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ON THE EXISTENCE OF WEIERSTRASS POINTS WITH A CERTAIN SEMIGROUP GENERATED BY 4 ELEMENTS

By

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Introduction

Let X be a smooth, proper 1-dimensional algebraic variety (of genus$\geqq 2$) over an algebraically closed field k of characteristic 0, and let P be a point of X. Then a positive integer ν is called a *gap* at P if $h^0(X, \mathcal{O}_X((\nu-1)P))=h^0(X, \mathcal{O}_X(\nu P))$, and G_P denotes the set of gaps at P. If we denote by N and H_P respectively the additive semigroup of non-negative integers and the complement of G_P in N, then H_P is a semigroup. A subsemigroup H of N whose complement is finite is called a *numerical semigroup*. The following problem is fundamental and is a long-standing problem.

Is there a pair (X, P) with X a smooth, proper 1-dimensional algebraic variety over k and P its point, such that $H=H_P$?

Using the deformation theory on algebraic varieties with G_m-action, Pinkham [7] constructed a moduli space $\mathcal{M}_H$ which classifies the set of isomorphic classes of pairs (X, P) consisting of a smooth, proper 1-dimensional algebraic variety X together with its point P such that $H_P=H$. But he did not claim that $\mathcal{M}_H$ is non-empty. Using the Pinkham's construction of $\mathcal{M}_H$, some mathematicians showed that for some H, $\mathcal{M}_H$ is non-empty. To state their results we prepare some notation. Let $M(H)=\{a_1, \cdots, a_n\}$ be the minimal set of generators for the semigroup H, which is uniquely determined by H. I_H denotes the kernel of the k-algebra homomorphism $\varphi: k[X]=k[X_1, \cdots, X_n]\rightarrow k[t]$ defined by $\varphi(X_i)=t^{a_i}$ where $k[X]$ and $k[t]$ are polynomial rings over k, and $\mu(H)$ denotes the least number of generators for the ideal I_H. When we set $C_H=\mathrm{Spec}\, k[X]/I_H$, we denote by $T^1_{C_H}=\bigoplus_{l\in Z} T^1_{C_H}(l)$ the k-vector space of first order deformations of C_H with a natural graded structure. Moreover, $g(H)$ and $C(H)$ denote the cardinal number of the set $N-H$ and the least integer c with $c+N\subseteqq H$, respectively. Then $\mathcal{M}_H$ is non-empty in the following cases:

1) H is a complete intersection, i.e., $\mu(H)=n-1$,
2) H is a special almost complete intersection (Waldi [10]),

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3) H is negatively graded, i.e., $T^1_{C_H}(l)=0$ for $l>0$ (Pinkham [7], Rim-Vitulli [8]),

4) H is generated by 4 elements and is symmetric, i.e., $C(H)=2g(H)$ (Buchweitz [2], Waldi [9]).

In this paper we shall give some examples of numerical semigroups H generated by 4 elements with $\mathcal{M}_H\neq\emptyset$, because for any numerical semigroup H generated by 2 or 3 elements, 1) and 2) imply $\mathcal{M}_H\neq\emptyset$. Throughout the paper, we are devoted to a numerical semigroup H of torus embedding type (see Definition 1.1), roughly speaking, C_H is the fibre of a torus embedding. For such an H, we can prove that $\mathcal{M}_H$ is non-empty. In Section 2 we show that numerical semigroups H generated by 2 or 3 elements are of torus embedding type. When H is a neat numerical semigroup (see Definition 3.1) generated by 4 elements, we construct a torus embedding, any irreducible component of whose fibre over the origin is isomorphic to C_H, in Section 4. Moreover, if H is 1-neat (see Definition 4.10), we can show that H is of torus embedding type. Using this we can show that symmetric or almost symmetric numerical semigroups H generated by 4 elements are of torus embedding type.

Notation

Throughout this paper we will use the following notation without further warning. We denote by k an algebraically closed field and by N the additive semigroup of non-negative integers. For elements $a_1, \cdots, a_n, m$ and l of N, $\langle a_1, \cdots, a_n\rangle$ (resp. $(a_1, \cdots, a_n)$, resp. $[l, m]$) denotes the subsemigroup of N generated by $a_1, \cdots, a_n$ (resp. the greatest common measure of $a_1, \cdots, a_n$, resp. the set of integers which is larger than or equal to l, and which is smaller than or equal to m). For a weighted ring R and a homogeneous element f of R, $\partial(f)$ means the weight of f. Let H be a numerical semigroup, i.e., the subsemigroup of N whose complement in N is finite. Then $\mathcal{M}_H$ denotes the moduli space, which is obtained by Pinkham, consisting of isomorphic classes of pairs (X, P) with a smooth, proper 1-dimensional algebraic variety X over k and with its point P whose gaps are $N-H$. Moreover, we denote by $g(H)$ the cardinal number of the set $N-H$, by $C(H)$ the least integer c with $c+N\subseteqq H$ and by $M(H)=\{a_1, \cdots, a_n\}$ the minimal set of generators for the semigroup H. We set

$$\alpha_i=\operatorname{Min}\{\alpha\in N-\{0\}\mid \alpha a_i\in\langle a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n\rangle\}$$

for all $i=1, \cdots n$. For any non-zero element h of H let

$$L_h(H)=\{0=\omega_h(1)<\cdots<\omega_h(h)\}$$

be the set of the least elements of H in respective congruence classes mod h. φ_H denotes the k-algebra homomorphism from $k[X_1, \cdots, X_n]$ to $k[t]$ defined by sending X_i to t^{a_i}, hence assigning $\partial(X_i)=a_i$ for $1 \leqq i \leqq n$ and $\partial(c)=0$ for $c \in k^{\times}$, $k[X_1, \cdots, X_n]$ is made into a weighted k-algebra. We denote by I_H the kernel of φ_H, by $\mu(H)$ the least number of generators for the ideal I_H and by C_H the affine curve Spec $k[X_1, \cdots, X_n]/I_H$.

1. Numerical semigroups of torus embedding type.

In this paper we are concerned with the following numerical semigroups:

DEFINITION 1.1. A numerical semigroup H with $M(H)=\{a_1, \cdots, a_n\}$ is *of torus embedding type* if there exist a positive integer $m \geqq n$, homogeneous elements $g_i (1 \leqq i \leqq m)$ of $k[X]=k[X_1, \cdots, X_n]$ of weight >0, and a saturated subsemigroup S of $\boldsymbol{Z}^{m+1-n}$ which is generated by $b_1, \cdots, b_m$ and which generates a subgroup of rank $m+1-n$ of $\boldsymbol{Z}^{m+1-n}$ as a group, such that the kernel of the k-algebra homomorphism

$$\pi: k[Y]=k[Y_1, \cdots, Y_m] \longrightarrow k[S]=k[T^s]_{s \in S}$$

defined by $\pi(Y_i)=T^{b_i}$, is generated by homogeneous elements $F_k (1 \leqq k \leqq u)$ with $I_H=(F_1(g_1, \cdots, g_m), \cdots, F_u(g_1, \cdots, g_m))$ where the weight on $k[Y]$ is defined by $\partial(Y_i)=\partial(g_i)$ for $1 \leqq i \leqq m$ and $\partial(c)=0$ for $c \in k^{\times}$.

A sufficient condition that a numerical semigroup is of torus embedding type, which we will use, is the following:

LEMMA 1.2. *Let H be a numerical semigroup with $M(H)=\{a_1, \cdots, a_n\}$. Assume that there exist a positive integer $m \geqq n$, non-constant monomials $g_i (1 \leqq i \leqq m)$ in $k[X]=k[X_1, \cdots, X_n]$, and a saturated subsemigroup S of $\boldsymbol{Z}^{m+1-n}$ which is generated by $b_1, \cdots, b_m$ and which generates a subgroup of rank $m+1-n$ of $\boldsymbol{Z}^{m+1-n}$ as a group, such that if we let*

$$\pi: k[Y]=k[Y_1, \cdots, Y_m] \longrightarrow k[T^s]_{s \in S} \qquad (resp.\ \eta: k[Y] \rightarrow k[X])$$

be the k-algebra homomorphism defined by $\pi(Y_i)=T^{b_i}$ (resp. $\eta(Y_i)=g_i$), then the ideal I_H is generated by the elements of $\eta(\mathrm{Ker}\,\pi)$. Then H is of torus embedding type.

PROOF. When we define a weight on $k[Y]$ in virtue of $\partial(Y_i)=\partial(g_i)$ for $1 \leqq i \leqq m$ and $\partial(c)=0$ for $c \in k^{\times}$, it suffices to show that there exists a set $\{F_k\}_{1 \leqq k \leqq u}$ of homogeneous generators for the ideal Ker π, because the ideal I_H is generated by $\eta(F_k)$ $(1\leqq k\leqq u)$. Now by [5] we may take generators $F_k(1\leqq k\leqq u)$ of the ideal Ker π as follows:

$$F_k=\prod_{i=1}^{m}Y_i^{\nu_{ki}}-\prod_{i=1}^{m}Y_i^{\mu_{ki}}$$

where $\nu_{ki}\cdot\mu_{ki}=0$ for all $1\leqq k\leqq u$ and all $1\leqq i\leqq m$. If we put $g_i=X_1^{\gamma_{i1}}\cdots X_n^{\gamma_{in}}$ for all $1\leqq i\leqq m$, then we have

$$0=\varphi_H(\eta(F_k))=\varphi_H\left(\prod_{i=1}^{m}g_i^{\nu_{ki}}-\prod_{i=1}^{m}g_i^{\mu_{ki}}\right)$$

$$=t^{\sum_{i=1}^{m}\nu_{ki}\sum_{j=1}^{n}\gamma_{ij}a_j}-t^{\sum_{i=1}^{m}\mu_{ki}\sum_{j=1}^{n}\gamma_{ij}a_j},$$

which implies $\sum_{i=1}^{m}\nu_{ki}\sum_{j=1}^{n}\gamma_{ij}a_j=\sum_{i=1}^{m}\mu_{ki}\sum_{j=1}^{n}\gamma_{ij}a_j$. Therefore F_k's are homogeneous.

Q. E. D.

Here we give a few examples of numerical semigroups of torus embedding type.

EXAMPLE 1.3. (1) $H=\langle 3,7\rangle$ is of torus embedding type. In fact, let $a_1=3$ and $a_2=7$. If we set $n=m=2$, $g_1=X_1^7$, $g_2=X_2^3$ and $b_1=b_2=1$, then these satisfy the assumption of Lemma 1.2. In this case Ker π contains a homogeneous element $F_1=Y_1-Y_2$. See Lemma 2.3 for a generalization.

(2) $H=\langle 4,7,13\rangle$ is of torus embedding type. In fact, let $a_1=4$, $a_2=7$ and $a_3=13$. If we set $n=3$, $m=6$, $g_1=X_1^2$, $g_2=X_2$, $g_3=X_3$, $g_4=X_1^3$, $g_5=X_2^2$, $g_6=X_3$, $b_1=(1,0,0,0)$, $b_2=(0,1,0,0)$, $b_3=(0,0,1,0)$, $b_4=(-1,1,1,0)$, $b_5=(0,0,0,1)$ and $b_6=(-1,1,0,1)$, then these satisfy the assumption of Lemma 1.2. In this case we can see that Ker π contains homogeneous elements $F_k(1\leqq k\leqq 3)$ as follows:

$$F_1=Y_1Y_4-Y_2Y_3,\quad F_2=Y_2Y_5-Y_1Y_6\quad\text{and}\quad F_3=Y_3Y_6-Y_4Y_5.$$

See Proposition 2.5 for a generalization.

(3) $H=\langle 4,9,14,15\rangle$ is of torus embedding type. In fact, let $a_1=15$, $a_2=9$, $a_3=4$ and $a_4=14$. If we set $n=4$, $m=9$, $g_1=X_1$, $g_2=X_2$, $g_3=X_3^4$, $g_4=X_4$, $g_5=X_1$, $g_6=X_2$, $g_7=X_3$, $g_8=X_4$, $g_9=X_3$, $b_i=e_i$ $(1\leqq i\leqq 4)$, $b_5=(-1,0,1,1,0,0)$, $b_6=e_5$, $b_7=e_6$, $b_8=(0,1,0,0,1,-1)$ and $b_9=(1,1,-1,0,0,-1)$ where for any $i\in[1,6]$ we denote by $e_i\in \boldsymbol{Z}^6$ the vector whose i-th component equals to 1 and whose j-th component equals to 0 if $j\neq i$, then these satisfy the assumption of Lemma 1.2. In this case we can see that Ker π contains homogeneous elements $F_k(1\leqq k\leqq 6)$ as follows:

$$F_1=Y_1Y_5-Y_3Y_4,\quad F_2=Y_2Y_6-Y_7Y_8,\quad F_3=Y_3Y_7Y_9-Y_1Y_2,$$

$$F_4=Y_4Y_8-Y_5Y_6Y_9,\quad F_5=Y_1Y_8-Y_3Y_6Y_9 \quad\text{and}\quad F_6=Y_2Y_4-Y_5Y_7Y_9.$$

See Theorem 4.11 for a generalization.

(4) $H=\langle 5,8,9,11\rangle$ is of torus embedding type. In fact, let $a_1=5$, $a_2=8$, $a_3=9$ and $a_4=11$. If we set $n=4$, $m=9$, $g_i=X_i$ $(1\leqq i\leqq 4)$, $g_5=X_1^2$, $g_{4+i}=X_i$ $(2\leqq i\leqq 4)$, $g_9=X_1$, $b_i=e_i$ $(1\leqq i\leqq 6)$, $b_7=(0,1,-1,0,1,0)$, $b_8=(-1,1,0,0,0,1)$ and $b_9=(-1,0,1,1,-1,0)$ where e_i's are as in (3), then these satisfy the assumption of Lemma 1.2. In this case, $\operatorname{Ker}\pi$ contains homogeneous elements $F_k\,(1\leqq k\leqq 5)$ as follows:

$$F_1=Y_1Y_5Y_9-Y_3Y_4,\quad F_2=Y_2Y_6-Y_1Y_8,\quad F_3=Y_3Y_7-Y_2Y_5,$$

$$F_4=Y_4Y_8-Y_6Y_7Y_9 \quad\text{and}\quad F_5=Y_1Y_7Y_9-Y_2Y_4.$$

See Theorem 4.11 for a generalization. Now we get $g(H)=7$ and $C(H)=13$, which imply $C(H)=2g(H)-1$, i.e., H is almost symmetric (see Theorem 6.4).

In the remains of this section we assume that k is of characteristic 0. If H is of torus embedding type, then we can show $\mathcal{M}_H\neq\emptyset$. For this purpose we show the following:

PROPOSITION 1.4. *Let* $a_1,\cdots,a_n$ *be positive integers and let* $k[X]=k[X_1,\cdots,X_n]$ *be a polynomial ring on which the weight is defined by* $\partial(X_i)=a_i$ *for* $1\leqq i\leqq n$ *and* $\partial(c)=0$ *for* $c\in k^\times$. *Let* $k[Y]=k[Y_1,\cdots,Y_m]$ *and* $k[Y,W]=k[Y_1,\cdots,Y_m,W_1,\cdots,W_l]$ *be two polynomial rings. Let* r *be a non-negative integer with* $n-l\geqq r$, *let* J *be an ideal in* $k[Y]$ *such that* $R=k[Y]/J$ *is a Cohen-Macaulay domain of dimension* $m+l+r-n$ *and that the singular locus of* Spec R *has codimension larger than* r, *and let* $R[X]=R[X_1,\cdots,X_n]$. *Assume that there exist homogeneous elements* $g_i\,(1\leqq i\leqq m)$ *and* $h_j\,(1\leqq j\leqq l)$ *of* $k[X]$ *of weight* >0 *such that we have the fibre product*:

$$\begin{array}{ccc} \phi^{-1}(\textit{the origin}) & \longrightarrow & \operatorname{Spec} R[X] \\ \downarrow & & \downarrow \phi \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[Y,W] \end{array}$$

with $\dim\phi^{-1}(\textit{the origin})=r$, *where* ϕ *is the morphism which is induced by the* k*-algebra homomorphism* $\phi^*\colon k[Y,W]\to R[X]$ *defined by* $\phi^*(Y_i)=g_i-Y_i \bmod J$ *and* $\phi^*(W_j)=h_j$, *and such that the ideal* J *is homogeneous where the weight on* $k[Y]$ *is defined by* $\partial(Y_i)=\partial(g_i)$ *for* $1\leqq i\leqq m$ *and* $\partial(c)=0$ *for* $c\in k^\times$. *Then* ϕ *is flat and there exists a non-empty open subset* V *of* Spec $k[Y,W]$ *such that the* *restriction* $\phi^{-1}(V)\to V$ *is smooth.*

PROOF. We define a weight on $k[Y, W]$ as follows:

$$\partial(Y_i)=\partial(g_i)\,,\quad \partial(W_j)=\partial(h_j)\quad \text{and}\quad \partial(c)=0 \qquad \text{for}\quad c\in k^{\times}\,.$$

Since the ideal J in $k[Y]$ is homogeneous, ϕ is a G_m-equivariant morphism. For any $s\in \boldsymbol{Z}$, the closed subset

$$F_s=\{x\in \operatorname{Spec} R[X]\,|\,\dim_x \phi^{-1}(\phi(x))\geqq s\}$$

contains the origin if $F_s\neq\emptyset$, because ϕ is G_m-equivariant and the weights of Y_i, X_k are positive. ϕ is dominating in virtue of

$$\dim \operatorname{Spec} R[X]-\dim \operatorname{Spec} k[Y, W]=m+l+r-(m+l)=r$$

and

$$\dim \phi^{-1}(\text{the origin})=r\,,$$

which implies $\dim_x \phi^{-1}(\phi(x))\geqq r$ for all $x\in \operatorname{Spec} R[X]$. Moreover, in virtue of $\partial(Y_i)>0$ and $\partial(W_j)>0$ the map ϕ send the origin in $\operatorname{Spec} R[X]$ to the one in $\operatorname{Spec} k[Y, W]$. Assume that $F_{r+1}\neq\emptyset$. Since the origin belongs to F_{r+1}, we get

$$r+1\leqq\dim_{\text{the origin}} \phi^{-1}(\phi(\text{the origin}))=\dim_{\text{the origin}} \phi^{-1}(\text{the origin})$$

$$\leqq\dim \phi^{-1}(\text{the origin})=r\,,$$

a contradiction, which implies $F_{r+1}=\emptyset$. Therefore we get $\dim_x \phi^{-1}(\phi(x))=r$ for all $x\in \operatorname{Spec} R[X]$, i.e., ϕ is equidimensional. Since R is a Cohen-Macaulay domain, ϕ is flat ([3]). Let $Z_i\,(i\in I)$ be the irreducible components in the singular locus $\operatorname{Sing}(\operatorname{Spec} R[X])$ of $\operatorname{Spec} R[X]$ and let η be the generic point of $\operatorname{Spec} k[Y, W]$. Assume that $\phi^{-1}(\eta)\cap\operatorname{Sing}(\operatorname{Spec} R[X])\neq\emptyset$, i.e., there exists $i\in I$ such that $\phi^{-1}(\eta)\cap Z_i\neq\emptyset$. Since the restriction $Z_i\subset\operatorname{Spec} R[X]\to\operatorname{Spec} k[Y, W]$ is dominating, we have

$$0\leqq\dim Z_i-\dim \operatorname{Spec} k[Y, W]\leqq\dim \operatorname{Sing}(\operatorname{Spec} R[X])-\dim \operatorname{Spec} k[Y, W]$$

$$<\dim \operatorname{Spec} R[X]-r-\dim \operatorname{Spec} k[Y, W]=0\,,$$

a contradiction. Hence we get $\phi^{-1}(\eta)\cap\operatorname{Sing}(\operatorname{Spec} R[X])=\emptyset$, which implies that the set

$$\{y\in \operatorname{Spec} k[Y, W]\,|\,\phi^{-1}(y)\cap\operatorname{Sing}(\operatorname{Spec} R[X])=\emptyset\}$$

contains a non-empty open subset U. Then we have

$$\phi^{-1}(U)\subseteqq\operatorname{Spec} R[X]-\operatorname{Sing}(\operatorname{Spec} R[X])$$

Hence there is a non-empty open subset V in $\operatorname{Spec} k[Y, W]$ such that the restriction $\phi^{-1}(V)\to V$ is smooth, because the restriction $\phi^{-1}(U)\to\mathrm{Spec}\, k[Y, W]$ is a morphism of varieties with smooth $\phi^{-1}(U)$ over the algebraically closed field k of charcteristic 0 ([4]). *Q. E. D.*

Pinkham [7] showed the following:

REMARK 1.5. Let H be a numerical semigroup with $M(H)=\{a_1, \cdots, a_n\}$. Then we have $\mathcal{M}_H\neq\emptyset$ if and only if there exists a flat homogeneous homomorphism ϕ^*: $A=\bigoplus_{i\in Z} A_i\to B=\bigoplus_{i\in Z} B_i$ of affine graded k-algebras with $A_0\supseteqq k$ and $B_0\supseteqq k$ such that 1) C_H is the fibre of the morphism ϕ: Spec $B\to$Spec A associated to ϕ^* over a homogeneous k-rational point on Spec A, 2) A is a domain and the generic fibre of ϕ is smooth, and 3) $A_i=0$ for all $i<0$.

Combining Proposition 1.4 with Remark 1.5, we get the following:

COROLLARY 1.6. *Let H be a numerical semigroup with $M(H)=\{a_1, \cdots, a_n\}$ and let $k[X]$, $k[Y]$ and $k[Y, W]$ be polynomial rings as in* Proposition 1.4. *Let J be an ideal in $k[Y]$ such that $R=k[Y]/J$ is a normal Cohen-Macaulay domain of dimension $m+l+1-n$. Assume that there exist homogeneous elements $g_i(1\leqq i\leqq m)$ and $h_j(1\leqq j\leqq l)$ of $k[X]$ of weight >0 such that we have the fibre product*:

$$\begin{array}{ccc} C_H & \longrightarrow & \mathrm{Spec}\, R[X] \\ \downarrow & & \downarrow \phi \\ \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, k[Y, W] \\ [(0)] & \longmapsto & \textit{the origin} \end{array}$$

whe ϕ is the morphism induced by the k-algebra homomorphism ϕ^*: $k[Y, W]\to R[X]$ defined by $\phi^*(Y_i)=g_i-Y_i \bmod J$ and $\phi^*(W_j)=h_j$, and such that the ideal J is homogeneous where the weight on $k[Y]$ is defined by $\partial(Y_i)=\partial(g_i)$ for $1\leqq i\leqq m$ and $\partial(c)=0$ for $c\in k^\times$. Then we have $\mathcal{M}_H\neq\emptyset$.

If we apply Corollary 1.6 to numerical semigroups of torus embedding type, we see:

THEOREM 1.7. *For any numerical semigroup H of torus embedding type, we have $\mathcal{M}_H\neq\emptyset$.*

PROOF. We use the notation in Definition 1.1. Since S is a saturated subsemigroup of Z^{m+1-n} which is finitely generated and which generates a subgroup of rank $m+1-n$ of Z^{m+1-n} as a group, by [6] Spec $k[T^s]_{s\in S}$ is a normal affine equivariant embedding of $(G_m)^{m+1-n}$ and is a Cohen-Macaulay scheme. Hence $R=k[Y]/\mathrm{Ker}\,\pi$ is a normal Cohen-Macaulay domain of dimension $m+1-n$ and the ideal $J=\mathrm{Ker}\,\pi$ is generated by homogeneous elements $F_k (1\leqq k\leqq u)$. Since the ideal I_H is generated by the $F_k(g_1, \cdots, g_m)$'s, we have a fibre product:

$$\begin{array}{ccc} C_H & \longrightarrow & \mathrm{Spec}\, R[X] \\ \downarrow & & \downarrow \phi \\ \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, k[Y] \\ [(0)] & \longmapsto & \textit{the origin} \end{array}$$

where ϕ is the morphism induced by the k-algebra homomorphism $\phi^*: k[Y]\to R[X]$ defined by $\phi^*(Y_i)=g_i-Y_i \bmod J$. If we apply Corollary 1.6 to the case $l=0$, we obtain $\mathcal{M}_H\neq\emptyset$. *Q. E. D.*

2. Numerical semigroups generated by 2 or 3 elements.

In this section we will show that numerical semigroups generated by 2 or 3 elements are of torus embedding type. First we consider the following numerical semigroups:

DEFINITION 2.1. A numerical semigroup H with $M(H)=\{a_1, \cdots, a_n\}$ is called a *strictly complete intersection* if renumbering $a_1, \cdots, a_n$ the least common multiple of $(a_1, \cdots, a_{i-1})$ and a_i belongs to $\langle a_1, \cdots, a_{i-1}\rangle$ for $2\leqq i\leqq n$. In this case by [5] a set of generators for the ideal I_H is well-known.

REMARK 2.2. For a numerical semigroup H as in Definition 2.1 we have $\alpha_i=(a_1, \cdots, a_{i-1})/(a_1, \cdots, a_i)$ for $2\leqq i\leqq n$. If we set

$$\alpha_i a_i=\sum_{j=1}^{i-1}\alpha_{ij}a_j \qquad \text{with} \quad \alpha_{ij}\in N$$

for $2\leqq i\leqq n$, then the ideal I_H is generated by $f_2, \cdots, f_n$ where we set $f_i= X_i^{\alpha_i}-X_1^{\alpha_{i1}}\cdots X_{i-1}^{\alpha_{i\,i-1}}$.

LEMMA 2.3. *A numerical semigroup H which is a strictly complete intersection, is of torus embedding type.*

PROOF. We use the notation in Remark 2.2. The set

$$U=\{(i, j)\in N^2 \mid 2\leqq i\leqq n \quad \text{and} \quad 1\leqq j\leqq i-1\}$$

is a totally ordered set, where we define $(i, j)\leqq(i', j')$ if $i<i'$ or if $i=i'$ and $j\leqq j'$. If we set $P=\{(i, j)\in U \mid \alpha_{ij}\neq 0\}$ and $l={}^{\#}P$, then we have the isomorphism $\xi: P\rightarrow [1, l]$ of ordered sets. Let

$$\pi: k[Y_{ij}((i, j)\in P); Z_k(2\leqq k\leqq n)] \longrightarrow k[t_1, \cdots, t_l]$$

be the k-algebra homomorphism of polynomial rings, defined by $\pi(Y_{ij})=t_{\xi(i,j)}$ and $\pi(Z_k)=\prod_{j\in P(k)} t_{\xi(k,j)}$ where $P(k)=\{j\in[1, k-1] \mid (k, j)\in P\}$. We set

$$g_{\xi(i,j)}=X_j^{\alpha_{ij}} \quad \text{for } (i, j)\in P \quad \text{and} \quad g_{l+k-1}=X_k^{\alpha_k} \quad \text{for } 2\leqq k\leqq n\,.$$

Let $\eta: k[Y_{ij}; Z_k]\rightarrow k[X]=k[X_1, \cdots, X_n]$ (resp. $\zeta: k[t_1, \cdots, t_l]\rightarrow k[t]$) be the k-algebra homomorphism defined by $\eta(Y_{ij})=g_{\xi(i,j)}$ and $\eta(Z_k)=g_{l+k-1}$ (resp. $\zeta(t_{\xi(i,j)})=t^{\alpha_{ij}a_j}$). In virtue of $\varphi_H\circ\eta=\zeta\circ\pi$, we get $\eta(\operatorname{Ker}\pi)\subseteqq\operatorname{Ker}\varphi_H=I_H$. If we set $F_k=Z_k-\prod_{j\in P(k)} Y_{kj}$ for $2\leqq k\leqq n$, then $F_k\in\operatorname{Ker}\pi$ and $\eta(F_k)=f_k$. Therefore by Remark 2.2 the ideal I_H is generated by the elements of $\eta(\operatorname{Ker}\pi)$. By Lemma 1.2 H is of torus embedding type. *Q. E. D.*

COROLLARY 2.4. 1) *Numerical semigroups with* $M(H)=\{a_1, a_2\}$ *are of torus embedding type.*

2) *Symmetric numerical semigroups, i.e.,* $C(H)=2g(H)$, *with* $M(H)=\{a_1, a_2, a_3\}$ *are of torus embedding type.*

PROOF. It is trivial that numerical semigroups with $M(H)=\{a_1, a_2\}$ are are strictly complete intersections. Herzog [5] proved that numerical semigroups H with $M(H)=\{a_1, a_2, a_3\}$ are strictly complete intersections if and only if they are symmetric. *Q. E. D.*

In the non-symmetric case H with $M(H)=\{a_1, a_2, a_3\}$, H is also of torus embedding type in the following way: by [5] there exist positive integers $\alpha_{ij}<\alpha_j$ such that

$$\alpha_1 a_1=\alpha_{12}a_2+\alpha_{13}a_3\,, \quad \alpha_2 a_2=\alpha_{21}a_1+\alpha_{23}a_3 \quad \text{and} \quad \alpha_3 a_3=\alpha_{31}a_1+\alpha_{32}a_2\,,$$

in this case

$$\alpha_1=\alpha_{21}+\alpha_{31}\,, \quad \alpha_2=\alpha_{12}+\alpha_{32} \quad \text{and} \quad \alpha_3=\alpha_{13}+\alpha_{23}\,.$$

Moreover, Herzog showed that the ideal I_H is generated by

$$f_1=X_1^{\alpha_1}-X_2^{\alpha_{12}}X_3^{\alpha_{13}}\,, \quad f_2=X_2^{\alpha_2}-X_1^{\alpha_{21}}X_3^{\alpha_{23}} \quad \text{and} \quad f_3=X_3^{\alpha_3}-X_1^{\alpha_{31}}X_2^{\alpha_{32}}\,.$$

Let S be the subsemigroup of $\boldsymbol{Z}^4$ generated by

$$b_{21}=(1,0,0,0)\,,\quad b_{12}=(0,1,0,0)\,,\quad b_{13}=(0,0,1,0)\,,\quad b_{31}=(-1,1,1,0)\,,$$
$$b_{32}=(0,0,0,1)\quad\text{and}\quad b_{23}=(-1,1,0,1)\,.$$

Then it can be easily seen that $S=\Sigma \boldsymbol{R}_{+}b_{ij}\cap \boldsymbol{Z}^{4}$ where $\boldsymbol{R}_{+}$ is the set of non-negative real numbers. Hence S is saturated. When we let

$$\pi: k[Y_{ij}]_{1\leqq i\neq j\leqq 3} \longrightarrow k[T^{s}]_{s\in S} \qquad (\text{resp. } \eta: k[Y_{ij}]\to k[X_1, X_2, X_3])$$

be the k-algebra homomorphism defined by $\pi(Y_{ij})=T^{b_{ij}}$ (resp. $\eta(Y_{ij})=X_j^{\alpha_{ij}}$), there exists a k-algebra homomorphism $\zeta: k[T^{s}]_{s\in S}\to k[t]$ such that $\varphi_H\circ\eta=\zeta\circ\pi$, which implies $\eta(\mathrm{Ker}\,\pi)\subseteqq I_H$. Since

$$F_1=Y_{21}Y_{31}-Y_{12}Y_{13}\,,\quad F_2=Y_{12}Y_{32}-Y_{21}Y_{23}\quad\text{and}\quad F_3=Y_{13}Y_{23}-Y_{31}Y_{32}$$

belong to $\mathrm{Ker}\,\pi$ and we have $\eta(F_i)=f_i$ for $1\leqq i\leqq 3$, the ideal I_H is generated by the elements of $\eta(\mathrm{Ker}\,\pi)$, hence H is of torus embedding type. Therefore combining this with Corollary 2.4 2), we obtain the following:

PROPOSITION 2.5. *Numerical semigroups with $M(H)=\{a_1, a_2, a_3\}$ are of torus embedding type.*

3. Neat numerical semigroups.

Hereafter we are concerned with the following numerical semigroups:

DEFINITION 3.1. For a numerical semigroup H with $M(H)=\{a_1, \cdots, a_n\}$,

$$\mathfrak{R}:\begin{cases} \alpha_i a_i=\sum_{j\neq i}\alpha_{ij}a_j \quad \text{with} \quad 0\leqq\alpha_{ij}<\alpha_j\,, & \text{for} \quad 1\leqq i\leqq n\,,\\ \sum_{i\neq j}\alpha_{ij}=\alpha_j & \text{for} \quad 1\leqq j\leqq n\end{cases}$$

is called a *neat system of relations with respect to H and $\{a_1, \cdots, a_n\}$*. When H has a neat system of relations, it is called to be *neat*.

EXAMPLE 3.2. (1) $H=\langle 4,7,13\rangle$ is neat. In fact, let $a_1=4$, $a_2=7$ and $a_3=13$. Then

$$\mathfrak{R}: 5a_1=a_2+a_3\,,\quad 3a_2=2a_1+a_3\,,\quad 2a_3=3a_1+2a_2$$

is a neat system of relations.

(2) $H=\langle 4,9,14,15\rangle$ is neat. In fact, let $a_1=15$, $a_2=9$, $a_3=4$ and $a_4=14$. Then

$$\mathfrak{R}: 2a_1=4a_3+a_4\,,\quad 2a_2=a_3+a_4\,,\quad 6a_3=a_1+a_2\,,\quad 2a_4=a_1+a_2+a_3$$

is a neat system of relations.

(3) $H=\langle 10, 11, 13, 14\rangle$ is neat. In fact, let $a_1=10$, $a_2=11$, $a_3=14$ and $a_4=13$. Then

$$\mathfrak{R}: 4a_1=a_3+2a_4, \quad 3a_2=2a_1+a_4, \quad 3a_3=2a_1+2a_2, \quad 3a_4=a_2+2a_3$$

is a neat system of relations.

(4) $H=\langle 5, 7, 9, 11, 13\rangle$ is neat. In fact, let $a_1=5$, $a_2=7$, $a_3=9$, $a_4=11$ and $a_5=13$. Then

$$\mathfrak{R}: 4a_1=a_2+a_5, \quad 2a_2=a_1+a_3, \quad 2a_3=a_2+a_4, \quad 2a_4=a_3+a_5, \quad 2a_5=3a_1+a_4$$

is a neat system of relations.

In this section, let H be a neat numerical semigroup with $M(H)=\{a_1, \cdots, a_n\}$, and let $\mathfrak{R}$ be a neat system of relations with respect to H and $\{a_1, \cdots, a_n\}$. We can see easily:

REMARK 3.3. We put

$$P=P_{\mathfrak{R}}=\{(i, j)\in[1, n]^2 \mid i\neq j, \alpha_{ij}\neq 0\}, \quad P^i=\{j\in[1, n] \mid (i, j)\in P\}$$

for $1\leqq i\leqq n$ and $P_j=|i\in[1, n]|(i, j)\in P\}$ for $1\leqq j\leqq n$.

Then ${}^{\#}P^i\geqq 2$ and ${}^{\#}P_j\geqq 2$. Hence we have ${}^{\#}P\geqq 2n$, for

$$P=\bigcup_{1\leqq i\leqq n}\{(i, j)\mid j\in P^i\}=\bigcup_{1\leqq j\leqq n}\{(i, j)\mid i\in P_j\}.$$

Moreover, we make P into a totally ordered set by defining an order on it as follows: for a fixed $j\in[1, n]$ and any $1\leqq k\leqq {}^{\#}P_j$ we define inductively

$$i_j(k)=\operatorname{Min}\{i\in[1, n] \mid i\in P_j-\{i_j(1), \cdots, i_j(k-1)\}\}.$$

For any (i, j) and $(i', j')\in P$ with $i=i_j(k)$ and $i'=i_{j'}(k')$, we define $(i, j)\leqq(i', j')$ if $k<k'$ or if $k=k'$ and $j\leqq j'$.

DEFINITION 3.4. An element (i, j) of P has a *v-relation* (resp. an *h-relation*) if we have

$$i=\operatorname{Max}\{i'\in[1, n] \mid i'\in P_j\} \quad \text{and} \quad P^j(i, j)=\emptyset$$

where $P^j(i, j)=\{j'\in P^j \mid (j, j')>(i, j)\}$

(resp. $(i, j)=\operatorname{Max}\{(i, j') \mid j'\in P^i\}$ and $P_i(i, j)=\emptyset$

where $P_i(i, j)=\{i'\in P_i \mid (i', i)>(i, j)\}$).

v-relations and h-relations have the following properties:

LEMMA 3.5. 1) $(i_0, j_0)=\operatorname{Max} P$ *has a v-relation and an h-relation.*

2) *For any* $1\leqq l\leqq n$, *there exists* $i\in[1, n]$ *such that* (i, l) *has a v-relation or*

$j\in[1, n]$ such that (l, j) has an h-relation.

3) *We have $^{\#}Q\leqq n-1$ where*

$$Q=\{(i, j)\in P \mid (i, j) \text{ has either a } v\text{-relation or an } h\text{-relation}\}.$$

PROOF. 1) is trivial. We set

$$i=\mathrm{Max}\, P_l \quad \text{and} \quad (l, j)=\mathrm{Max}\{(l, j') \mid j'\in P^l\}.$$

Assume that (i, l) does not have a v-relation and that (l, j) does not have an h-relation. Then there exist $j'\in P^l(i, l)$ and $i'\in P_l(l, j)$, which imply

$$(i, l)\geqq(i', l)>(l, j)\geqq(l, j')>(i, l),$$

a contradiction. This proves 2). Let $l\in[1, n]$. If (i, l) has a v-relation, then we define $\zeta(l)=(i, l)$. If (l, j) has an h-relation, then we define $\zeta(l)=(l, j)$. Then the map $\zeta: [1, n]\to Q$ is well-defined. In fact, if (i, l) (resp. (i', l)) has a v-relation, then $i=\mathrm{Max}\, P_l=i'$. If (l, j) (resp. (l, j')) has an h-relation, then $(l, j)=\mathrm{Max}\{(l, k) \mid k\in P^l\}=(l, j')$, hence $j=j'$. If (i, l) (resp. (l, j)) has a v-relation (resp. an h-relation), then we have $(i, l)\leqq(l, j)\leqq(i, l)$, hence $l=j$, a contradiction. To prove 3) it suffices to show that ζ is surjective, because we have $\zeta(i_0)=(i_0, j_0)=\zeta(j_0)$. If $(i, j)\in Q$ has a v-relation (resp. an h-relation), then $\zeta(j)=(i, j)$ (resp., $\zeta(i)=(i, j)$). Hence ζ is surjective. *Q.E.D.*

Finally we define the subset P_H of $\mathcal{S}_n=\{(i, j)\in[1, n]^2 \mid i\neq j\}$ associated to a neat numerical semigroup H with $M(H)=\{a_1, \cdots, a_n\}$ as follows:

DEFINITION 3.6. We define an order on the set of subsets of $\mathcal{S}_n$ in the following way:

1) for any (i, j) and $(i', j')\in\mathcal{S}_n$, we define $(i, j)\leqq(i', j')$ if $i<i'$ or if $i=i'$ and $j\leqq j'$,

2) for two subsets P and P' of $\mathcal{S}_n$ with $^{\#}P={}^{\#}P'={}^{\#}\mathcal{S}_n-r$, we define $P\leqq P'$ if there exists $0\leqq q\leqq r$ such that

$$(i_1, j_1)=(i'_1, j'_1), \cdots, (i_q, j_q)=(i'_q, j'_q) \quad \text{and} \quad (i_{q+1}, j_{q+1})<(i'_{q+1}, j'_{q+1})$$

where

$$\mathcal{S}_n-P=\{(i_1, j_1)<\cdots<(i_r, j_r)\} \quad \text{and} \quad \mathcal{S}_n-P'=\{(i'_1, j'_1)<\cdots<(i'_r, j'_r)\},$$

3) for two subsets P and P' of $\mathcal{S}_n$ we define $P\leqq P'$ if $^{\#}P<{}^{\#}P'$ or if $^{\#}P={}^{\#}P'$ and $P\leqq P'$.

Then the set of subsets of $\mathcal{S}_n$ becomes a totally ordered set. Using this order, we define the subset P_H of $\mathcal{S}_n$:

$P_H=\mathrm{Min}\{P_{H,\{a_{\sigma(1)},\cdots,a_{\sigma(n)}\}}\mid \sigma$ runs over the set of permutations of $[1, n]\}$

where

$P_{H,\{a_1,\cdots,a_n\}}=\mathrm{Min}\{P_{\mathfrak{R}}\mid \mathfrak{R}$ runs over the set of neat systems of relations with respect to H and $\{a_1, \cdots, a_n\}\}$.

4. Neat numerical semigroups generated by 4 elements.

In this section, we are devoted to neat numerical semigroups H with $M(H)=\{a_1, a_2, a_3, a_4\}$. In the case $^{\#}M(H)=4$ we can explain v-relations and h-relations in detail.

LEMMA 4.1. *Let $\mathfrak{R}$ be a neat system of relations with respect to H and $\{a_1, a_2, a_3, a_4\}$. Then*

1) *$(i, j)\in P_{\mathfrak{R}}$ has a v-relation and an h-relation if and only if $(i, j)=\mathrm{Max}\, P_{\mathfrak{R}}$,*

2) *we have $^{\#}Q=3$ where*

$$Q=\{(i, j)\in P_{\mathfrak{R}}\mid (i, j) \text{ has either a } v\text{-relation or an } h\text{-relation}\}.$$

PROOF. To check 1), by Lemma 3.5 1) it suffices to show the "only if" part. For brevity, we put $P=P_{\mathfrak{R}}$. Let us take $(i, j)\in P$ which has a v-relation and an h-relation. Then for any $k\in[1,4]$ the following hold:

a) if $(i, k)\in P$, then $(i, k)\leqq(i, j)$, b) if $(j, k)\in P$, then $(j, k)<(i, j)$, c) if $(k, i)\in P$, then $(k, i)<(i, j)$, d) if $(k, j)\in P$, then $(k, j)\leqq(i, j)$.

From now on we will see that for $(k, l)\in P$ with $k, l\in[1,4]-\{i, j\}$, $(k, l)<(i, j)$. The case $i=1$ does not occur, because (i, j) has a v-relation. Moreover, since for $k=1$ we have $(k, l)<(i, j)$, we may assume $j=1$ or $l=1$.

(A) $j=1$. Then $i=3$ or 4, because $i\geqq i_1(2)\geqq 3$.

1) $i=3$. Then $(i_3(2), 3)<(3, 1)=(i_1(2), 1)$, a contradiction.

2) $i=4$. Then $(k, l)=(2, 3)$ or $(3, 2)$. If $(k, l)=(2, 3)$, then

$$(k, l)\leqq(i_3(2), 3)<(i_4(2), 4)<(4, 1)=(i, j).$$

If $(k, l)=(3, 2)$, then

$$(k, l)\leqq(i_2(2), 2)<(i_4(2), 4)<(4, 1)=(i, j).$$

(B) $l=1$. Then $k=2$ or 3 or 4.

1) $k=2$. Then $(k, l)=(i_1(1), 1)<(i, j)$.

2) $k=3$. Then $(k, l)\leqq(i_1(2), 1)<(i_j(2), j)\leqq(i, j)$.

3) $k=4$. Then $(i, j)=(2, 3)$ or $(3, 2)$. If $i=i_j(3)$, then

$$(k, l)=(4, 1)\leqq(i_1(3), 1)<(i_j(3), j)=(i, j).$$

Assume $i=i_j(2)$. Then

$$(i, j)=(i_j(2), j)<(i_4(2), 4)<(i, j),$$

because $i_4(2)=2$ or 3. This is a contradiction. Hence we have $(i, j)=\mathrm{Max}\, P$.

By the proof of Lemma 3.5 3), we can define a surjective map $\zeta: [1, 4]\to Q$ by sending l to (i_l, l) (resp. (l, i_l)) if (i_l, l) has a v-relation (resp. if (l, i_l) has an h-relation). Let l and l' be two distinct elements of $[1, 4]$ such that $\zeta(l)=\zeta(l')$. Then $\zeta(l)=\zeta(l')$ has a v-relation and an h-relation. Hence if we set $(i, j)=\mathrm{Max}\, P$, by 1) we get $\{l, l'\}=\{i, j\}$. So $\zeta(k)$, $\zeta(k')$ and $\zeta(i)$ are distinct where we set $[1, 4]=\{i, j, k, k'\}$. Therefore we obtain $^{\#}Q=3$, because ζ is surjective.

Q. E. D.

From now on, we will construct a torus embedding $T_H\times A_k^4$, any irreducible component of whose fibre over the origin of Spec $k[Y_{ij}]_{(i,j)\in P_H}$ is isomorphic to C_H. First let $\mathfrak{R}$ be a neat system of relations with respect to H and $\{a_1, a_2, a_3, a_4\}$, i.e., $\alpha_i a_i=\sum_{j\neq i}\alpha_{ij}a_j$ for $1\leqq i\leqq 4$ and $\alpha_j=\sum_{i\neq j}\alpha_{ij}$ for $1\leqq j\leqq 4$, with $0\leqq\alpha_{ij}<\alpha_j$, and let Y_{ij}, $(i, j)\in P_{\mathfrak{R}}$, (resp. $t_1, \cdots, t_{m-3}$) be independent variables over k where we put $m={}^{\#}P_{\mathfrak{R}}$. Q denotes the set of $(i, j)\in P_{\mathfrak{R}}$ which has either a v-relation or an h-relation. For brevity, we put $P=P_{\mathfrak{R}}$, and let the order on Q (resp. $P-Q$) be induced by that on P defined in Definition 3.3. Then by Lemma 4.1 2) the set Q consists of three elements

$$(i', j')<(i'', j'')<(i_0, j_0),$$

and there exists a unique isomorphism $\xi: P-Q\to[1, m-3]$ of ordered sets. Now we will define a k-algebra homomorphism

$$\pi: k[Y_{ij}]_{(i,j)\in P}\longrightarrow k[t_1^{\pm 1}, \cdots, t_{m-3}^{\pm 1}]$$

inductively as follows:

1) $\pi_1: k[Y_{ij}]_{(i,j)\in P<(i'',j'')}\to k[t_1^{\pm 1}, \cdots, t_{m-3}^{\pm 1}]$ is defined by

$$\pi_1(Y_{ij})=t_{\xi(ij)} \quad \text{if } (i, j)<(i', j'),$$

$$\pi_1(Y_{i'j'})=\begin{cases} \prod_{i\in P_{j'}-\{i'\}} t_{\xi(ij')}^{-1} \prod_{j\in P j'} t_{\xi(j'j)} & \text{if } (i', j') \text{ has a } v\text{-relation},\\ \prod_{j\in P^{i'}-\{j'\}} t_{\xi(i'j)}^{-1} \prod_{i\in P_{i'}} t_{\xi(ii')} & \text{if } (i', j') \text{ has an } h\text{-relation}, \end{cases}$$

and

$$\pi_1(Y_{ij})=t_{\xi(ij)} \quad \text{if } (i', j')<(i, j)<(i'', j''),$$

2) $\pi_2: k[Y_{ij}]_{(i,j)\in P<(i_0,j_0)}\to k[t_1^{\pm 1}, \cdots, t_{m-3}^{\pm 1}]$ is defined by

$$\pi_2(Y_{ij})=\pi_1(Y_{ij}) \quad \text{if } (i, j)<(i'', j''),$$

$$\pi_2(Y_{i''j''})=\begin{cases}\prod_{i\in P_{j''}-\{i''\}}\pi_1(Y_{ij''})^{-1}\prod_{j\in P^{j''}}\pi_1(Y_{j''j}) & \text{if } (i'', j'') \text{ has a } v\text{-relation},\\ \prod_{j\in P^{i''}-\{j''\}}\pi_1(Y_{i''j})^{-1}\prod_{i\in P_{i''}}\pi_1(Y_{ii''}) & \text{if } (i'', j'') \text{ has an } h\text{-relation},\end{cases}$$

and

$$\pi_2(Y_{ij})=t_{\xi(ij)} \quad \text{if } (i'', j'')<(i, j)<(i_0, j_0),$$

3) $\pi: k[Y_{ij}]_{(i,j)\in P}\to k[t_1^{\pm 1}, \cdots, t_{m-3}^{\pm 1}]$ is defined by

$$\pi(Y_{ij})=\pi_2(Y_{ij}) \quad \text{if } (i, j)<(i_0, j_0)$$

and

$$\pi(Y_{i_0j_0})=\prod_{i\in P_{j_0}-\{i_0\}}\pi_2(Y_{ij_0})^{-1}\prod_{j\in P^{j_0}}\pi_2(Y_{j_0j}).$$

We note that

$$\prod_{i\in P_{j_0}-\{i_0\}}\pi_2(Y_{ij_0})^{-1}\prod_{j\in P^{j_0}}\pi_2(Y_{j_0j})=\prod_{j\in P^{i_0}-\{j_0\}}\pi_2(Y_{i_0j})^{-1}\prod_{i\in P_{i_0}}\pi_2(Y_{ii_0}).$$

DEFINITION 4.2. If we canonically identify $k[t_1^{\pm 1}, \cdots, t_{m-3}^{\pm 1}]$ with the semigroup k-algebra $k[T^b]_{b\in Z^{m-3}}$, in the above situation for any $(i, j)\in P$ there exists a unique $b_{ij}\in Z^{m-3}$ such that $\pi(Y_{ij})=T^{b_{ij}}$. Then the subsemigroup S of Z^{m-3} generated by $b_{ij}((i, j)\in P)$ is called the *semigroup associated to* P and the surjective k-algebra homomorphism $\pi: k[Y_{ij}]_{(i,j)\in P}\to k[T^s]_{s\in S}$ is called the *homomorphism associated to* P.

LEMMA 4.3. *Let* $\eta: k[Y_{ij}]_{(i,j)\in P}\to k[X]=k[X_1, X_2, X_3, X_4]$ *be the k-algebra homomorphism defined by sending* Y_{ij} *to* $X_j^{a_{ij}}$. *Then we have* $I_H\supseteqq\eta(\operatorname{Ker}\pi)$.

PROOF. The k-algebra homomorphism $\zeta': k[T^{b_{ij}}]_{(i,j)\in P-Q}\to k[t^h]_{h\in H}$ defined by $\zeta'(T^{b_{ij}})=t^{a_{ij}a_j}$ extends uniquely to the k-algebra homomorphism $\zeta: k[T^s]_{s\in S}\to k[t^h]_{h\in H}$. Moreover,

$$\varphi_H\circ\eta(Y_{ij})=\varphi_H(X_j^{a_{ij}})=t^{a_{ij}a_j}$$

and

$$\zeta\circ\pi(Y_{ij})=\zeta(T^{b_{ij}})=t^{a_{ij}a_j},$$

hence $\varphi_H\circ\eta=\zeta\circ\pi$, which implies $I_H=\operatorname{Ker}\varphi_H\supseteqq\eta(\operatorname{Ker}\pi)$. *Q.E.D.*

Let us recall the definition of P_H in Definition 3.6 which is determined by a neat numerical semigroup H. In our case $M(H)=\{a_1, a_2, a_3, a_4\}$, elementary computations show the following:

PROPOSITION 4.4. P_H *is one of the following*:

(1) *the case* $^{\#}P_H=12$, *then* $P_H=\mathcal{S}_4=\{(i, j)\in[1, 4]^2 \mid i\neq j\}$,

(2) *the case* $^{\#}P_H=11$, *then* $P_H=\mathcal{S}_4-\{(1, 2)\}$,

(3) *the case* $^{\#}P_H=10$, *then* $P_H=\mathcal{S}_4-(\{(1, 2)\}\cup G)$ *where* G *is one of the following*:

a) $\{(2, 1)\}$, b) $\{(2, 3)\}$, c) $\{(3, 4)\}$,

(4) *the case* $^{\#}P_H=9$, *then* $P_H=\mathcal{S}_4-(\{(1, 2)\}\cup G)$ *where* G *is one of the following*:

a) $\{(2, 1), (3, 4)\}$, b) $\{(2, 3), (3, 1)\}$, c) $\{(2, 3), (3, 4)\}$,

(5) *the case* $^{\#}P_H=8$, *then* $P_H=\mathcal{S}_4-(\{(1, 2)\}\cup G)$ *where* G *is one of the following*:

a) $\{(2, 1), (3, 4), (4, 3)\}$ *and* b) $\{(2, 3), (3, 4), (4, 1)\}$.

DEFINITION-PROPOSITION 4.5. Let S_H be the semigroup associated to P_H. Then the subsemigroup S_H of $\boldsymbol{Z}^{m-3}$ is saturated and generates $\boldsymbol{Z}^{m-3}$ as a group. Therefore $T_H=\mathrm{Spec}\, k[Y_{ij}]_{(i,j)\in P_H}/\mathrm{Ker}\,\pi$, which is isomorphic to $\mathrm{Spec}\, k[T^s]_{s\in S_H}$, is called the *torus embedding associated to the neat numerical semigroup* H with $M(H)=\{a_1, a_2, a_3, a_4\}$.

PROOF. By the construction of S_H, S_H generates $\boldsymbol{Z}^{m-3}$ as a group. For any $i\in[1, m-3]$ we denote by $e_i\in\boldsymbol{Z}^{m-3}$ the vector whose i-th component equals to 1 and whose j-th component equals to 0 if $j\neq i$. Let

$$\sigma:[1, m]\longrightarrow P_H=\{(i, j)\in[1, 4]^2 \mid i\neq j,\ \alpha_{ij}\neq 0\}$$

be the isomorphism of ordered sets, and for brevity we set $b_i=b_{\sigma(i)}$ for all $i\in[1, m]$. Let the situation be as in Proposition 4.4. Then

(1) $b_i=e_i$ $(1\leqq i\leqq 8)$, $b_9=(-1, 1, 1, 1, -1, 0, 0, 0, 0)$, $b_{10}=(1, -1, 0, 0, 0, -1, 1, 1, 0)$, $b_{11}=e_9$, $b_{12}=(0, 0, 1, 0, -1, -1, 1, 0, 1)$,

(2) $b_i=e_i$ $(1\leqq i\leqq 7)$, $b_8=(-1, 1, 0, 0, 0, 1, -1, 0)$, $b_9=(-1, 0, 1, 1, -1, 0, 0, 0)$, $b_{10}=e_8$, $b_{11}=(0, -1, 1, 0, -1, 0, 1, 1)$,

(3) a) $b_i=e_i$ $(1\leqq i\leqq 4)$, $b_5=(-1, 0, 1, 1, 0, 0, 0)$, $b_6=e_5$, $b_7=e_6$, $b_8=(0, 1, 0, 0, 1, -1, 0)$, $b_9=e_7$, $b_{10}=(-1, -1, 1, 0, 0, 1, 1)$,

b) $b_i=e_i$ $(1\leqq i\leqq 7)$, $b_8=(-1, 1, 0, 0, 0, 1, 0)$, $b_9=(-1, 0, 1, 1, -1, 0, 0)$, $b_{10}=(0, -1, 1, 0, -1, 0, 1)$,

c) $b_i=e_i$ $(1\leqq i\leqq 7)$, $b_8=(-1, 1, 0, 0, 0, 1, -1)$, $b_9=(-1, 0, 1, 1, -1, 0, 0)$, $b_{10}=(0, 1, -1, 0, 1, 0, -1)$,

(4) a) $b_i=e_i$ $(1\leqq i\leqq 4)$, $b_5=(-1, 0, 1, 1, 0, 0)$, $b_6=e_5$, $b_7=e_6$, $b_8=(0, 1, 0, 0, 1, -1)$, $b_9=(1, 1, -1, 0, 0, -1)$,

b) $b_i=e_i$ $(1\leqq i\leqq 4)$, $b_5=(-1, 0, 1, 1, 0, 0)$, $b_6=e_5$, $b_7=e_6$, $b_8=(-1, 1, 0, 0, 1, 0)$, $b_9=(0, -1, 1, 0, 0, 1)$,

c) $b_i=e_i$ $(1\leqq i\leqq 6)$, $b_7=(0, 1, -1, 0, 1, 0)$, $b_8=(-1, 1, 0, 0, 0, 1)$, $b_9=(-1, 0, 1, 1, -1, 0)$,

(5) a) $b_i=e_i$ $(1\leqq i\leqq 4)$, $b_5=(-1,0,1,1,0)$, $b_6=e_5$, $b_7=(1,1,-1,0,0)$, $b_8=(-1,0,1,0,1)$,

b) $b_i=e_i$ $(1\leqq i\leqq 4)$, $b_5=(-1,0,1,1,0)$, $b_6=e_5$, $b_7=(-1,1,0,1,0)$, $b_8=(-1,1,0,0,1)$.

By computation the subsemigroups S_H of $\boldsymbol{Z}^{m-3}$ generated by $b_1, \cdots, b_m$ are saturated. For example, we check the case (4) c). It suffices to show that $\sum_{i=1}^{9} \boldsymbol{R}_+ b_i \cap \boldsymbol{Z}^6 \subseteqq S_H$ where $\boldsymbol{R}_+$ is the set of non-negative real numbers. Let us take $z=\sum_{i=1}^{9} \lambda_i b_i \in \boldsymbol{Z}^6$ with $\lambda_i \in \boldsymbol{R}_+$, and set $\lambda_i = m_i + \beta_i$ with $m_i \in \boldsymbol{N}$ and $0 \leqq \beta_i < 1$ for $1\leqq i\leqq 9$. Hence it suffices to show that $y=\sum_{i=1}^{9} \beta_i b_i \in S_H$. Now we get

$$y=(\beta_1-\beta_8-\beta_9,\ \beta_2+\beta_7+\beta_8,\ \beta_3-\beta_7+\beta_9,\ \beta_4+\beta_9,\ \beta_5+\beta_7-\beta_9,\ \beta_6+\beta_8)\in \boldsymbol{Z}^6,$$

hence

$$\beta_1-\beta_8-\beta_9=-1 \text{ or } 0,\ \beta_2+\beta_7+\beta_8=0 \text{ or } 1 \text{ or } 2,\ \beta_3-\beta_7+\beta_9=0 \text{ or } 1,$$

$$\beta_4+\beta_9=0 \text{ or } 1,\ \beta_5+\beta_7-\beta_9=0 \text{ or } 1, \text{ and } \beta_6+\beta_8=0 \text{ or } 1.$$

First assume $\beta_1-\beta_8-\beta_9=0$. Since $e_i \in S_H$ for all $1\leqq i\leqq 6$, we get $y\in S_H$. Secondly assume $\beta_1-\beta_8-\beta_9=-1$. Then we have $\beta_8>0$ and $\beta_9>0$, which imply $\beta_2+\beta_7+\beta_8=1$ or 2, $\beta_4+\beta_9=1$ and $\beta_6+\beta_8=1$. Then $y\in S_H$, because $(-1,1,0,1,0,1)=b_4+b_8\in S_H$. Therefore S_H is saturated. The other cases work similarly.

Q.E.D.

For our purposes it is necessary to investigate generators of the ideal I_H. When H is a neat numerical semigroup with $M(H)=\{a_1, a_2, a_3, a_4\}$, the following Lemma gives us a set of generators for I_H.

LEMMA 4.6. *Let H be a numerical semigroup with* $M(H)=\{a_1, a_2, a_3, a_4\}$, *such that for any* $1\leqq i\leqq 4$

$$\alpha_i a_i = \alpha_{ij} a_j + \alpha_{ik} a_k + \alpha_{il} a_l \quad \textit{with } \alpha_{ij}>0,\ \alpha_{ik}>0 \textit{ and } \alpha_{il}\geqq 0$$

where i, j, k and l are distinct. For any $1\leqq i\leqq 4$ *we denote* $X_i^{\alpha_i}-X_j^{\alpha_{ij}}X_k^{\alpha_{ik}}X_l^{\alpha_{il}}$ *by* f_i. *Set*

$$A_1=\{f_1, f_2, f_3, f_4\},\quad A_2=\{X_1^{\beta_1}X_2^{\beta_2}-X_3^{\beta_3}X_4^{\beta_4}\in I_H \mid 0<\beta_i<\alpha_i\},$$

$$A_3=\{X_1^{\beta_1}X_3^{\beta_3}-X_2^{\beta_2}X_4^{\beta_4}\in I_H \mid 0<\beta_i<\alpha_i\},\quad A_4=\{X_1^{\beta_1}X_4^{\beta_4}-X_2^{\beta_2}X_3^{\beta_3}\in I_H \mid 0<\beta_i<\alpha_i\}.$$

Moreover, for any $2\leqq i\leqq 4$ *we put*

$$A_i^*=\{X_1^{\beta_1}X_i^{\beta_i}-X_j^{\beta_j}X_k^{\beta_k}\in A_i \mid \textit{for any } X_1^{\gamma_1}X_i^{\gamma_i}-X_j^{\gamma_j}X_k^{\gamma_k}\in A_i, \textit{ different}$$
$$\textit{from } X_1^{\beta_1}X_i^{\beta_i}-X_j^{\beta_j}X_k^{\beta_k},\ \gamma_1\leqq\beta_1 \textit{ and } \gamma_i\leqq\beta_i \textit{ do not hold}\}.$$

Then 1) *the ideal* I_H *is generated by the elements of the set* $A_1\cup A_2^*\cup A_3^*\cup A_4^*$, 2) *if* $\alpha_i a_i\neq\alpha_j a_j$ *for* $i\neq j$, *then* $\mu(H)$ *is equal to* $4+{}^*A_2^*+{}^*A_3^*+{}^*A_4^*$.

PROOF. 1) Let (A') (resp. (A), resp. (A^*)) be the ideal generated by the set

$$A'=A_1\cup\{X_i^{\gamma_i}X_j^{\gamma_j}-X_k^{\gamma_k}X_l^{\gamma_l}\in I_H\mid \gamma_i,\gamma_j,\gamma_k,\gamma_l>0 \text{ and } (i, j, k, l) \text{ is a permutation of } [1, 4]\}$$

(resp. the set $A=A_1\cup A_2\cup A_3\cup A_4$, resp. the set $A^*=A_1\cup A_2^*\cup A_3^*\cup A_4^*$).

First we show: $I_H=(A')$, that is, $g=X_i^{\lambda_i}-X_j^{\nu_j}X_k^{\nu_k}X_l^{\nu_l}\in I_H$, with $\lambda_i\geqq\alpha_i$ and a permutation (i, j, k, l) of $[1, 4]$, belongs to (A'), i.e., $g=f+\left(\prod_{s=1}^{4}X_s^{\kappa_s}\right)h$ with $f\in(A')$ and $\partial h<\partial g$ if $h\neq 0$. If we set $\lambda_i=\alpha_i q+r$ with $q>0$ and $0\leqq r<\alpha_i$, then

$$G=g-X_i^r(X_i^{\alpha_i q}-X_j^{\alpha_{ij}q}X_k^{\alpha_{ik}q}X_l^{\alpha_{il}q})=X_i^rX_j^{\alpha_{ij}q}X_k^{\alpha_{ik}q}X_l^{\alpha_{il}q}-X_j^{\nu_j}X_k^{\nu_k}X_l^{\nu_l}.$$

Then we can write $G=f+\left(\prod_{s=1}^{4}X_s^{\kappa_s}\right)h$ with $f\in(A')$ and $\partial h<\partial g$ if $h\neq 0$.

Secondly we see: $I_H=(A)$, that is, $g=X_i^{\gamma_i}X_j^{\gamma_j}-X_k^{\gamma_k}X_l^{\gamma_l}\in I_H$, with $\gamma_i,\gamma_j,\gamma_k,\gamma_l>0$ and a permutation (i, j, k, l) of $[1, 4]$, belongs to (A). We may assume that $\gamma_i=\alpha_i q+r$ with $q>0$ and $0\leqq r<\alpha_i$. Hence we have

$$G=g-X_i^rX_j^{\gamma_j}(X_i^{\alpha_i q}-X_j^{\alpha_{ij}q}X_k^{\alpha_{ik}q}X_l^{\alpha_{il}q})=X_i^rX_j^{\gamma_j+\alpha_{ij}q}X_k^{\alpha_{ik}q}X_l^{\alpha_{il}q}-X_k^{\gamma_k}X_l^{\gamma_l}.$$

Then we can write $G=\left(\prod_{s=1}^{4}X_s^{\kappa_s}\right)h$ with $\partial h<\partial g$ if $h\neq 0$.

Lastly we check: $I_H=(A^*)$. Let us take $g=X_1^{\gamma_1}X_i^{\gamma_i}-X_j^{\gamma_j}X_k^{\gamma_k}\in A_i$ such that there exists $g_i=X_1^{\beta_1}X_i^{\beta_i}-X_j^{\beta_j}X_k^{\beta_k}\in A_i^*$ with $\gamma_1\geqq\beta_1$, $\gamma_i\geqq\beta_i$ and $(\gamma_1, \gamma_i)\neq(\beta_1, \beta_i)$. Then

$$G=g-X_1^{\gamma_1-\beta_1}X_i^{\gamma_i-\beta_i}g_i=X_1^{\gamma_1-\beta_1}X_i^{\gamma_i-\beta_i}X_j^{\beta_j}X_k^{\beta_k}-X_j^{\gamma_j}X_k^{\gamma_k}=X_j^{\kappa_j}X_k^{\kappa_k}\cdot h$$

with $\partial h<\partial g$.

2) It suffices to show that the images of elements of $A_1\cup A_2^*\cup A_3^*\cup A_4^*$ in $I_H/(X_1, X_2, X_3, X_4)I_H$ are linearly independent over k. By the assumptions $\alpha_i a_i\neq\alpha_j a_j$ and the minimality of α_i, the weights of elements of $A_1\cup A_2\cup A_3\cup A_4$ are distinct. For brevity, the ideal (X_1, X_2, X_3, X_4) (resp. $X_1^{\beta_1}X_i^{\beta_i}-X_j^{\beta_j}X_k^{\beta_k}\in A_i^*$) is denoted by (X) (resp. $g_{\beta_1\beta_i}^{(i)}$). Let

$$\sum_{i=1}^{4}c_if_i+\sum c_{\beta_1\beta_2}^{(2)}g_{\beta_1\beta_2}^{(2)}+\sum c_{\beta_1\beta_3}^{(3)}g_{\beta_1\beta_3}^{(3)}+\sum c_{\beta_1\beta_4}^{(4)}g_{\beta_1\beta_4}^{(4)}\in(X)I_H,$$

with c_i, $c_{\beta_1\beta_2}^{(2)}$, $c_{\beta_1\beta_3}^{(3)}$, $c_{\beta_1\beta_4}^{(4)}\in k$. First assume that $c_i\neq 0$. Since the ideal $(X)I_H$ is homogeneous, we get $c_if_i\in(X)I_H$, which has an expression:

$$c_if_i=\sum_{m=1}^{4}h_mf_m+\sum h_{\beta_1\beta_2}^{(2)}g_{\beta_1\beta_2}^{(2)}+\sum h_{\beta_1\beta_3}^{(3)}g_{\beta_1\beta_3}^{(3)}+\sum h_{\beta_1\beta_4}^{(4)}g_{\beta_1\beta_4}^{(4)}$$

with h_m, $h^{(2)}_{\beta_1\beta_2}$, $h^{(3)}_{\beta_1\beta_3}$, $h^{(4)}_{\beta_1\beta_4}\in(X)$. If we substitute 0 for X_j, all j different from i, then we get $c_iX_i^{\alpha_i}=cX_i^{\beta+\alpha_i}$ with $c\in k$ and $\beta>0$, a contradiction. Hence $c_i=0$ for all $i=1,\cdots,4$. Secondly assume that $c^{(i)}_{\beta_1\beta_i}\neq 0$. Then $c^{(i)}_{\beta_1\beta_i}g^{(i)}_{\beta_1\beta_i}\in(X)I_H$, which has an expression:

$$c^{(i)}_{\beta_1\beta_i}g^{(i)}_{\beta_1\beta_i}=\sum h^{(2)}_{\beta_1\beta_2}g^{(2)}_{\beta_1\beta_2}+\sum h^{(3)}_{\beta_1\beta_3}g^{(3)}_{\beta_1\beta_3}+\sum h^{(4)}_{\beta_1\beta_4}g^{(4)}_{\beta_1\beta_4}$$

because of $g^{(i)}_{\beta_1\beta_i}\in A_i$ and the minimality of α_j. Substituting 0 for X_j and X_k, where $(1, i, j, k)$ is a permutation of $[1, 4]$, we obtain

$$c^{(i)}_{\beta_1\beta_i}X_1^{\beta_1}X_i^{\beta_i}=\sum_{(\gamma_1,\gamma_i)\neq(\beta_1,\beta_i)} h^{(i)}_{\gamma_1\gamma_i}(X_1, 0, X_i, 0)X_1^{\gamma_1}X_i^{\gamma_i}\,,$$

hence there exists $(\lambda_1, \lambda_i)\in N^2$, $\neq(0, 0)$ such that

$$\beta_1a_1+\beta_ia_i=(\gamma_1+\lambda_1)a_1+(\gamma_i+\lambda_i)a_i\,.$$

If $\beta_1\geqq\gamma_1+\lambda_1$, in virtue of $\alpha_1>\beta_1$ we have $\beta_1=\gamma_1+\lambda_1$ and $\beta_i=\gamma_i+\lambda_i$, which contradict $g^{(i)}_{\beta_1\beta_i}\in A_i^*$. If $\beta_1<\gamma_1+\lambda_1$, we have

$$(\beta_i-\gamma_i-\lambda_i)a_i=(\gamma_1+\lambda_1-\beta_1)a_1\,,$$

which contradicts the minimality of α_i. Hence we get $c^{(i)}_{\beta_1\beta_i}=0$. *Q.E.D.*

For a neat system $\mathfrak{R}$: $\alpha_ia_i=\sum\alpha_{ij}a_j$ for $1\leqq i\leqq 4$ and $\alpha_j=\sum\alpha_{ij}$ for $1\leqq j\leqq 4$, of relations with respect to H with $M(H)=\{a_1, a_2, a_3, a_4\}$, the following holds:

LEMMA 4.7. *We have*

$$D=\begin{vmatrix}\alpha_1 & -\alpha_{12} & -\alpha_{13}\\ -\alpha_{21} & \alpha_2 & -\alpha_{23}\\ -\alpha_{31} & -\alpha_{32} & \alpha_3\end{vmatrix}>0\,.$$

PROOF. Since we have $\alpha_j=\sum_{i\neq j}\alpha_{ij}$ for $1\leqq j\leqq 4$, we obtain

$$D=\begin{vmatrix}\alpha_1 & -\alpha_{12} & -\alpha_{13}\\ -\alpha_{21} & \alpha_2 & -\alpha_{23}\\ \alpha_{41} & \alpha_{42} & \alpha_{43}\end{vmatrix}=\alpha_{41}\begin{vmatrix}-\alpha_{12} & -\alpha_{13}\\ \alpha_2 & -\alpha_{23}\end{vmatrix}-\alpha_{42}\begin{vmatrix}\alpha_1 & -\alpha_{13}\\ -\alpha_{21} & -\alpha_{23}\end{vmatrix}$$

$$+\alpha_{43}\begin{vmatrix}\alpha_1 & -\alpha_{12}\\ -\alpha_{21} & \alpha_2\end{vmatrix}$$

$$=\alpha_{41}(\alpha_{12}\alpha_{23}+\alpha_2\alpha_{13})+\alpha_{42}(\alpha_1\alpha_{23}+\alpha_{21}\alpha_{13})+\alpha_{43}\{\alpha_1(\alpha_{32}+\alpha_{42})+(\alpha_{31}+\alpha_{41})\alpha_{12}\}$$

If $\alpha_{43}>0$, then $D>0$ because of $\alpha_{43}\alpha_1(\alpha_{32}+\alpha_{42})>0$. If $\alpha_{43}=0$, then $\alpha_{41}>0$ and $\alpha_{13}>0$, hence we get $D>0$. *Q.E.D.*

Hereafter we are in the following situation, which is similar to that in Corollary 1.6: let $P=P_H$ be as in Definition 3.6 and let $T_H=\mathrm{Spec}\ k[Y_{ij}]_{(i,j)\in P}/\mathrm{Ker}\ \pi$ be the torus embedding associated to the neat numerical semigroup H with $M(H)=\{a_1, a_2, a_3, a_4\}$. Let us consider the fibre product:

$$\begin{array}{ccc} \psi^{-1}(O) & \longrightarrow & T_H\times A_k^4\cong \mathrm{Spec}(k[Y_{ij}]/J)[X_1, X_2, X_3, X_4] \\ \big\downarrow & & \big\downarrow \psi \\ \mathrm{Spec}\ k & \longrightarrow & \mathrm{Spec}\ k[Y_{ij}]_{(i,j)\in P} \end{array}$$

where O and J are respectively the origin of $\mathrm{Spec}\ k[Y_{ij}]$ and the ideal $\mathrm{Ker}\ \pi$, and ψ is the morphism corresponding to the k-algebra homomorphism $\psi^*: k[Y_{ij}] \to (k[Y_{ij}]/J)[X_1, X_2, X_3, X_4]$ by sending Y_{ij} to $X_j^{\alpha_{ij}}-Y_{ij} \bmod J$. If J_0 is the ideal in $k[X]=k[X_1, X_2, X_3, X_4]$ generated by the set $\eta(J)$ where $\eta: k[Y_{ij}]\to k[X]$ is the k-algebra homomorphism defined by $\eta(Y_{ij})=X_j^{\alpha_{ij}}$, then $\psi^{-1}(O)$ is isomorphic to $\mathrm{Spec}\ k[X]/J_0$.

PROPOSITION 4.8. *C_H is an irreducible component in $\psi^{-1}(O)=\mathrm{Spec}\ k[X]/J_0$.*

PROOF. We use the notation in Lemma 4.6. Since

$$F_i=\prod_{j\in P_i} Y_{ji}-\prod_{j\in P^i} Y_{ij}\in J$$

for all i implies $(f_1, f_2, f_3, f_4)\subseteqq J_0$ and by Lemma 4.3 we have $I_H\supseteqq J_0$, we will check that the ideal I_H is minimal prime over (f_1, f_2, f_3, f_4). Let $\mathfrak{p}$ be any prime ideal in $k[X]$ with $(f_1, f_2, f_3, f_4)\subseteqq \mathfrak{p}\subseteqq I_H$. Let us take

$$g=X_1^{\beta_1}X_2^{\beta_2}-X_3^{\beta_3}X_4^{\beta_4}\in A_2, \qquad \text{hence}\quad \beta_1a_1+\beta_2a_2-\beta_3a_3=\beta_4a_4.$$

By Lemma 4.7, there exists a positive integer μ such that

$$\mu(\beta_1, \beta_2, -\beta_3)=\nu_1(\alpha_1, -\alpha_{12}, -\alpha_{13})+\nu_2(-\alpha_{21}, \alpha_2, -\alpha_{23})+\nu_3(-\alpha_{31}, -\alpha_{32}, \alpha_3)$$

with $\nu_i\in \boldsymbol{Z}$, which implies $\mu\beta_4=\nu_1\alpha_{14}+\nu_2\alpha_{24}+\nu_3\alpha_{34}$. Since $\beta_i>0$ for $1\leqq i\leqq 4$, this case is divided into the following:

1) $\nu_1>0$, $\nu_2>0$, $\nu_3\geqq 0$, 2) $\nu_1>0$, $\nu_2>0$, $\nu_3<0$,

3) $\nu_1>0$, $\nu_2<0$, $\nu_3<0$, 4) $\nu_1\leqq 0$, $\nu_2>0$, $\nu_3<0$.

If $\nu_1>0$, $\nu_2>0$ and $\nu_3\geqq 0$, then

$$\begin{aligned} & X_1^{\nu_2\alpha_{21}+\nu_3\alpha_{31}}X_2^{\nu_1\alpha_{12}+\nu_3\alpha_{32}}X_3^{\nu_3\alpha_3}(X_1^{\mu\beta_1}X_2^{\mu\beta_2}-X_3^{\mu\beta_3}X_4^{\mu\beta_4}) \\ &\quad =X_2^{\nu_2\alpha_2}X_3^{\nu_3\alpha_3}(X_1^{\nu_1\alpha_1}-X_2^{\nu_1\alpha_{12}}X_3^{\nu_1\alpha_{13}}X_4^{\nu_1\alpha_{14}}) \\ &\qquad +X_2^{\nu_1\alpha_{12}}X_3^{\nu_1\alpha_{13}+\nu_3\alpha_3}X_4^{\nu_1\alpha_{14}}(X_2^{\nu_2\alpha_2}-X_1^{\nu_2\alpha_{21}}X_3^{\nu_3\alpha_{23}}X_4^{\nu_4\alpha_{24}}) \\ &\qquad +X_1^{\nu_2\alpha_{21}}X_2^{\nu_1\alpha_{12}}X_3^{\nu_1\alpha_{13}+\nu_2\alpha_{23}}X_4^{\nu_1\alpha_{14}+\nu_2\alpha_{24}}(X_3^{\nu_3\alpha_3}-X_1^{\nu_3\alpha_{31}}X_2^{\nu_3\alpha_{32}}X_4^{\nu_3\alpha_{34}}) \\ &\quad \in(f_1, f_2, f_3)\subseteqq \mathfrak{p}\subseteqq I_H. \end{aligned}$$

Since

$$X_1^{\nu_2\alpha_{21}+\nu_3\alpha_{31}}X_2^{\nu_1\alpha_{12}+\nu_3\alpha_{32}}X_3^{\nu_3\alpha_3}(X_1^{(\mu-1)\beta_1}X_2^{(\mu-1)\beta_2}+\cdots+X_3^{(\mu-1)\beta_3}X_4^{(\mu-1)\beta_4})\notin I_H\,,$$

we get $g=X_1^{\beta_1}X_2^{\beta_2}-X_3^{\beta_3}X_4^{\beta_4}\in\mathfrak{p}$. The other cases work similarly. For $g\in A_3\cup A_4$, the proof of $g\in\mathfrak{p}$ is similar. By Lemma 4.6 $\mathfrak{p}$ coincides with I_H, hence we get our desired result. *Q.E.D.*

If $\psi^{-1}(O)$ and C_H are respectively regarded as the algebraic subsets $V(J_0)$ and $V(I_H)$ of the affine space $\boldsymbol{A}_k^4$, we see:

PROPOSITION 4.9. 1) *For any* $x=(x_1, x_2, x_3, x_4)\in\psi^{-1}(O)$, *different from the origin, we have* $x_i\neq 0$ *for any* $1\leqq i\leqq 4$.

2) *For any* $x=(x_1, x_2, x_3, x_4)\in\psi^{-1}(O)$, *different from the origin, we have* $x^{-1}=(x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1})\in\psi^{-1}(O)$.

3) *Any irreducible component in* $\psi^{-1}(O)$ *is isomorphic to* C_H.

PROOF. In the proof we use the notation in Lemma 4.6.

1) If $x_i=0$ for some i, x must be the origin of $\boldsymbol{A}_k^4$, because J_0 contains the ideal (f_1, f_2, f_3, f_4).

2) We may take generators $F_k(1\leqq k\leqq u)$ of the ideal J as follows:

$$F_k=\prod_{(i,j)\in P}Y_{ij}^{\nu_{ij}}-\prod_{(i,j)\in P}Y_{ij}^{\mu_{ij}}$$

with $\nu_{ij}\mu_{ij}=0$. In virtue of $x\in\psi^{-1}(O)=V(J_0)=V(\eta(J))$, we have

$$\prod x_j^{\nu_{ij}\alpha_{ij}}-\prod x_j^{\mu_{ij}\alpha_{ij}}=0\,,$$

which implies

$$\prod(x_j^{-1})^{\nu_{ij}\alpha_{ij}}-\prod(x_j^{-1})^{\mu_{ij}\alpha_{ij}}=0\,.$$

This means $x^{-1}\in\psi^{-1}(O)$.

3) For any $x=(x_1, x_2, x_3, x_4)\in\psi^{-1}(O)$, different from the origin, let $\varphi_x: k[X]\to k[X]/J_0$ be the k-algebra homomorphism defined by $\varphi_x(X_i)=x_iX_i+J_0$. Then $\operatorname{Ker}\varphi_x$ contains the ideal J_0, because

$$\begin{aligned}\varphi_x(\eta(F_k))&=\prod(x_jX_j)^{\alpha_{ij}\nu_{ij}}-\prod(x_jX_j)^{\alpha_{ij}\mu_{ij}}+J_0\\&=\prod x_j^{\alpha_{ij}\nu_{ij}}(\prod(X_j)^{\alpha_{ij}\nu_{ij}}-\prod(X_j)^{\alpha_{ij}\mu_{ij}})+J_0\\&=\prod x_j^{\alpha_{ij}\nu_{ij}}\eta(F_k)+J_0=J_0\,.\end{aligned}$$

Therefore φ_x induces the homomorphism $\bar{\varphi}_x: k[X]/J_0\to k[X]/J_0$, which is an isomorphism by 2). Since J_0 is homogeneous, $\psi^{-1}(O)$ has a natural $\boldsymbol{G}_m$-action. Then we see that for any $x\in\psi^{-1}(O)$, different from the origin, we have

$$\psi_{x^{-1}}(\text{the closure of } G_m\cdot x)=C_H$$

where $\psi_{x^{-1}}$ is the automorphism of $\psi^{-1}(O)$ corresponding to $\varphi_{x^{-1}}$. Using Proposition 4.8 any irreducible component in $\psi^{-1}(O)$ is isomorphic to C_H. *Q.E.D.*

Lastly, for our purpose we classify neat numerical semigroups H with $M(H)=\{a_1, a_2, a_3, a_4\}$ as follows:

DEFINITION 4.10. In virtue of $(a_1, a_2, a_3, a_4)=1$ and Lemma 4.7, there exists a unique positive integer ν such that

$$\nu a_4=\begin{vmatrix} \alpha_1 & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_2 & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_3 \end{vmatrix}=D.$$

Then the numerical semigroup H is called to be *ν-neat.*

Our main result in this section is the following:

THEOREM 4.11. *1-neat numerical semigroups H are of torus embedding type, hence if the characteristic of k is 0, then we get $\mathcal{M}_H\neq\emptyset$.*

PROOF. Let the situation be as in Proposition 4.4. Since $a_4=D$, by computation we get:

(1) $L_{a_3}(H)=\{\beta_1a_1+\beta_2a_2+\beta_4a_4\,|\,\beta_i\in \boldsymbol{N}$ and $(\beta_1, \beta_2, \beta_4)$ satisfies one of the following: 1) $\beta_1<\alpha_{21}+\alpha_{31}$, $\beta_2<\alpha_{32}$, $\beta_4<\alpha_4$, 2) $\beta_1<\alpha_{31}$, $\alpha_{32}\leqq\beta_2<\alpha_2$, $\beta_4<\alpha_{14}+\alpha_{34}$, 3) $\alpha_{21}+\alpha_{31}\leqq\beta_1<\alpha_1$, $\beta_2<\alpha_{32}$, $\beta_4<\alpha_{24}+\alpha_{34}$, 4) $\alpha_{31}\leqq\beta_1<\alpha_1$, $\alpha_{32}\leqq\beta_2<\alpha_{32}+\alpha_{42}$, $\beta_4<\alpha_{34}$, 5) $\beta_1<\alpha_{31}$, $\alpha_{32}\leqq\beta_2<\alpha_{12}+\alpha_{32}$, $\alpha_{14}+\alpha_{34}\leqq\beta_4<\alpha_4$, 6) $\alpha_{31}\leqq\beta_1<\alpha_{31}+\alpha_{41}$, $\alpha_{32}+\alpha_{42}\leqq\beta_2<\alpha_2$, $\beta_4<\alpha_{34}\}$,

(2) $L_{a_3}(H)=\{\beta_1a_1+\beta_2a_2+\beta_4a_4\,|\,\beta_i\in \boldsymbol{N}$ and $(\beta_1, \beta_2, \beta_4)$ satisfies one of the following: 1) $\beta_1<\alpha_{21}+\alpha_{31}$, $\beta_2<\alpha_{32}$, $\beta_4<\alpha_4$, 2) $\beta_1<\alpha_{31}$, $\alpha_{32}\leqq\beta_2<\alpha_2$, $\beta_4<\alpha_{14}+\alpha_{34}$, 3) $\alpha_{21}+\alpha_{31}\leqq\beta_1<\alpha_1$, $\beta_2<\alpha_{32}$, $\beta_4<\alpha_{24}+\alpha_{34}$, 4) $\alpha_{31}\leqq\beta_1<\alpha_1$, $\alpha_{32}\leqq\beta_2<\alpha_2$, $\beta_4<\alpha_{34}\}$,

(3) a) $L_{a_3}(H)=\{\beta_1a_1+\beta_2a_2+\beta_4a_4\,|\,\beta_i\in \boldsymbol{N}$ and $(\beta_1, \beta_2, \beta_4)$ satisfies one of the following: 1) $\beta_1<\alpha_1$, $\beta_2<\alpha_2$, $\beta_4<\alpha_{34}$, 2) $\beta_1<\alpha_{31}$, $\beta_2<\alpha_{32}$, $\alpha_{34}\leqq\beta_4<\alpha_4$, 3) $\alpha_{31}\leqq\beta_1<\alpha_1$, $\beta_2<\alpha_{32}$, $\alpha_{34}\leqq\beta_4<\alpha_{24}+\alpha_{34}$, 4) $\beta_1<\alpha_{31}$, $\alpha_{32}\leqq\beta_2<\alpha_2$, $\alpha_{34}\leqq\beta_4<\alpha_{14}+\alpha_{34}\}$,

b) $L_{a_1}(H)=\{\beta_2a_2+\beta_3a_3+\beta_4a_4\,|\,\beta_i\in \boldsymbol{N}$ and $(\beta_2, \beta_3, \beta_4)$ satisfies one of the following: 1) $\beta_2<\alpha_2$, $\beta_3<\alpha_3$, $\beta_4<\alpha_{14}$, 2) $\beta_2<\alpha_2$, $\beta_3<\alpha_{13}$, $\alpha_{14}\leqq\beta_4<\alpha_{14}+\alpha_{34}$, 3) $\beta_2<\alpha_{32}$, $\beta_3<\alpha_{13}$, $\alpha_{14}+\alpha_{34}\leqq\beta_4<\alpha_4\}$,

c) $L_{a_3}(H)=\{\beta_1a_1+\beta_2a_2+\beta_4a_4\,|\,\beta_i\in \boldsymbol{N}$ and $(\beta_1, \beta_2, \beta_4)$ satisfies one of the following: 1) $\beta_1<\alpha_{21}+\alpha_{31}$, $\beta_2<\alpha_{32}$, $\beta_4<\alpha_4$, 2) $\beta_1<\alpha_{31}$, $\alpha_{32}\leqq\beta_2<\alpha_2$, $\beta_4<\alpha_{14}$, 3) $\alpha_{21}+\alpha_{31}\leqq\beta_1<\alpha_1$, $\beta_2<\alpha_{32}$, $\beta_4<\alpha_{24}\}$,

(4) a) $L_{a_3}(H)=\{\beta_1a_1+\beta_2a_2+\beta_4a_4 \mid \beta_i\in N$ and $(\beta_1, \beta_2, \beta_4)$ satisfies one of the following: 1) $\beta_1<\alpha_{31}, \beta_2<\alpha_{32}, \beta_4<\alpha_4$, 2) $\beta_1<\alpha_{31}, \alpha_{32}\leqq\beta_2<\alpha_2, \beta_4<\alpha_{14}$, 3) $\alpha_{31}\leqq\beta_1<\alpha_1, \beta_2<\alpha_{32}, \beta_4<\alpha_{24}\}$,

b) $L_{a_4}(H)=\{\beta_1a_1+\beta_2a_2+\beta_3a_3 \mid \beta_i\in N$ and $(\beta_1, \beta_2, \beta_3)$ satisfies one of the following: 1) $\beta_1<\alpha_1, \beta_2<\alpha_2, \beta_3<\alpha_{43}$, 2) $\beta_1<\alpha_1, \beta_2<\alpha_{42}, \alpha_{43}\leqq\beta_3<\alpha_3$, 3) $\beta_1<\alpha_{41}, \alpha_{42}\leqq\beta_2<\alpha_2, \alpha_{43}\leqq\beta_3<\alpha_3\}$,

c) $L_{a_1}(H)=\{\beta_2a_2+\beta_3a_3+\beta_4a_4 \mid \beta_i\in N$ and $(\beta_2, \beta_3, \beta_4)$ satisfies one of the following: 1) $\beta_2<\alpha_2, \beta_3<\alpha_3, \beta_4<\alpha_{14}$, 2) $\beta_2<\alpha_{32}, \beta_3<\alpha_{13}, \alpha_{14}\leqq\beta_4<\alpha_4\}$,

(5) a) $L_{a_1}(H)=\{\beta_2a_2+\beta_3a_3+\beta_4a_4 \mid \beta_i\in N$ and $(\beta_2, \beta_3, \beta_4)$ satisfies one of the following: 1) $\beta_2<\alpha_2, \beta_3<\alpha_{13}, \beta_4<\alpha_{14}$, 2) $\beta_2<\alpha_{42}, \alpha_{13}\leqq\beta_3<\alpha_3, \beta_4<\alpha_{14}$, 3) $\beta_2<\alpha_{32}, \beta_3<\alpha_{13}, \alpha_{14}\leqq\beta_4<\alpha_4\}$,

b) $L_{a_1}(H)=\{\beta_2a_2+\beta_3a_3+\beta_4a_4 \mid \beta_i\in N$ and $(\beta_2, \beta_3, \beta_4)$ satisfies one of the following: 1) $\beta_2<\alpha_2, \beta_3<\alpha_3, \beta_4<\alpha_{14}$, 2) $\beta_2<\alpha_{32}, \beta_3<\alpha_{13}, \alpha_{14}\leqq\beta_4<\alpha_4\}$.

Using the above and Lemma 4.6 we get $J_0=I_H$. For example, in the case (4) c) we will show that $J_0=I_H$. It suffices to show that $J_0\supseteqq I_H$. We use the notation in Lemma 4.6. Assume that $A_2\neq\emptyset$, i.e., take

$$X_1^{\beta_1}X_2^{\beta_2}-X_3^{\beta_3}X_4^{\beta_4}\in A_2\,, \qquad \text{hence} \quad \beta_1a_1+\beta_2a_2=\beta_3a_3+\beta_4a_4\,.$$

Then 1) implies $\beta_4\geqq\alpha_{14}$, hence by 2) we get $\beta_3\geqq\alpha_{13}$. Therefore we have

$$\begin{aligned}\beta_1a_1+\beta_2a_2&=(\beta_3-\alpha_{13})a_3+(\beta_4-\alpha_{14})a_4+\alpha_{13}a_3+\alpha_{14}a_4\\&=(\beta_3-\alpha_{13})a_3+(\beta_4-\alpha_{14})a_4+\alpha_1a_1\,,\end{aligned}$$

which implies

$$\beta_2a_2=(\alpha_1-\beta_1)a_1+(\beta_3-\alpha_{13})a_3+(\beta_4-\alpha_{14})a_4\,.$$

Since $0<\beta_2<\alpha_2$, this contradicts the minimality of α_2, hence $A_2=\emptyset$, which implies $A_2^*=\emptyset$. Now we have

$$g_3=X_1^{\alpha_{21}+\alpha_{41}}X_3^{\alpha_{43}}-X_2^{\alpha_{32}}X_4^{\alpha_{14}}\in A_3\,.$$

Take $X_1^{\beta_1}X_3^{\beta_3}-X_2^{\beta_2}X_4^{\beta_4}\in A_3$, different from g_3. Then 1) implies $\beta_4\geqq\alpha_{14}$, hence by 2) we get $\beta_2\geqq\alpha_{32}$. Therefore we get

$$A_3^*=\{g_3=X_1^{\alpha_{21}+\alpha_{41}}X_3^{\alpha_{43}}-X_2^{\alpha_{32}}X_4^{\alpha_{14}}\}\,.$$

Lastly 1) implies $A_4=\emptyset$. Hence by Lemma 4.6 the ideal I_H is generated by f_1, f_2, f_3, f_4 and g_3. Since we have

$$\pi(Y_{21}Y_{41}Y_{43}-Y_{32}Y_{14})=t_1t_1^{-1}t_5^{-1}t_3t_4t_3^{-1}t_5t_2-t_2t_4=0$$

and

$$\eta(Y_{21}Y_{41}Y_{43}-Y_{32}Y_{14})=X_1^{\alpha_{21}+\alpha_{41}}X_3^{\alpha_{43}}-X_2^{\alpha_{32}}X_4^{\alpha_{14}}=g_3\,,$$

we get $I_H \subseteqq J_0$. The other cases work similarly. Using Lemma 1.2, H is of torus embedding type. *Q. E. D.*

REMARK 4.12. 1) By calculation, any neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$ and $g(H) \leqq 8$ is 1-neat.

2) For a ν-neat numerical semigroup H with $\nu \geqq 2$, $\psi^{-1}(O) = \mathrm{Spec}\, k[X]/J_0$ does not necessarily coincide with $C_H = \mathrm{Spec}\, k[X]/I_H$. For example, let H be the numerical semigroup with $M(H) = \{10, 11, 14, 13\}$. Then $g(H) = 16$ and H is 2-neat. Using Lemma 4.6, I_H is generated by

$$f_1 = X_1^4 - X_3 X_4^2, \quad f_2 = X_2^3 - X_1^2 X_4, \quad f_3 = X_3^3 - X_1^2 X_2^2, \quad f_4 = X_4^3 - X_2 X_3^2,$$

$$f_5 = X_1^3 X_2 - X_3^2 X_4, \quad f_6 = X_1 X_3 - X_2 X_4 \quad \text{and} \quad f_7 = X_1 X_4^2 - X_2^2 X_3,$$

hence $\mu(H) = 7$. But J_0 is generated by f_1, f_2, f_3, f_4 and $X_1^2 X_3^2 - X_2^2 X_4^2$. More explicitly, as an algebraic subset of A_k^4 we have $V(J_0) \supsetneqq V(I_H)$, because $(-1, 1, 1, 1) \in V(J_0) - V(I_H)$.

5. Symmetric numerical semigroups generated by 4 elements.

In this section, we always assume that H is a numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$. Then using Bresinsky's result [1] we will show that any symmetric H is of torus embedding type, in this case if H is not a complete intersection then it is 1-neat. In the symmetric case, a set of generators for the ideal I_H is given by the following, which is due to Bresinsky:

REMARK 5.1. Let H be symmetric, i. e., $2g(H) = C(H)$.

(1) When H is a complete intersection, renumbering a_1, a_2, a_3, a_4 we may assume that $X_1^{\alpha_1} - X_2^{\alpha_2} \in I_H$.

a) The case $X_3^{\alpha_3} - X_4^{\alpha_4} \in I_H$. Then $(a_1, a_2)(a_3, a_4) \in \langle a_1, a_2 \rangle \cap \langle a_3, a_4 \rangle$, hence we put

$$(a_1, a_2)(a_3, a_4) = \beta_1 a_1 + \beta_2 a_2 = \beta_3 a_3 + \beta_4 a_4 .$$

In this case,

$$I_H = (f_1 = X_1^{\alpha_1} - X_2^{\alpha_2},\ f_2 = X_3^{\alpha_3} - X_4^{\alpha_4},\ f_3 = X_1^{\beta_1} X_2^{\beta_2} - X_3^{\beta_3} X_4^{\beta_4}) .$$

b) The case $X_3^{\alpha_3} - X_4^{\alpha_4} \notin I_H$. Then H is a strictly complete intersection.

(2) If H is not a complete intersection, renumbering a_1, a_2, a_3, a_4 we have

$$I_H = (f_1 = X_1^{\alpha_1} - X_3^{\alpha_{13}} X_4^{\alpha_{14}},\ f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}} X_4^{\alpha_{24}},\ f_3 = X_3^{\alpha_3} - X_1^{\alpha_{31}} X_2^{\alpha_{32}},$$

$$f_4 = X_4^{\alpha_4} - X_2^{\alpha_{42}} X_3^{\alpha_{43}},\ f_5 = X_1^{\alpha_{21}} X_3^{\alpha_{43}} - X_2^{\alpha_{32}} X_4^{\alpha_{14}})$$

where

$$0<\alpha_{ij}<\alpha_j, \quad \alpha_1=\alpha_{21}+\alpha_{31}, \quad \alpha_2=\alpha_{32}+\alpha_{42}, \quad \alpha_3=\alpha_{13}+\alpha_{43}, \quad \alpha_4=\alpha_{14}+\alpha_{24}.$$

In this case,

$$a_1=\alpha_2\alpha_3\alpha_{14}+\alpha_{32}\alpha_{13}\alpha_{24}, \quad a_2=\alpha_{21}\alpha_3\alpha_4+\alpha_{31}\alpha_{43}\alpha_{24}, \quad a_3=\alpha_1\alpha_{32}\alpha_4+\alpha_{31}\alpha_{42}\alpha_{14}$$

and

$$a_4=\alpha_1\alpha_2\alpha_{43}+\alpha_{21}\alpha_{42}\alpha_{13},$$

hence H is 1-neat.

PROPOSITION 5.2. *Any symmetric H is of torus embedding type.*

PROOF. In virtue of Lemma 2.3 and Theorem 4.11, it suffices to show that in the case of Remark 5.1 (1) a) H is of torus embedding type. Renumbering a_1 and a_2 (resp. a_3 and a_4), we may assume that $\beta_1\neq 0$ and $\beta_3\neq 0$, hence the following four cases occur:

$$1)\ \beta_2\neq 0 \text{ and } \beta_4\neq 0, \quad 2)\ \beta_2\neq 0 \text{ and } \beta_4=0, \quad 3)\ \beta_2=0 \text{ and } \beta_4\neq 0$$

and

$$4)\ \beta_2=0 \text{ and } \beta_4=0.$$

For the case 1), let

$$\pi: k[Z, Y]=k[Z_1, \cdots, Z_4, Y_1, \cdots, Y_4] \longrightarrow k[t_1^{\pm 1}, \cdots, t_5^{\pm 1}]$$

$$(\text{resp. } \eta: k[Z, Y] \longrightarrow k[X]=k[X_1, \cdots, X_4])$$

be the k-algebra homomorphism defined by $\pi(Z_i)=t_1$ for $i=1, 2$, $\pi(Z_j)=t_2$ for $j=3, 4$, $\pi(Y_k)=t_{2+k}$ for $k=1, 2, 3$ and $\pi(Y_4)=t_3t_4t_5^{-1}$ (resp. $\eta(Z_i)=X_i^{\alpha_i}$ and $\eta(Y_i)=X_i^{\beta_i}$ for $1\leqq i\leqq 4$). Then we see easily that $I_H\supseteqq\eta(\text{Ker}\,\pi)$. Moreover, since $F_1=Z_1-Z_2$, $F_2=Z_3-Z_4$ and $F_3=Y_1Y_2-Y_3Y_4\in\text{Ker}\,\pi$, we have $I_H=(\eta(F_1), \eta(F_2), \eta(F_3))$, which is generated by the set $\eta(\text{Ker}\,\pi)$. Using Lemma 1.2, H is of torus embedding type. The other cases 2), 3), 4) work similarly. *Q. E. D.*

6. Almost symmetric numerical semigroups generated by 4 elements.

In the last section we will give another examples of 1-neat numerical semigroups, which are called to be *almost symmetric*, i. e., $C(H)=2g(H)-1$. In this section we are devoted to proving that any almost symmetric numerical semigroup H with $M(H)=\{a_1, a_2, a_3, a_4\}$ is 1-neat. First we investigate the properties of almost symmetric H with $M(H)=\{a_1, \cdots, a_n\}$.

LEMMA 6.1. *Let H be a numerical semigroup with $M(H)=\{a_1, \cdots, a_n\}$ and h be its element.*

0) *For any* $1 \leqq i \leqq h$ *there exists a unique* $1 \leqq h_i \leqq h$ *such that* $\omega_h(h)-\omega_h(i) \equiv \omega_h(h_i) \bmod h$.

1) *H is almost symmetric if and only if there exists a unique* $i_0 \in [2, h-1]$ *such that* $2\omega_h(i_0)=\omega_h(h)+h$ *and that* $\omega_h(i)+\omega_h(h_i)=\omega_h(h)$ *for all* $i \neq i_0$.

PROOF. The definition of $L_h(H)=\{\omega_h(1)< \cdots <\omega_h(h)\}$ means 0). We see easily:

$$g(H)=\sum_{i=1}^{h}[\omega_h(i)/h] \quad \text{and} \quad C(H)-g(H)=\sum_{i=1}^{h}[(\omega_h(h)-\omega_h(i))/h]$$

where [ ] is the Gauss symbol. For any $1 \leqq i \leqq h$ there exists a unique $n_i \in N$ such that $\omega_h(h)-\omega_h(i)=\omega_h(h_i)-n_i h$. Hence H is almost symmetric if and only if $\sum_{i=1}^{h} n_i=1$. This implies 1). *Q.E.D.*

PROPOSITION 6.2. *Let H be an almost symmetric numerical semigroup with* $M(H)=\{a_1, \cdots, a_n\}$, *and let j, k be two distinct element of* $[1, n]$ *such that* $\alpha_j a_j=\sum_{l \neq j}\alpha_{jl}a_l$ *with* $\alpha_{jk} \geqq 1$.

1) *If* $\alpha_{jk} \geqq 2$, *then* $\omega_{a_k}(a_k)-(\alpha_j-1)a_j \in L_{a_k}(H)$.

2) *We have*

$$\omega_{a_k}(a_k)=\begin{cases} \sum_{l \in [1,n]-\{k,j\}} \beta_l a_l+(\alpha_j-1)a_j & \text{if } \omega_{a_k}(a_k)-(\alpha_j-1)a_j \in L_{a_k}(H). \\ \sum_{l \in [1,n]-\{k,j\}} \alpha_{jl}a_l+(\alpha_j-2)a_j & \text{otherwise.} \end{cases}$$

PROOF. 1) Since $(\alpha_j-1)a_j \in L_{a_k}(H)$, by Lemma 6.1 it suffices to show that

$$(\alpha_j-1)a_j \neq \omega_{a_k}(i_0) \qquad \text{where} \quad 2\omega_{a_k}(i_0)=\omega_{a_k}(a_k)+a_k .$$

Assume $(\alpha_j-1)a_j=\omega_{a_k}(i_0)$. Then

$$\omega_{a_k}(a_k)+a_k=2(\alpha_j-1)a_j=(\alpha_j-2)a_j+\sum_{l \neq j}\alpha_{jl}a_l .$$

Hence we have

$$\omega_{a_k}(a_k)-a_k=(\alpha_j-2)a_j+(\alpha_{jk}-2)a_k+\sum_{l \in [1,n]-\{j,k\}}\alpha_{jl}a_l .$$

This contradicts $\omega_{a_k}(a_k)-a_k \notin H$.

2) In view of $\alpha_{jk} \geqq 1$, if $\omega_{a_k}(a_k)-(\alpha_j-1)a_j \in L_{a_k}(H)$, then

$$\omega_{a_k}(a_k)=\sum_{l \in [1,n]-\{k,j\}}\beta_l a_l+(\alpha_j-1)a_j .$$

If $\omega_{a_k}(a_k)-(\alpha_j-1)a_j \notin L_{a_k}(H)$, we have

$$2(\alpha_j-1)a_j=2\omega_{a_k}(i_0)=\omega_{a_k}(a_k)+a_k ,$$

hence

$$\omega_{a_k}(a_k)=\alpha_j a_j+(\alpha_j-2)a_j-a_k=\sum_{l\in[1,n]-\{j,k\}}\alpha_{jl}a_l+(\alpha_j-2)a_j+(\alpha_{jk}-1)a_k$$

$$=\sum_{l\in[1,n]-\{j,k\}}\alpha_{jl}a_l+(\alpha_j-2)a_j. \qquad Q.E.D.$$

For the remainder of this section we assume that H is a numerical semigroup with $M(H)=\{a_1, a_2, a_3, a_4\}$.

PROPOSITION 6.3. *Let H be almost symmetric and let $k\in[1, 4]$ such that for any $j\in[1, 4]$, different from k, we have $\alpha_j a_j=\sum_{l\neq j}\alpha_{jl}a_l$ with $\alpha_{jk}\geqq 1$.*

1) *For any $j\in[1, 4]$, different from k, the following are eqivalent:*

a) $\omega_{a_k}(a_k)=\sum_{l\in[1,4]-\{k,j\}}\beta_l a_l+(\alpha_j-2)a_j$,

b) $\omega_{a_k}(a_k)-(\alpha_j-1)a_j\notin L_{a_k}(H)$.

In this case, $\alpha_{jk}=1$ and $\beta_l=\alpha_{jl}$ for $l\in[1, 4]-\{k, j\}$.

2) *We have*

$$\omega_{a_k}(a_k)=(\alpha_i-1)a_i+(\alpha_l-1)a_l+(\alpha_j-2)a_j$$

and

$$L_{a_k}(H)=\{\beta_i a_i+\beta_l a_l+\beta_j a_j \mid 0\leqq\beta_i<\alpha_i,\ 0\leqq\beta_l<\alpha_l,\ 0\leqq\beta_j<\alpha_j-1\}\cup\}(\alpha_j-1)a_j\}$$

for some permutation (k, i, l, j) of $[1, 4]$.

PROOF. 1) Proposition 6.2 2) implies b)$\Rightarrow$a). By the assumption we have $\beta_l<\alpha_l$ for $l\in[1, 4]-\{k, j\}$, which induces $\beta_l=\alpha_{jl}$. Assume that $\omega_{a_k}(a_k)-(\alpha_j-1)a_j\in L_{a_k}(H)$. Then we have

$$\sum_{l\in[1,4]-\{k,j\}}\beta_l a_l+(\alpha_j-2)a_j=\sum_{l\in[1,4]-\{k,j\}}\beta'_l a_l+(\alpha_j-1)a_j.$$

This is a contradiction.

2) Renumbering $a_1, \cdots, a_4$, we may assume $k=1$. Now assume $\omega_{a_1}(a_1)-(\alpha_j-1)a_j\in L_{a_1}(H)$ for all $j\in[2, 4]$. Then by Proposition 6.2 and the assumption, we get

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_4-1)a_4,$$

which implies

$$L_{a_1}(H)=\{\beta_2 a_2+\beta_3 a_3+\beta_4 a_4 \mid 0\leqq\beta_i<\alpha_i\}.$$

This contradicts Lemma 6.1 1). Hence there exists a unique $j\in[2, 4]$ such that $2(\alpha_j-1)a_j=\omega_{a_1}(a_1)+a_1$, which implies

$$\omega_{a_1}(a_1)=\sum_{l\in[2,4]-\{j\}}\beta_l a_l+(\alpha_j-2)a_j.$$

Therefore we get

$$\omega_{a_1}(a_1)=(\alpha_i-1)a_i+(\alpha_l-1)a_l+(\alpha_j-2)a_j$$

for some permutation (i, l, j) or $(2, 3, 4)$. Hence we have

$$L_{a_1}(H)\supseteqq\{\beta_i a_i+\beta_l a_l+\beta_j a_j \mid 0\leqq\beta_i<\alpha_i,\ 0\leqq\beta_l<\alpha_l,\ 0\leqq\beta_j<\alpha_j-1\}\cup\{(\alpha_j-1)a_j\}\,.$$

Assume $z=\gamma_i a_i+\gamma_l a_l+(\alpha_j-1)a_j\in L_{a_1}(H)$ with $(\gamma_i, \gamma_l)\neq(0, 0)$. Since $\omega_{a_1}(a_1)-z\in L_{a_1}(H)$, we put

$$\omega_{a_1}(a_1)-z=\nu_i a_i+\nu_l a_l+\nu_j a_j$$

where $\nu_i<\alpha_i$, $\nu_l<\alpha_l$ and $\nu_j<\alpha_j$, hence

$$(\alpha_i-1-\gamma_i)a_i+(\alpha_l-1-\gamma_l)a_l-a_j=\nu_i a_i+\nu_l a_l+\nu_j a_j\,,$$

which implies $\nu_j+1=0$, a contradiction. *Q. E. D.*

By tedious computations using Proposition 6.3 we can give generators of the ideal I_H in the case of almost symmetric H.

THEOREM 6.4. *Let H be almost symmetric. Then renumbering a_1, a_2, a_3, a_4 the ideal I_H is generated by*

$$f_1=X_1^{\alpha_1}-X_3^{\alpha_{13}}X_4^{\alpha_{14}}\,,\quad f_2=X_2^{\alpha_2}-X_1^{\alpha_{21}}X_4^{\alpha_{24}}\,,\quad f_3=X_3^{\alpha_3}-X_1^{\alpha_{31}}X_2^{\alpha_{32}}\,,$$

$$f_4=X_4^{\alpha_4}-X_1^{\alpha_{41}}X_2^{\alpha_{42}}X_3^{\alpha_{43}}\quad\text{and}\quad g=X_1^{\alpha_{21}+\alpha_{41}}X_3^{\alpha_{43}}-X_2^{\alpha_{32}}X_4^{\alpha_{14}}$$

where $0<\alpha_{ij}<\alpha_j$, $\alpha_1=\alpha_{21}+\alpha_{31}+\alpha_{41}$, $\alpha_2=\alpha_{32}+\alpha_{42}$, $\alpha_3=\alpha_{13}+\alpha_{43}$ and $\alpha_4=\alpha_{14}+\alpha_{24}$, which imply $\mu(H)=5$. More explicitly we obtain $\alpha_{13}=1$, $\alpha_{14}=\alpha_4-1$, $\alpha_{24}=1$, $\alpha_{31}=\alpha_1-\alpha_{21}-1$, $\alpha_{32}=1$, $\alpha_{41}=1$, $\alpha_{42}=\alpha_2-1$ and $\alpha_{43}=\alpha_3-1$. Hence using Proposition 6.3 2). *We can show that H is 1-neat.*

PROOF. For any $i\in[1, 4]$, let $f_i\in I_H$ be a polynomial of the type $X_i^{\alpha_i}-\prod_{j\in[1,4]-\{i\}}X_j^{\alpha_{ij}}$. First, assume that there exist two distinct $i, j\in[1, 4]$ with $X_i^{\alpha_i}-X_j^{\alpha_j}\in I_H$. Then renumbering $a_1, \cdots, a_4$ we may assume $i=1$ and $j=2$. They are divided into the four cases:

1) $X_i^{\alpha_i}-X_j^{\alpha_j}\notin I_H$ for all $\{i, j\}\neq\{1, 2\}$,

2) $X_1^{\alpha_1}-X_3^{\alpha_3}\in I_H$ and $X_1^{\alpha_1}-X_4^{\alpha_4}\notin I_H$,

3) $X_3^{\alpha_3}-X_4^{\alpha_4}\in I_H$ and $X_1^{\alpha_1}-X_3^{\alpha_3}\notin I_H$,

4) $X_1^{\alpha_1}-X_3^{\alpha_3}\in I_H$ and $X_1^{\alpha_1}-X_4^{\alpha_4}\in I_H$.

The case 1). Then $f_3=X_3^{\alpha_3}-X_1^{\alpha_{31}}X_2^{\alpha_{32}}X_4^{\alpha_{34}}$ and $f_4=X_4^{\alpha_4}-X_1^{\alpha_{41}}X_2^{\alpha_{42}}X_3^{\alpha_{43}}$. These are divided into the following:

a) $\alpha_{31}>0,\ \alpha_{32}>0,\ \alpha_{41}>0,\ \alpha_{42}>0$, b) $\alpha_{31}>0,\ \alpha_{32}>0,\ \alpha_{41}>0,\ \alpha_{42}=0$,

c) $\alpha_{31}>0,\ \alpha_{32}=0,\ \alpha_{41}>0,\ \alpha_{42}=0$, d) $\alpha_{31}>0,\ \alpha_{32}=0,\ \alpha_{41}=0,\ \alpha_{42}>0$.

a) Then we have

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_4-2)a_4=\alpha_{42}a_2+\alpha_{43}a_3+(\alpha_4-2)a_4,$$

$$\omega_{a_2}(a_2)=(\alpha_1-1)a_1+(\alpha_3-1)a_3+(\alpha_4-2)a_4=\alpha_{41}a_1+\alpha_{43}a_3+(\alpha_4-2)a_4,$$

which imply $\alpha_1=\alpha_2=2$, hence $a_1=a_2$, a contradiction.

b) Similarly, we get $a_1=a_2$, a contradiction.

c) We have

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_i-1)a_i+(\alpha_j-2)a_j$$

with $\{i, j\}=\{3, 4\}$. This is a contradiction.

d) We get

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+\beta_4a_4, \quad \omega_{a_2}(a_2)=(\alpha_1-1)a_1+(\alpha_4-1)a_4+\beta_3a_3,$$

which implies $\beta_4=\alpha_4-1$. Hence we have

$$L_{a_1}(H)=\{\beta_2a_2+\beta_3a_3+\beta_4a_4 \mid 0\leqq\beta_i<\alpha_i\},$$

which implies $C(H)=2g(H)$, a contradiction.

The case 2). Then $f_4=X_4^{\alpha_4}-X_1^{\alpha_{41}}X_2^{\alpha_{42}}X_3^{\alpha_{43}}$, where we may assume $\alpha_{41}>0$. In the similar manner to 1) a), we get $a_1=a_2$, a contradiction.

The case 3). We have $\omega_{a_3}(a_3)=\gamma_1a_1+\gamma_2a_2+(\alpha_4-1)a_4$. Set $d=(a_3, a_4)$ and $H'=\langle d, a_1, a_2\rangle$. Then $L_d(H')\subseteqq L_{a_3}(H)$. If $\nu_1a_1+\nu_2a_2+\nu_4a_4=\mu_1a_1+\mu_2a_2+\mu_4a_4$ with $\nu_4<\alpha_4$ and $\mu_4<\alpha_4$, then $\nu_4=\mu_4$. Using this, for any $\omega'\in\langle a_1, a_2\rangle$ with $\omega_{a_3}(a_3)-\omega'\in L_{a_3}(H)$ we have

$$\omega_{a_3}(a_3)-\omega'=\mu_1a_1+\mu_2a_2+(\alpha_4-1)a_4$$

with $\mu_1, \mu_2\in N$. Hence if $\omega'\in L_d(H')$ with $\omega_{a_3}(a_3)-\omega'\in L_{a_3}(H)$, then for any $\nu_4\in[0, \alpha_4-1]$ we get $\omega'+\nu_4a_4\in L_{a_3}(H)$. Therefore we can see:

$$L_{a_3}(H)=\{\omega'+\nu_4a_4 \mid \omega'\in L_d(H'),\ 0\leqq\nu_4<\alpha_4\} \quad \text{and} \quad \omega_{a_3}(a_3)=\omega_d(d)+(\alpha_4-1)a_4.$$

Since we have $\omega_d(d)-\omega'\in L_d(H')$ for any $\omega'\in L_d(H')$, we get $\omega_{a_3}(a_3)-\omega\in L_{a_3}(H)$ for any $\omega\in L_{a_3}(H)$, i.e., $C(H)=2g(H)$, a contradiction.

The case 4). Then H is a complete intersection ([1]), which implies $C(H)=2g(H)$, a contradiction.

Secondly, assume: each f_i contains st least three variables and there exists $j\in[1, 4]$ such that the variable X_j appears only in the f_j. Then we may assume that

$$f_1=X_1^{\alpha_1}-X_2^{\alpha_{12}}X_3^{\alpha_{13}}X_4^{\alpha_{14}}, \quad f_2=X_2^{\alpha_2}-X_3^{\alpha_{23}}X_4^{\alpha_{24}}, \quad f_3=X_3^{\alpha_3}-X_2^{\alpha_{32}}X_4^{\alpha_{34}},$$

and

$$f_4=X_4^{\alpha_4}-X_2^{\alpha_{42}}X_3^{\alpha_{43}}$$

with $\alpha_{13}>0$, $\alpha_{14}>0$. Hence we get

$$\omega_{a_3}(a_3)=(\alpha_2-1)a_2+(\alpha_4-1)a_4+(\alpha_1-2)a_1,$$

$$\omega_{a_4}(a_4)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_1-2)a_1,$$

which imply $\alpha_4 a_4=\alpha_3 a_3=\alpha_{32}a_2+\alpha_{34}a_4$, a contradiction.

Thirdly, assume: each f_i contains at least three variables and there exists $j\in[1, 4]$ such that the variable X_j appears twice in the f_i's. Then we may assume that

$$f_1=X_1^{\alpha_1}-X_2^{\alpha_{12}}X_3^{\alpha_{13}}X_4^{\alpha_{14}}, \quad f_2=X_2^{\alpha_2}-X_1^{\alpha_{21}}X_3^{\alpha_{23}}X_4^{\alpha_{24}}, \quad f_3=X_3^{\alpha_3}-X_2^{\alpha_{32}}X_4^{\alpha_{34}}$$

and

$$f_4=X_4^{\alpha_4}-X_2^{\alpha_{42}}X_3^{\alpha_{43}}.$$

The case $\alpha_{12}>0$. Then we have

$$\omega_{a_2}(a_2)=(\alpha_3-1)a_3+(\alpha_4-1)a_4+(\alpha_1-2)a_1,$$

$$\omega_{a_3}(a_3)=(\alpha_4-1)a_4+(\alpha_i-1)a_i+(\alpha_j-2)a_j,$$

$$\omega_{a_4}(a_4)=(\alpha_3-1)a_3+(\alpha_j-1)a_j+(\alpha_i-2)a_i,$$

with $\{i, j\}=\{1, 2\}$. If $j=1$ (resp. 2), then $(\alpha_4-\alpha_{34})a_4=a_1+(\alpha_{32}-1)a_2$ (resp. $(\alpha_3-\alpha_{43})a_3=a_1+(\alpha_{42}-1)a_2$), a contradiction. The case $\alpha_{12}=0$. We have

$$\omega_{a_3}(a_3)=(\alpha_1-1)a_1+(\alpha_4-1)a_4+(\alpha_2-2)a_2,$$

$$\omega_{a_4}(a_4)=(\alpha_1-1)a_1+(\alpha_3-1)a_3+(\alpha_2-2)a_2,$$

which implies $\alpha_4 a_4=\alpha_3 a_3$, a contradiction.

Lastly, assume: each f_i contains at least three variables and all the variables X_j appear at least three times in the f_i's. Renumbering $a_1, \cdots, a_4$, these are divided into the 10 cases in Proposition 4.4.

The case (1). Then we may assume:

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_4-2)a_4,$$

$$\omega_{a_4}(a_4)=(\alpha_i-1)a_i+(\alpha_j-1)a_j+(\alpha_k-2)a_k.$$

Using $\omega_{a_1}(a_1)-a_1=\omega_{a_4}(a_4)-a_4$, this is a contradiction.

The case (2). We have

$$\omega_{a_3}(a_3)=(\alpha_1-1)a_1+(\alpha_i-1)a_i+(\alpha_j-2)a_j,$$

$$\omega_{a_4}(a_4)=(\alpha_k-1)a_k+(\alpha_l-1)a_l+(\alpha_m-2)a_m.$$

This is a contradiction.

The case (3) a). We have

$$\omega_{a_3}(a_3)=(\alpha_1-1)a_1+(\alpha_2-1)a_2+(\alpha_4-2)a_4\,,$$

$$\omega_{a_4}(a_4)=(\alpha_1-1)a_1+(\alpha_2-1)a_2+(\alpha_3-2)a_3\,.$$

Then $(\alpha_4-1)a_4=(\alpha_3-1)a_3$, a contradiction.

The case (3) b). We have

$$\omega_{a_4}(a_4)=(\alpha_1-1)a_1+(\alpha_2-1)a_2+(\alpha_3-2)a_3\,,$$

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_4-2)a_4\,.$$

Moreover,

$$\omega_{a_3}(a_3)=(\alpha_1-1)a_1+\gamma_2a_2+\gamma_4a_4 \quad \text{or} \quad (\alpha_1-2)a_1+\alpha_{14}a_4\,.$$

Using $\omega_{a_4}(a_4)-a_4=\omega_{a_3}(a_3)-a_3=\omega_{a_1}(a_1)-a_1$, this is a contradiction.

The case (3) c). We have

$$\omega_{a_1}(a_1)=(\alpha_3-1)a_3+(\alpha_i-1)a_i+(\alpha_j-2)a_j\,,$$

$$\omega_{a_3}(a_3)=(\alpha_1-1)a_1+(\alpha_j-1)a_j+(\alpha_i-2)a_i$$

This is a contradiction.

The case (4) a). We have

$$\omega_{a_3}(a_3)=(\alpha_1-1)a_1+(\alpha_2-1)a_2+(\alpha_4-2)a_4$$

and

$$\omega_{a_1}(a_1)=(\alpha_3-1)a_3+\gamma_2a_2+\gamma_4a_4 \quad \text{or} \quad (\alpha_3-2)a_3+\alpha_{32}a_2\,.$$

This is a contradiction.

The case (4) b). $\omega_{a_4}(a_4)=(\alpha_i-1)a_i+(\alpha_j-1)a_j+(\alpha_l-2)a_l$, a contradiction.

The case (5) a). We have

$$\omega_{a_1}(a_1)=(\alpha_3-1)a_3+\gamma_2a_2+\gamma_4a_4 \quad \text{or} \quad (\alpha_3-2)a_3+\alpha_{32}a_2$$

Moreover,

$$\omega_{a_1}(a_1)=(\alpha_4-1)a_4+\beta_2a_2+\beta_3a_3 \quad \text{or} \quad (\alpha_4-2)a_4+\alpha_{42}a_2$$

This is a contradiction.

The case (5) b). We have

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+\gamma_4a_4 \quad \text{or} \quad (\alpha_2-1)a_2+(\alpha_3-2)a_3\,,$$

$$\omega_{a_2}(a_2)=(\alpha_3-1)a_3+(\alpha_4-1)a_4+\gamma_1a_1 \quad \text{or} \quad (\alpha_3-1)a_3+(\alpha_4-2)a_4\,,$$

$$\omega_{a_3}(a_3)=(\alpha_4-1)a_4+(\alpha_1-1)a_1+\gamma_2a_2 \quad \text{or} \quad (\alpha_4-1)a_4+(\alpha_1-2)a_1$$

and

$$\omega_{a_4}(a_4)=(\alpha_1-1)a_1+(\alpha_2-1)a_2+\gamma_3a_3 \quad \text{or} \quad (\alpha_1-1)a_1+(\alpha_2-2)a_2\,.$$

If we renumber $a_1, \cdots, a_4$, each latter case is reduced to the case (4) c). For example, let $\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-2)a_3$. If $\omega_{a_2}(a_2)=(\alpha_3-1)a_3+(\alpha_4-1)a_4+\gamma_1 a_1$, then $\alpha_2 a_2=(\gamma_1+1)a_1+a_3+(\alpha_4-1)a_4$, whose case is reduced to (4) c). If $\omega_{a_2}(a_2)=(\alpha_3-1)a_3+(\alpha_4-2)a_4$, then $\alpha_2 a_2=a_1+a_3+(\alpha_4-2)a_4$. If $\alpha_4=2$, we replace f_2 by $X_2^2-X_1X_3$, which is reduced to the third case, a contradiction. Hence we have $\alpha_4\geqq 3$, whose case is reduced to (4) c). Therefore for any $i\in[1, 4]$, $\omega_{a_i}(a_i)$ is equal to the former case. Then we see:

$$\alpha_{21}+\alpha_{31}=\alpha_1, \quad \alpha_{32}+\alpha_{42}=\alpha_2, \quad \alpha_{13}+\alpha_{43}=\alpha_3 \quad \text{and} \quad \alpha_{14}+\alpha_{24}=\alpha_4 .$$

Using $\omega_{a_1}(a_1)-a_1=\omega_{a_4}(a_4)-a_4$ we obtain

$$\begin{aligned}\omega_{a_1}(a_1)&=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_{14}-1)a_4\\ &=(\alpha_{32}-1)a_2+(\alpha_{13}-1)a_3+(\alpha_4+\alpha_{14}-1)a_4 ,\end{aligned}$$

which implies

$$L_{a_1}(H)\supseteqq\{\beta_2a_2+\beta_3a_3+\beta_4a_4\,|\,\beta_i\in N \text{ and } (\beta_2, \beta_3, \beta_4) \text{ satisfies one of the following:}$$
$$1)\ \beta_2<\alpha_2,\ \beta_3<\alpha_3,\ \beta_4<\alpha_{14},\quad 2)\ \beta_2<\alpha_{32},\ \beta_3<\alpha_{13},\ \alpha_{14}\leqq\beta_4<\alpha_4\} .$$

Since there exists a positive integer ν such that

$$a_1=\nu^{-1}(\alpha_2\alpha_3\alpha_{14}+\alpha_{32}\alpha_{13}\alpha_{24}) ,$$

the above inclusion becomes the equality, hence for any $\omega\in L_{a_1}(H)$ we have $\omega_{a_1}(a_1)-\omega\in L_{a_1}(H)$, i. e., $C(H)=2g(H)$, a contradiction.

Therefore, if H is almost symmetric, renumbering $a_1, \cdots, a_4$ it is reduced to the case (4) c), i. e.,

$$f_1=X_1^{\alpha_1}-X_3^{\alpha_{13}}X_4^{\alpha_{14}}, \quad f_2=X_2^{\alpha_2}-X_1^{\alpha_{21}}X_4^{\alpha_{24}}, \quad f_3=X_3^{\alpha_3}-X_1^{\alpha_{31}}X_2^{\alpha_{32}}$$

and

$$f_4=X_4^{\alpha_4}-X_1^{\alpha_{41}}X_2^{\alpha_{42}}X_3^{\alpha_{43}} .$$

Moreover,

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_4-2)a_4=\alpha_{42}a_2+\alpha_{43}a_3+(\alpha_4-2)a_4 ,$$

which implies $\alpha_{41}=1$, $\alpha_{42}=\alpha_2-1$ and $\alpha_{43}=\alpha_3-1$. Now

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1-\alpha_{13})a_3+(\alpha_4-2-\alpha_{14})a_4+\alpha_1a_1 ,$$

which implies $\alpha_{14}=\alpha_4-1$. Moreover, we get

$$\begin{aligned}\omega_{a_4}(a_4)&=(\alpha_1-1)a_1+(\alpha_2-1)a_2+(\alpha_3-1-\alpha_{13})a_3\\ &=(\alpha_1-1-\alpha_{21}-\alpha_{31}-\alpha_{41})a_1+(\alpha_2-1-\alpha_{42}+\alpha_2-\alpha_{32})a_2\\ &\quad+(\alpha_3-\alpha_{43})a_3+(\alpha_4-\alpha_{24})a_4 .\end{aligned}$$

If $\alpha_2 \geqq \alpha_{32}$, then $\alpha_1 \leqq \alpha_{21}+\alpha_{31}+\alpha_{41}$. If $\alpha_2 < \alpha_{32}$, then we have

$$\omega_{a_4}(a_4)=(\alpha_1-1)a_1+(\alpha_2-1)a_2+(\alpha_3-2)a_3$$
$$=(\alpha_{21}+\alpha_{31})a_1+(\alpha_{32}-\alpha_2)a_2+(\alpha_3-2)a_3\,,$$

which implies $\alpha_{21}+\alpha_{31}=\alpha_1-1$, hence $\alpha_1 \leqq \alpha_{21}+\alpha_{31}+\alpha_{41}$. Since

$$\alpha_1 a_1+\alpha_2 a_2+\alpha_3 a_3+\alpha_4 a_4=(\alpha_{21}+\alpha_{31}+\alpha_{41})a_1+(\alpha_{32}+\alpha_{42})a_2$$
$$+(\alpha_{13}+\alpha_{43})a_3+(\alpha_{14}+\alpha_{34})a_4\,,$$

we have

$$\alpha_1=\alpha_{21}+\alpha_{31}+\alpha_{41}\,,\quad \alpha_2=\alpha_{32}+\alpha_{42}\,,\quad \alpha_3=\alpha_{13}+\alpha_{43} \text{ and } \alpha_4=\alpha_{14}+\alpha_{34}\,,$$

which imply

$$\alpha_{41}=1\,,\quad \alpha_{31}=\alpha_1-\alpha_{21}-1\,,\quad \alpha_{32}=1\,,\quad \alpha_{42}=\alpha_2-1\,,$$
$$\alpha_{13}=1\,,\quad \alpha_{43}=\alpha_3-1\,,\quad \alpha_{14}=\alpha_4-1\,,\quad \alpha_{34}=1\,.$$

Since we have

$$L_{a_1}(H)=\{\beta_2 a_2+\beta_3 a_3+\beta_4 a_4 \,|\, 0 \leqq \beta_2 < \alpha_2,\ 0 \leqq \beta_3 < \alpha_3,\ 0 \leqq \beta_4 < \alpha_4-1\} \cup \{(\alpha_4-1)a_4\}\,,$$

H is 1-neat. *Q.E.D.*

Conversely, by simple calculations we get the following:

THEOREM 6.5. *Let* $\alpha_i > 1$ *for* $1 \leqq i \leqq 4$ *and let* $0 < \alpha_{21} < \alpha_1 - 1$. *If* $a_1=\alpha_2\alpha_3(\alpha_4-1)+1$, $a_2=\alpha_{21}\alpha_3\alpha_4+(\alpha_1-\alpha_{21}-1)(\alpha_3-1)+\alpha_3$, $a_3=\alpha_1\alpha_4+(\alpha_1-\alpha_{21}-1)(\alpha_2-1)(\alpha_4-1)-\alpha_4+1$, $a_4=\alpha_1\alpha_2(\alpha_3-1)+\alpha_{21}(\alpha_2-1)+\alpha_2$ *and* $(a_1, a_2, a_3, a_4)=1$, *then* $H=\langle a_1, a_2, a_3, a_4\rangle$ *is an almost symmetric numerical semigroup with* $M(H)=\{a_1, a_2, a_3, a_4\}$ *and the ideal* I_H *is generated by*

$$f_1=X_1^{\alpha_1}-X_3X_4^{\alpha_4-1}\,,\quad f_2=X_2^{\alpha_2}-X_1^{\alpha_{21}}X_4\,,\quad f_3=X_3^{\alpha_3}-X_1^{\alpha_1-\alpha_{21}-1}X_2\,,$$
$$f_4=X_4^{\alpha_4}-X_1X_2^{\alpha_2-1}X_3^{\alpha_3-1} \quad \textit{and} \quad g=X_1^{\alpha_{21}+1}X_3^{\alpha_3-1}-X_2X_4^{\alpha_4-1}\,.$$

PROOF. By the assumption, we have

$$\alpha_1 a_1=a_3+(\alpha_4-1)a_4\,,\quad \alpha_2 a_2=\alpha_{21}a_1+a_4 \quad \text{and} \quad \alpha_3 a_3=(\alpha_1-\alpha_{21}-1)a_1+a_2\,,$$

which imply $\alpha_4 a_4=a_1+(\alpha_2-1)a_2+(\alpha_3-1)a_3$. Using the relations, we get

$$L_{a_1}(H)=\{\beta_2 a_2+\beta_3 a_3+\beta_4 a_4 \,|\, 0 \leqq \beta_2 < \alpha_2,\ 0 \leqq \beta_3 < \alpha_3,\ 0 \leqq \beta_4 < \alpha_4-1\} \cup \{(\alpha_4-1)a_4\}$$

and

$$\omega_{a_1}(a_1)=(\alpha_2-1)a_2+(\alpha_3-1)a_3+(\alpha_4-2)a_4\,,$$

which show that H is almost symmetric. Moreover, since we have

$$L_{a_4}(H)=\{\beta_1 a_1+\beta_2 a_2+\beta_3 a_3 \mid 0\leqq\beta_1<\alpha_1,\ 0\leqq\beta_2<\alpha_2,\ 0\leqq\beta_3<\alpha_3-1\}$$
$$\cup\{\beta_1 a_1+\beta_2 a_2+(\alpha_3-1)a_3 \mid 0\leqq\beta_1\leqq\alpha_{21},\ 0\leqq\beta_2<\alpha_2-1\}$$
$$\cup\{(\alpha_2-1)a_2+(\alpha_3-1)a_3\}\,,$$

we get $a_1\notin\langle a_2, a_3, a_4\rangle$, $a_2\notin\langle a_1, a_3, a_4\rangle$, $a_3\notin\langle a_1, a_2, a_4\rangle$, $a_4\notin\langle a_1, a_2, a_3\rangle$. Using the above relations, we get

$$L_{a_2}(H)=\{\beta_1 a_1+\beta_3 a_3+\beta_4 a_4 \mid 0\leqq\beta_1<\alpha_{21},\ 0\leqq\beta_3<\alpha_3,\ 0\leqq\beta_4<\alpha_4\}$$
$$\cup\{\beta_1 a_1+\beta_3 a_3 \mid \alpha_{21}\leqq\beta_1<\alpha_1,\ 0\leqq\beta_3<\alpha_3-1\}\cup\{\alpha_{21}a_1+(\alpha_3-1)a_3\}\,.$$

The complete descriptions of $L_{a_1}(H)$, $L_{a_2}(H)$ and $L_{a_4}(H)$ show that the above relations are minimal. *Q.E.D.*

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