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ON A TYPE OF REAL HYPERSURFACES IN COMPLEX PROJECTIVE SPACE

By

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Abstract. We give a classification of real hypersurfaces in complex projective space under assumptions that the structure vector ξ is principal, the focal map has constant rank and that $\nabla_{\xi}C=0$, where C is the Weyl conformal curvature tensor of the real hypersurface.

1. Introduction

Let $M^{n}(c)$ denote an n-dimensional complex space form with constant holomorphic sectional curvature c. It is well known that a complete and simply connected complex space form is either complex projective space PC^{n}, complex Euclidean space C^{n} or complex hyerbolic space HC^{n}, according as $c>0$, $c=0$ or $c<0$.

In this paper we consider a real hypersurface M of PC^{n}. The induced almost contact metric structure and the Weyl conformal curvature tensor of the real hypersurface M in PC^{n} are respectively denoted by (φ,ξ,η,g) and C. Many differential geometers have studied M by using the structure (φ,ξ,η,g). Typical examples of real hypersurfaces in complex projective space PC^{n} are homogeneous ones and one of the first researches is the classification of these by Takagi [12]. He proved that all homogeneous hypersurfaces of PC^{n} could be divided into six types which are said to be A_1,A_2,B,C,D and E (see Theorem A). This result was generalized by Kimura [4], who classified real hypersurfaces of PC^{n} with constant principal curvatures and for which the structure vector ξ is principal. Now, there exist many studies of real hypersurfaces in PC^{n}. Some hypersurfaces in PC^{n} are characterized by conditions on the shape operator (or principal curvatures) and one of the structure tensors. On the other hand, some

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studies about the nonexistence of real hypersurfaces under natural linear conditions imposed on the Ricci tensor S or ∇S or the Weyl conformal curvature tensor C or ∇C have been made by many researchers. Many results for real hypersurfaces of complex projective space have been obtained by Cecil and Ryan [1], Kimura [4], [5], Kon [7], S. Maeda [8], [9], Okumura [11], Takagi [12], [13] and so on (for more details see [8]). In particular, it is well known that there exist no Einstein real hypersurfaces M in PC^n for $n \geq 3$ (cf. [7]). Also $PC^n (n \geq 3)$ does not admit real hypersurfaces M with parallel Ricci tensor S [2]. Recently S. Maeda [9], classified real hypersurfaces M in PC^n satisfying $\nabla_\xi S = 0$ (that is the Ricci tensor S is parallel in the direction of the structure vector $\xi = -JN$, where N is a unit normal vector field on M) under the conditions that ξ is a principal curvature vector of M and that the focal map has constant rank on M.

On the other hand U. H. Ki, H. Nakagawa and Y. J. Suh [3] have proved that PC^n does not admit real hypersurfaces M with harmonic Weyl tensor C. So PC^n does not admit real hypersurfaces M with parallel Weyl tensor C (that is $\nabla_X C = 0$ for each vector X tangent to M). This is perhaps natural since $\nabla C = 0$ is not a conformal invariant. However one might impose a weaker condition utilizing some additional structure eventhough one might not have conformal invariance. Thus we investigate real hypersurfaces M by using the condition $\nabla_\xi C = 0$ (on the derivative of C) which is weaker than $\nabla C = 0$.

The purpose of this paper is to classify real hypersurfaces M in PC^n satisfying $\nabla_\xi C = 0$ under the condition that ξ is a principal curvature vector of M and that the focal map has constant rank on M.

THEOREM. *Let M be a real hypersurface of $PC^n (n \geq 3)$ on which ξ is a principal curvature vector with principal curvature $\alpha = 2\cot 2r$ and the focal map ϕ_r has constant rank. If for the Weyl conformal curvature tensor C we have $\nabla_\xi C = 0$, then M is locally congruent to one of the following:*

(1) *a homogeneous real hypersurface which lies on a tube of radius r over a totally geodesic $PC^k (1 \leq k \leq n-1)$, where $0 < r < \pi/2$,*

(2) *a non-homogeneous real hypersurface whch lies on a tube of radius $\pi/4$ over a Kaehler submanifold $\tilde{N}$ with nonzero principal curvature $\neq \pm 1$.*

(3) *a non-homogeneous real hypersurface which lies on a tube of radius r over a k-dimensional Kaehler submanifold $\tilde{N}$ on which the rank of each shape operator is not greater than 2 with nonzero principal curvature $\neq \pm\sqrt{(n-k-1)/(k-1)}$ and $\cot^2 r = (n-k-1)/(k-1)$, where $k = 2, \cdots, n-1$.*

REMARK 1. Since case (3) in the Theorem is a new example which is different from case (7) in Maeda's theorem in [9], it is essential to guarantee the existence of the Kaehler submanifold $\tilde{N}^k (k \geq 2)$ such that the rank of each shape operator is not greater than 2 in PC^n. The following example $\tilde{N}^{n-1}$ is a complex hypersurface (with singularity) in PC^n such that the rank of each shape operator is not greater than 2 in PC^n.

EXAMPLE. Let γ be a non-totally-geodesic complex curve in PC^n and let $\phi_{\pi/2}(\gamma)$ be a tube of radius $\pi/2$ around the curve γ, that is $\phi_{\pi/2}(\gamma) = \bigcup_{x\in\gamma} \{exp_x(\pi/2)\upsilon, \upsilon$ is a unit normal vector of γ at $x\}$. Then $\phi_{\pi/2}(\gamma)$ is an $(n$-$1)$-dimensional complex hypersurface in PC^n (with singularity). Let $\pm cot\theta$ be the eigenvalues of the shape operator A_υ with respect to a unit normal vector υ of γ. Then the principal curvatures of $\phi_{\pi/2}(\gamma)$ at $exp_x(\pi/2)\upsilon$ are given by (see Proposition 3.1 in [1]) $cot(\pi/2+\theta)$ with multiplicity 1, $cot(\pi/2-\theta)$ with multiplicity 1 and 0 with multiplicity $2n-4$.

2. Preliminaries.

First we briefly describe the basic properties of real hypersurfaces of a complex projective space. Let M be an orientable real hypersurface of $PC^n (n \geq 3)$ with the Fubini-Study metric of constant holomorphic sectional curvature 4. On a neighborhood of each point of M, we denote by N a local unit normal vector field of M in PC^n. It is well known that M admits an almost contact metric structure induced from the complex structure J on PC^n. Namely, for the Riemannian metric g of M induced from the Fubini-Study metric $\tilde{g}$ of PC^n, we define a tensor field φ of type (1,1), a vector fiels ξ and a 1-form η on M by $g(\varphi X, Y) = \tilde{g}(JX, Y), g(\xi, X) = \eta(X) = \tilde{g}(JX, N)$ for any vector fields X, Y on M. Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \ g(\xi, \xi) = 1, \ \varphi(\xi) = 0. \tag{2.1}$$

The Riemannian connections $\tilde{\nabla}$ of PC^n and ∇ of M are related by the following formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \tilde{\nabla}_X N = -AX \tag{2.2}$$

where A is the shape operator of M in PC^n.

Now it follows from (2.2) that the structure (φ, ξ, η, g) satisfies

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \ \nabla_X \xi = \varphi AX. \tag{2.3}$$

Let $\tilde{R}$ and R be the curvature tensors of PC^n and M, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, namely PC^n is of constant holomorphic sectional curvature 4, the Gauss and Codazzi equations are respectively

$$(2.4)\quad \begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y \\ &\quad -2g(\varphi X,Y)\varphi Z + g(AY,Z)AX - g(AX,Z)AY, \\ (\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\varphi Y - 2g(\varphi X,Y)\xi \end{aligned}$$

By (2.1), (2.3) and (2.4) we get

$$(2.5)\qquad QX = (2n+1)X - 3\eta(X)\xi + hAX - A^2X$$

where $h = traceA$ and Q denotes the Ricci operator of M defined from the Ricci tensor S, i.e. $S(X,Y) = g(QX,Y)$. The Weyl conformal curvature tensor C of M is given by

$$(2.6)\quad \begin{aligned} C(X,Y)Z &= R(X,Y)Z + \frac{1}{2n-3}[g(QX,Z)Y - g(QY,Z)X + g(X,Z)QY \\ &\quad - g(Y,Z)QX] - \frac{\tau}{2(n-1)(2n-3)}(g(X,Z)Y - g(Y,Z)X) \end{aligned}$$

where τ is the scalar curvature of M.

An eigenvector X of the shape operator A is called a principal curvature vector and an eigenvalue λ is called a principal curvature. From now on, we assume that the structure vector field ξ is principal, and α is the principal curvature associated with ξ, that is, $A\xi = \alpha\xi$. Then it has been shown that α is constant (see [14]). Also for a principal curvature vector X orthogonal to ξ and the associated principal curvature λ we have (see [10])

$$(2.7)\qquad AX = \lambda X \ \ and \ \ A\varphi X = \frac{\alpha\lambda + 2}{2\lambda - \alpha}\varphi X$$

Now we recall without proof the following in order to prove our Theorem.

THEOREM A ([12]). *Let M be a homogeneous real hypersurface of PC^n. Then M is a tube of radius r over one of the following Kaehler submanifolds:*

(A_1) *hyperplane PC^{n-1}, where $0 < r < \pi/2$,*
(A_2) *totally geodesic $PC^k (1 \le k \le n-2)$, where $0 < r < \pi/2$,*
(B) *complex quadric Q_{n-1}, where $0 < r < \pi/4$,*
(C) *$PC^1 \times PC^{(n-1)/2}$, where $0 < r < \pi/4$ and n (≥ 5) is odd,*
(D) *complex Grasmannian $G_{2,5}(C)$, where $0 < r < \pi/4$ and $n = 9$,*
(E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$ and*

$n=15$.

THEOREM B ([4]). *Let M be a real hypersurface of PC^n. Then M has constant principal curvatures and ξ is a principal vector if and only if M is locally congruent to a homogeneous real hypersurface.*

THEOREM C ([6]). *Let M be a real hypersurface of PC^n. If $\nabla_\xi A=0$, then ξ is a principal curvature vector; in addition, except for the null set on which the focal map ϕ_r degenerates, M is locally congruent to one of the following:*

(i) *a homogeneous real hypersurface which lies on a tube of radius r over a totally geodesic $PC^k (1\le k\le n-1)$, where $0<r<\pi/2$.*

(ii) *a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kaehler submanifold N with nonzero principal curvatures $\ne \pm 1$.*

THEOREM D ([1]). *Let M be a connected orientable real hypersurface (with unit normal vector N) in PC^n on which ξ is a principal curvature vector with principal curvature $\alpha = 2cot2r$ and the focal map ϕ_r has constant rank on M. Then the following hold:*

(i) *M is a tube of radius r around a certain Kaehler submanfild $\tilde{N}$ in PC^n.*

(ii) *For $x\in M$, let $cot\theta$ be a principal curvature of the shape operator at $exp_x rN$ of $\tilde{N}$, N being the inward normal at x. Then the real hypersurface M has a principal curvature equal to $cot(\theta-r)$ at x.*

REMARK 2. For later use, we note that from the Theorem A, the homogeneous real hypersurfaces M of type A_1, A_2, B, C, D, and E have distinct principal curvatures ξ_i with multiplicities $m(\xi_i)$ which we can read as follows (cf. [12]).

A_1: $\xi_1 = cotr,\ m(\xi_1)=2(n-1),\ \xi_2 = 2cot2r,\ m(\xi_2)=1$

A_2: $\xi_1 = cotr,\ m(\xi_1)=2k,\ \xi_2=-tanr,\ m(\xi_2)=2(n-k-1)$,
$\xi_3 = 2cot2r,\ m(\xi_3)=1$

B: $\xi_1 = cot(r-(\pi/4)),\ m(\xi_1)=n-1,\ \xi_2 = -tan(r-(\pi/4)),\ m(\xi_2)=n-1$,
$\xi_3 = 2cot2r,\ m(\xi_3)=1$

C: $\xi_i = cot(r-(\pi i/4))\ (i=1,2,3,4),\ m(\xi_i)=n-3$, for $i=2,4$
and $m(\xi_i)=2$, for $i=1,3$ $\xi_5 = 2cot2r,\ m(\xi_5)=1$

D: $\xi_i = cot(r-(\pi i/4)),\ m(\xi_i)=4\ (i=1,2,3,4)$,
$\xi_5 = 2cot2r,\ m(\xi_5)=1$ and $\dim M = 17$

E: $\xi_i = cot(r-(\pi i/4)),\ (i=1,2,3,4),\ m(\xi_i)=8$ for $i=2,4$ and
$m(\xi_i)=6$ for $i=1,3$, $\xi_5 = 2cot2r$, and $m(\xi_5)=1$ and $\dim M = 29$.

It is easy to see that if ξ is a principal curvature vector with principal curvature α, then

$$(\nabla_\xi A)X = \alpha\varphi AX - A\varphi AX + \varphi X. \tag{2.8}$$

Indeed, from (2.4) for $Y = \xi$ we have $(\nabla_\xi A)X = \alpha\nabla_X\xi - A\nabla_X\xi - \varphi X$ and then using (2.3) we obtain (2.8).

Finally we complete our preliminaries with the following two lemmas:

LEMMA 1. *If ξ is a principal curvature vector and $\nabla_\xi C = 0$, then $\xi\tau = 0$.*

PROOF. From (2.6) by using (2.4) and (2.5) we get

$$\begin{aligned} C(X,Y)Z = &\frac{1}{2n-3}\left(\frac{\tau}{2(n-1)} - 2n - 5\right)(g(Y,Z)X - g(X,Z)Y) + g(\varphi Y,Z)\varphi X \\ &- g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z + g(AY,Z)AX - g(AX,Z)AY \\ &+ \frac{1}{2n-3}[3\eta(Z)(\eta(Y)X - \eta(X)Y) + h(g(AX,Z)Y - g(AY,Z)X) \\ &+ g(A^2Y,Z)X - g(A^2X,Z)Y + 3(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi \\ &+ h(g(X,Z)AY - g(Y,Z)AX) + g(Y,Z)A^2X - g(X,Z)A^2Y] \end{aligned}$$

We note that the condition $\nabla_\xi C = 0$ is equivalent to

$$\nabla_\xi(C(X,Y)Z - C(\nabla_\xi X,Y)Z - C(X,\nabla_\xi C)Z - C(X,Y)\nabla_\xi Z = 0. \tag{2.9}$$

Now for simplicity we put

$$U_X = \alpha\varphi AX - A\varphi AX + \varphi X, \quad V_X = U_{AX} + AU_X. \tag{2.10}$$

Then by a straightforward calculation and using (2.3) and (2.8) we obtain

$$\begin{aligned} (\nabla_\xi C)(X,Y,Z) = &\frac{1}{2(n-1)(2n-3)}(\xi\tau)(g(Y,Z)X - g(X,Z)Y) \\ &+ g(U_Y,Z)AX - g(U_X,Z)AY + g(AY,Z)U_X - g(AX,Z)U_Y \\ &+ \frac{1}{2n-3}[h(g(U_X,Z)Y - g(U_Y,Z)X) + g(V_Y,Z)X - g(V_X,Z)Y \\ &+ h(g(X,Z)U_Y - g(Y,Z)U_X) + g(Y,Z)V_X - g(X,Z)V_Y] \end{aligned} \tag{2.11}$$

If we choose X orthogonal to ξ and $AX = \lambda X$, then

$$U_X = (\alpha\lambda - \lambda\mu + 1)\varphi X, \quad V_X = (\lambda + \mu)(\alpha\lambda - \lambda\mu + 1)\varphi X \tag{2.12}$$

where $\mu=(\alpha\lambda+2)/(2\lambda-\alpha)$.

Therefore putting $Z=\xi$ in (2.11) we obtain

$$\frac{1}{2(n-1)(2n-3)}(\xi\tau)\eta(Y)X + (\alpha\lambda-\lambda\mu+1)\left(\alpha+\frac{1}{2n-3}(\lambda+\mu-h)\right)\eta(Y)\varphi X=0. \tag{2.13}$$

Thus $\xi\tau=0$.

We notice that from (2.13) we also have

$$(\alpha\lambda-\lambda\mu+1)\left(\alpha+\frac{1}{2n-3}(\lambda+\mu-h)\right)=0. \tag{2.14}$$

LEMMA 2. *If ξ is a principal curvature vector with principal curvature $\alpha=0$, then $\xi\tau=0$ and $\nabla_\xi C=0$.*

PROOF. From (2.5) we have $\tau=4(n^2-1)+h^2-trA^2$. Thus $\xi\tau=2h(\xi h)-tr\nabla_\xi A^2$. But from [9. Lemma 2] we know that $\xi h=0$. Also $\alpha=0$ implies $\nabla_\xi A=0$ (see [9, Lemma 1]). Thus we obtain $\xi\tau=0$.

Now from (2.10) and (2.8) we get $U_\xi=0$ and $U_X=0$ for X orthogonal to ξ such that $AX=\lambda X$. Thus finally we have $U_X=V_X=0$ for all X. Then from (2.11) we obtain $\nabla_\xi C=0$.

3. Proof of Theorem:

From the fact that the principal curvature α of the principal curvature vector ξ is constant, our discussion is divided into two cases:

(i) $\alpha=0$ and (ii) $\alpha\neq 0$.

(i) $\alpha=0$.

In this case we have $\nabla_\xi A=0$. Hence by virtue of Theorem C we find that M is locally congruent to a homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a totally geodesic $PC^k(1\leq k\leq n-1)$, or congruent to a non-homogeneous real hypersurface which lies on a tube of radius $\pi/4$ over a Kaehler submanifold $\tilde{N}$ with nonzero principal curvatures $\neq\pm1$. Thus M is of case (1) with $r=(\pi/4)$ or of case (2) in the Theorem. From Lemma 2 we have that these examples satisfy $\nabla_\xi C=0$.

(ii) $\alpha\neq 0$.

We will follow the method of [9] and we will prove that M cannot be homogeneous of type B, C, D, or E.

From Lemma 1 and the relations (2.11) and (2.14) we have that for any principal curvature vector X orthogonal to ξ, the principal curvature λ must satisfy the following equation for λ

$$(3.1)\qquad (\lambda^2-\alpha\lambda-1)[2\lambda^2-2(h-(2n-3)\alpha)\lambda+h\alpha+2-(2n-3)\alpha^2]=0\,.$$

Since ξ is a principal curvature vector and the focal map ϕ_r has constant rank on M, our hypersurface M is a tube (of radius r) over a certain (k-dimensional) Kaehler submanifold $\tilde{N}$ in PC^n. So we may put $\alpha=2cot2r(=cotr-tanr)$ (cf. Theorem D). Now from (3.1) we have $\lambda^2-\alpha\lambda-1=0$ which gives $\lambda=cotr$ and $\lambda=-tanr$, or

$$(3.2)\qquad 2\lambda^2-2(h-(2n-3)\alpha)\lambda+h\alpha+2-(2n-3)\alpha^2=0\,.$$

We denote by $\lambda_1,\lambda_2(\neq cotr,-tanr)$ the solutions of (3.2).

Since

$$(3.3)\qquad \lambda_1+\lambda_2=h-(2n-3)\alpha$$

we have

$$(3.4)\qquad \frac{\alpha\lambda_1+2}{2\lambda_1-\alpha}=\lambda_2$$

Now denote by V_λ the eigenspace of A associated with the eigenvalue λ and by $m(\lambda)$ the multiplicity of λ. Then by using (2.7) and (3.4) we obtain

$$\varphi V_{cotr}=V_{cotr},\ \varphi V_{-tanr}=V_{-tanr}\ \text{and}\ \varphi V_{\lambda_1}=V_{\lambda_2}\,.$$

Thus the real hypersurface M has at most five distinct principal curvatures $2cot2r$ (with multiplicity 1) $cotr$ (with multiplicity $2n-2k-2$), $-tanr$ (with multiplicity $2k-2m$), λ_1 (with multiplicity $m\geq 0$) and λ_2 (with multiplicity $m\geq 0$). Hence

$$(3.5)\qquad h=(2n-2k-1)cotr-(2k-2m+1)tanr+m(\lambda_1+\lambda_2)\,.$$

Using (3.3), (3.4) and (3.5) we obtain

$$(3.6)\qquad (2n-2k-1)cotr-(2k-2m+1)tanr+(m-1)\left(\lambda_1+\frac{\alpha\lambda_1+2}{2\lambda_1-\alpha}\right)-(2n-3)\alpha=0\,.$$

Now for the multiplicity m of the principal curvature λ_1, namely for the integer $m=m(\lambda_1)$ we distinguish three cases: $m=0$, $m=1$ and $m\geq 2$.

We shall prove that $m<2$.

Suppose for the moment that $m\geq 2$. From (3.6) we have that λ_1 = *constant*.

Thus our manifold M is homogeneous (cf. Theorem B) and from the Remark 2 we conclude that M is of type B, C, D or E. We will check one by one that these cases cannot occur.

Let M be of type B (namely M is a tube of radius r). Then M has three distinct constant principal curvatures $\mu_1=(1+x)/(1-x)$, $\mu_2=(x-1)/(x+1)$, $\alpha=(x-1/x)$, where $x = cotr$, with $m(\mu_1)=n-1$, $m(\mu_2)=n-1$ and $m(\alpha)=1$.

Thus

$$h=(n-1)\frac{4x}{1-x^2}+\frac{x^2-1}{x}.$$

On the other hand, from (3.3) we have

$$h=\frac{4x}{1-x^2}+(2n-3)\frac{x^2-1}{x}.$$

From the last two relations we obtain

$$(n-2)\frac{4x}{1-x^2}=2(n-2)\frac{x^2-1}{x} \quad or \quad x^4+1=0, \text{ impossible.}$$

Now let M be of type C (which is also a tube of radius r). Let $x = cotr$. Then M has five distinct constant principal curvatures $\mu_1=(1+x)/(1-x)$ with $m(\mu_1)=2$, $\mu_2=(x-1)/(x+1)$ with $m(\mu_2)=2$, $\mu_3=x$ with $m(\mu_3)=n-3$, $\mu_4=(-1/x)$ with $m(\mu_4)=n-3$ and $\alpha=(x-1/x)$ with $m(\alpha)=1$ (cf. Remark 2). Since $\varphi V_{\mu_1}=V_{\mu_2}$, $\varphi V_{\mu_3}=V_{\mu_3}$ and $\varphi V_{\mu_4}=V_{\mu_4}$, the condition $\nabla_\xi C=0$ is equivalent to $h=\mu_1+\mu_2+(2n-3)\alpha$. Then from this we obtain

$$\frac{1+x}{1-x}+\frac{x-1}{x+1}+(n-2)\left(x-\frac{1}{x}\right)=(2n-3)\frac{x^2-1}{x}$$

or

$$(n-1)x^4-2(n-3)x^2+n-1=0.$$

But this is impossible because the discriminant of this equation is negative.

Let M be of type D (which is a tube of radius r). Then M has five distinct constant principal curvatures $\mu_1=(1+x)/(1-x)$ with $m(\mu_1)=4$, $\mu_2=(x-1)/(x+1)$ with $m(\mu_2)=4$, $\mu_3=x$ with $m(\mu_3)=4$, $\mu_4=-1/x$ with $m(\mu_4)=4$ and $\alpha=(x-1/x)$ with $m(\alpha)=1$, where $x = cotr$ and dim$M$ = 17 (cf. Remark 2). We have again as in case of type C, that $\varphi V_{\mu_1}=V_{\mu_2}$, $\varphi V_{\mu_3}=V_{\mu_3}$ and $\varphi V_{\mu_4}=V_{\mu_4}$. Thus the condition $\nabla_\xi C=0$ is equivalent to $h=\mu_1+\mu_2+(2n-3)\alpha$. This becomes $(n-4)x^4-2(n-7)x^2+n-4=0$. From this we get $n\leq 5$ or equivalently $M\leq 9$, a contradiction.

Finally, let M be of type E (which is a tube of radius r). Then as above M has the same five distinct constant principal curvatures $\mu_1, \mu_2, \mu_3, \mu_4$ and α but with multiplicity $m(\mu_1)=m(\mu_2)=6$, $m(\mu_3)=m(\mu_4)=8$ and $m(\alpha)=1$ (cf. Remark 2). By virtue of the discussion in cases of type C or D we have only to solve the equation $h-\mu_1-\mu_2-(2n-3)\alpha=0$. Namely we have the equation $(n-6)x^4-2(n-11)x^2+(n-6)=0$. But in our case dim$M$ = 29, or equivalently n = 15. Thus we have $9x^4-8x^2+9=0$, which is impossible. This completes the proof of the assertion that $m<2$.

We will examine now the cases m = 0 and m = 1 separately. Let m = 0. Our real hypersurface M has three distinct principal curvatures and it is of case (1) with $0<r(\neq\pi/4)<\pi/2$ in the Theorem. Now let m = 1. Our real hypersurface M has at most five distinct principal curvatures $2cor2r$ with $m(2cot2r)=1$, $cotr$ with $m(cotr)$ = $2n-2k-2$, $-tanr$ with $m(-tanr)$ = $2k-2$, λ_1 with $m(\lambda_1)=1$ and λ_2 with $m(\lambda_2)=1$. Since the multiplicities of the principal curvatures of M do not match with the multiplicities of any homogeneous real hypersurface (cf. Remark 2), the manifold M is not homogeneous. Hence both λ_1 and λ_2 are not constant (cf. Theorem B). Moreover, Theorem D shows that λ_1 and λ_2 can be expressed as: $\lambda_1=cot(r-\theta)$ and $\lambda_2=cot(r+\theta)$, where $cot\theta$ is a principal curvature of the Kaehler submanifold $\tilde{N}$. In addition equation (3.6) yields that $cot^2r=(n-k-1)/(k-1)$. Hence the manifold M is of case (3) in the Theorem.

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