\alpha)\Sigma_\lambda |c_\lambda|^2\}.\end{aligned} \tag{4.4}$$

PROOF. As M is 4-orthogonal, $c=0$ by Lemma 1. Using (2.11), (2.1) and that $c=0$, we have

$$d(\Sigma_\lambda |a_\lambda|^2)=\Sigma_\lambda(\bar{a}_\lambda a_{\lambda,1}\phi+a_\lambda \bar{a}_{\lambda,1}\bar{\phi}), \tag{4.5}$$

from which we get (4.1). By (4.5),

$$d^c(\Sigma_\lambda |a_\lambda|^2)=i\Sigma_\lambda(a_\lambda \bar{a}_{\lambda,1}\bar{\phi}-\bar{a}_\lambda a_{\lambda,1}\phi). \tag{4.6}$$

By taking the exterior derivative of (2.11) and using (2.11), (2.9), (2.2), (2.7), (2.8), (2.3), (2.4), we get

$$\begin{aligned}d(a_{\lambda,1}\phi)=&-2ia_{\lambda,1}\phi\wedge\rho-\Sigma_\mu a_{\mu,1}\phi\wedge\omega_{\lambda\mu}+\{-Ka_\lambda\\&+\sec^2(\alpha/2)a_\lambda(\Sigma_\mu |a_\mu|^2)-\operatorname{cosec}^2(\alpha/2)c_\lambda(\Sigma_\mu a_\mu \bar{c}_\mu)+\cos(\alpha)a_\lambda\}\phi\wedge\bar{\phi}.\end{aligned} \tag{4.7}$$

Because of $dd^c(\Sigma_\lambda |a_\lambda|^2)=(i/2)\Delta(\Sigma_\lambda |a_\lambda|^2)\phi\wedge\bar{\phi}$, by taking the exterior derivative of (4.6) and using (2.11), (4.7), (2.1), we get (4.2). The equations (4.3) and (4.4) can be shown similarly.

Now we prove the following theorem which includes Theorem 1 and the fact (II) in the introduction.

THEOREM 3. *Let M be a minimal surface with constant Kähler angle in CP^3. Assume that M is 4-orthogonal. Then M is either holomorphic, anti-holomorphic, totally real or of constant curvature.*

PROOF. We use the notation in Section 2. Assume that M is neither holomorphic, anti-holomorphic nor totally real. As $d\alpha = a\phi + \bar{a}\bar{\phi}$ by (2.5) and α is constant, $a = 0$. As M is 4-orthogonal, $c = 0$ by Lemma 1. So we have by (2.10) and (2.13),

$$|a_3|^2 = \frac{1}{2}\cos^2(\alpha/2)(1+3\cos(\alpha)-K), \tag{4.8}$$

$$|c_3|^2 = \frac{1}{2}\sin^2(\alpha/2)(1-3\cos(\alpha)-K) \tag{4.9}$$

The equations (4.1) and (4.3) are written as $|d(|a_3|^2)|^2 = 4|a_3|^2|a_{3,1}|^2$ and $|d(|c_3|^2)|^2 = 4|c_3|^2|c_{3,2}|^2$, respectively. So (4.2) and (4.4) are rewritten as

$$|a_3|^2\Delta(|a_3|^2) = |d(|a_3|^2)|^2 + 4|a_3|^4\{K - \sec^2(\alpha/2)|a_3|^2 + \operatorname{cosec}^2(\alpha/2)|c_3|^2 - \cos(\alpha)\}, \tag{4.10}$$

$$|c_3|^2\Delta(|c_3|^2) = |d(|c_3|^2)|^2 + 4|c_3|^4\{K + \sec^2(\alpha/2)|a_3|^2 - \operatorname{cosec}^2(\alpha/2)|c_3|^2 + \cos(\alpha)\}, \tag{4.11}$$

respectively. Inserting (4.8) (4.9) into (4.10), (4.11), and noting that $\cos(\alpha) \neq 0$, we get

$$\Delta K = \frac{8}{3}\{5K^2 - 7K + 2(1+9\cos^2(\alpha))\} =: P(K),$$

$$|dK|^2 = \frac{4}{3}\{7K^3 - 18K^2 + 3(5-21\cos^2(\alpha))K - 4(1-9\cos^2(\alpha))\} =: Q(K).$$

If K is not constant, then

$$QK + (P-Q')(P-\frac{1}{2}Q') + Q(P' - \frac{1}{2}Q'') = 0, \tag{4.12}$$

where the prime denotes the differentiation with respect to K (see [7, Lemma 3.3]). By the computation we can find that (4.12) is a nontrivial equation for K. So K must be constant, which is a contradiction. Hence K is constant, and the proof is complete.

PROOF OF THEOREM 1. As M is superconformal, M is neither holomorphic nor anti-holomorphic. In the case where M does not lie fully in CP^3, the theorem is included in the fact (I) in the introduction. So we assume that M lies fully in CP^3. As M is a superconformal minimal surface lying fully in CP^3, M is 4-orthogonal but not pseudo-holomorphic. Non-pseudo-holomorphic minimal surfaces with constant curvature and Kahler angle in CP^n are totally real (see [10]). Hence by Theorem 3, M is totally real.

PROOF OF THEOREM 2. We use the notation in Section 2 and 3. In the case where M does not lie fully in CP^4, the theorem is included in the fact (I) and (II) in the introduction. So we assume that M lies fully in CP^4. Assume that M is neither holomorphic, anti-holomorphic nor totally real. By the hypothesis and Lemma 1, we have $a = c = 0$.

If ∂_1 is the zero-map, that is, $\sum_\lambda |a_\lambda|^2 = 0$ (see Section 3), then by (2.10) and (2.13), we can see that K is constant. Similarly, if $\bar{\partial}_{-1}$ is the zero-map, that is, $\sum_\lambda |c_\lambda|^2 = 0$, then K is constant.

If neither ∂_1 nor $\bar{\partial}_{-1}$ is the zero-map, then by the hypothesis and Lemma 3, we may choose the frame so that $a_4 = c_3 = a_{4,1} = c_{3,2} = 0$. So we have by (2.10) and (2.13),

$$|a_3|^2 = \frac{1}{2}\cos^2(\alpha/2)(1+3\cos(\alpha)-K), \tag{4.13}$$

$$|c_4|^2 = \frac{1}{2}\sin^2(\alpha/2)(1-3\cos(\alpha)-K) \tag{4.14}$$

The equations (4.1) and (4.3) are written as $|d(|a_3|^2)|^2 = 4|a_3|^2|a_{3,1}|^2$ and $|d(|c_4|^2)|^2 = 4|c_4|^2|c_{4,2}|^2$; respectively. So, noting that $a_4 = c_3 = a_{4,1} = c_{3,2} = 0$, (4.2) and (4.4) are rewritten as

$$|a_3|^2\Delta(|a_3|^2) = |d(|a_3|^2)|^2 + 4|a_3|^4\{K - \sec^2(\alpha/2)|a_3|^2 - \cos(\alpha)\}, \tag{4.15}$$

$$|c_4|^2\Delta(|c_4|^2) = |d(|c_4|^2)|^2 + 4|c_4|^4\{K - \operatorname{cosec}^2(\alpha/2)|c_4|^2 + \cos(\alpha)\}, \tag{4.16}$$

respectively. Inserting (4.13), (4.14) into (4.15), (4.16), and noting that $\cos(\alpha) \neq 0$, we get

$$\Delta K = \frac{2}{3}\{23K^2 - 34K + 11 + 45\cos^2(\alpha)\} =: R(K),$$

$$|dK|^2 = \frac{4}{3}\{7K^3 - 18K^2 + 3(5 - 21\cos^2(\alpha))K - 4(1 - 9\cos^2(\alpha))\} =: S(K).$$

If K is not constant, then

$$SK + (R - S')(R - \frac{1}{2}S') + S(R' - \frac{1}{2}S'') = 0, \tag{4.17}$$

where the prime denotes the differentiation with respect to K (see [7, Lemma 3.3]). By the computation we can find that (4.17) is a nontrivial equation for K. So K must be constant, which is contradiction. Hence K is constant, and the proof is complete.

5. Remark 2 (iii).

Let $\boldsymbol{RP}^n$ be the n-dimensional real projective space of constant curvature 1, which we regard as a totally geodesic submanifold in $\boldsymbol{CP}^n$ through the standard inclusion $i:\boldsymbol{RP}^n \to \boldsymbol{CP}^n$. Let $\pi: S^n \to \boldsymbol{RP}^n$ be the natural projection, where S^n denotes the n-dimensional unit sphere.

For a minimal surface M in S^n,$(i\circ\pi)$ (M) is a totally real minimal surface in $\boldsymbol{CP}^n$, which lies in $\boldsymbol{RP}^n$ (see [3]). So if M is a minimal surface in S^n such that $(i\circ\pi)$ (M) is pseudo-holomorphic in $\boldsymbol{CP}^n$ (for example, M is a minimal 2-sphere in S^n), then $(i\circ\pi)$ (M) is a pseudo-holomorphic totally real minimal surface in $\boldsymbol{CP}^n$, which lies in $\boldsymbol{RP}^n$.

Conversely, if M is a pseudo-holomorphic totally real minimal surface in $\boldsymbol{CP}^n$, then by Theorem 3.6 of [3], there is a holomorphic isometry g of $\boldsymbol{CP}^n$ such that $g(M)\subset \boldsymbol{RP}^n$. So, up to congruence, pseudo-holomorphic totally real minimal surfaces in $\boldsymbol{CP}^n$ are constructed as above.

If M is a minimal surface in S^3 which is not totally geodesic, then $(i\circ\pi)$ (M) is a superconformal totally real minimal surface lying fully in $\boldsymbol{CP}^3$, which lies in $\boldsymbol{RP}^3$ (see the last Remark in [3]). Conversely, if M is a superconformal totally real minimal surface lying fully in $\boldsymbol{CP}^3$, then by Theorem 3.6 of [3] together with Theorem 2.2 of [5], there is a holomorphic isometry g of $\boldsymbol{CP}^3$ such that $g(M)$ $\subset \boldsymbol{RP}^3$. So, up to congruence, superconformal totally real minimal surfaces lying fully in $\boldsymbol{CP}^3$ are constructed as above.

As noted in Section 1, every minimal surface in $\boldsymbol{CP}^2$ is either pseudo-holomorphic or superconformal. If M is a pseudo-holomorphic totally real minimal surface in $\boldsymbol{CP}^2$, then as mentioned above, there is a holomorphic isometry g of $\boldsymbol{CP}^2$ such that $g(M)$ $\subset RP^2$. So, totally real minimal surfaces in $\boldsymbol{CP}^2$ with Gaussian curvature not identically 1, are superconformal, which are intrinsically characterized in Theorem 3.8 of [7]. Hence, superconformal totally real minimal surfaces in $\boldsymbol{CP}^2$ are constructed through Theorem 3.8 of [7].

References

[1] J. Bolton, G. R. Jensen, M. Rigoli and L. M. Woodward, On conformal minimal immersions of S^2 into CP^n, Math. Ann. **279** (1988), 599–620.

[2] J. Bolton, F. Pedit and L. M. Woodward, Minimal surfaces and the affine Toda field model, J.reine angew. Math. **459** (1995), 119–150.

[3] J. Bolton and L. M. Woodward, Congruence theorems for harmonic maps from a Riemann surface into CP^n and S^n, J. London Math. Soc. **45** (1992), 363–376.

[4] J. Bolton and L. M. Woodward, Minimal surfaces in CP^n with constant curvature and Kähler angle, Math. Proc. Camb. Phil. Soc. **112** (1992), 287–296.

[5] J. Bolton and L. M. Woodward, The affine Toda equations and minimal surfaces, Harmonic Maps and Integrable Systems, (A. Fordy and J. C. Wood, eds.), Aspects of Math. E23, Vieweg, 1994, 59–82.

[6] S. S. Chern and J. G. Wolfson, Minimal surfaces by moving frames, Amer. J. Math. **105** (1983), 59–83.,

[7] J. H. Eschenburg, I. V. Guadalupe and R. A. Tribuzy, The fundamental equations of minimal surfaces in CP^2, Math. Ann. **270** (1985), 571–598.

[8] T. Ogata, Curvature pinching theorem for minimal surfaces with constant Kaehler angle in complex projective spaces, Tohoku Math. J. **43** (1991), 361–374.

[9] T. Ogata, Curvature pinching theorem for minimal surfaces with constant Kaehler angle in complex projective spaces, II, Tohoku Math. J. **45** (1993), 271–283.

[10] Y. Ohnita, Minimal surfaces with constant curvature and Kähler angle in complex space forms, Tsukuba J. Math. **13** (1989), 191–207.

[11] J. G. Wolfson, Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds, J. Diff. Geom. **27** (1988), 161–178.

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