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TWO THEOREMS ON THE EXISTENCE OF INDISCERNIBLE SEQUENCES

By

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§0. Introduction.

In this paper we shall state two theorems (Theorem A and Theorem B below) concerning the existence of indiscernible sequences which realize a given type. When we know the existence of a set $A=(a_i)_{i<\omega}$ which realizes a given infinite type p, it will be convenient to assume that A is an indiscernible sequence. Of course this is not always the case. But the type p satisfies a certain condition, we can assume A to be an indiscernible sequence. The reader will find some such conditions in this paper. Our results generalize the following fact:

FACT. The following two conditions on a type $p(x_0, \cdots, x_i, \cdots)_{i<\omega}$ are equivalent:

i) There is an indiscernible sequence $(a_i)_{i<\omega}$ which realizes $p(x_i)_{i<\omega}$.

ii) There is a sequence $(a_i)_{i<\omega}$ such that $(a_{f(i)})_{i<\omega}$ realizes $p(x_i)_{i<\omega}$, whenever f is an increasing function on ω.

Our results in this paper will be used to investigate the number $\kappa_{inp}(T)$ of independent partions of T, in the sequel [3] to this paper. In §1 below, we shall state Theorem A and Theorem B, whose proofs will be given in §2.

§1. Theorems.

We use the usual standard notions in Shelah [2]. But some of them will be explained below. Let T be a fixed complete theory formulated in a first order language $L(T)$, and $\mathfrak{C}$ a model of T with sufficiently large saturation (cf. p. 7 in [2]). We use $\alpha, \beta, \gamma, \cdots$ for ordinals and $m, n, i, j, k, \cdots$ for natural numbers. $\bar{a}$, $\bar{b}$, and $\bar{a}_\alpha^i$ are used to denote finite tuples of elements in $\mathfrak{C}$. $\bar{x}$, $\bar{y}$, and $\bar{x}_\alpha^i$ are used to denote finite sequence of variables. We use capitals $A_\alpha, B_\alpha, \cdots (X_\alpha, Y_\alpha, \cdots)$ to denote (distinct) ω-sequences of (distinct) k-tuples of (distinct) elements in $\mathfrak{C}$ (variables). Therefore, A_α has the form $(\bar{a}_\alpha^i)_{i<\omega}$, where $\bar{a}_\alpha^i$ is a tuple of elements of $\mathfrak{C}$, whose length is k. For such an A_α,

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$\cup A_\alpha$ means the set $\{a: a$ is an element of some $\bar{a}^i_\alpha\}$. The set of increasing functions on ω is denoted by $\mathcal{F}$. If $f \in \mathcal{F}$ and $A=(\bar{a}^i)_{i<\omega}$, then A^f is the sequence $(\bar{a}^{f(i)})_{i<\omega}$. Similarly, $\cup X_\alpha$ and X^f will be used. Also we assume that $\cup X_\alpha$, $\cup X_\beta, \cdots, \cup Y_\alpha, \cdots$ are all disjoint.

To state our results, we require two notions, "strongly consistent" and "almost strongly consistent", on a set $p(X_\alpha)_{\alpha<\kappa}$ of formulas of $L(T)$ with variables in $\bigcup_{\alpha<\kappa} \cup X_\alpha$ defined by:

i) $p(X_\alpha)_{\alpha<\kappa}$ is strongly consistent if $\bigcup_{F \in {}^\kappa\mathcal{F}} p(X_\alpha^{F(\alpha)})_{\alpha<\kappa}$ is consistent with T;

ii) $p(X_\alpha)_{\alpha<\kappa}$ is almost strongly consistent if $\bigcup_{F \in {}^\kappa\mathcal{F}} p(Y_{F\restriction\alpha}^{F(\alpha)})_{\alpha<\kappa}$ is consistent with T, where $F\restriction\alpha$ is the restriction of F to α, and Y_G $(G \in {}^{\kappa>}\mathcal{F})$ are sequences of new variables.

Then our results are:

THEOREM A. *The following two conditions on a type $p(X_\alpha)_{\alpha<\kappa}$ are equivalent:*

a) *$p(X_\alpha)_{\alpha<\kappa}$ is strongly consistent.*

b) *There is a sequence $(A_\alpha)_{\alpha<\kappa}$ with the properties* i) *$(A_\alpha)_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$ and* ii) *$A_\alpha=(\bar{a}^i_\alpha)_{i<\omega}$ is an indiscernible sequence over $\bigcup_{\beta\neq\alpha} \cup A_\beta$ for each $\alpha<\kappa$.*

THEOREM B. *The following two conditions on a type $p(X_\alpha)_{\alpha<\kappa}$ are equivalent:*

c) *$p(X_\alpha)_{\alpha<\kappa}$ is almost strongly consistent.*

d) *There is a sequence $(A_\alpha)_{\alpha<\kappa}$ with the properties* i) *$(A_\alpha)_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$ and* ii) *$A_\alpha=(\bar{a}^i_\alpha)_{i<\omega}$ is an indiscernible sequence over $\bigcup_{\beta<\alpha} \cup A_\beta$ for each $\alpha<\kappa$.*

§2. Proofs.

The implication b)$\Rightarrow$a) is trivial, because the sequence $(A_\alpha)_{\alpha<\kappa}$ realizes every $p(X_\alpha^{F(\alpha)})_{\alpha<\kappa}$ $(F \in {}^\kappa\mathcal{F})$. a)$\Rightarrow$b) and c)$\Rightarrow$d) will be proved by iterated use of Ramsey's theorem.

a)$\Rightarrow$b): Let's define the set $r(X, Y)$ by

$$\begin{aligned} r(X, Y)=\{\phi(\bar{x}^{i_1}{}^\frown\cdots{}^\frown\bar{x}^{i_n}:\bar{y}) \longleftrightarrow \phi(\bar{x}^{j_1}{}^\frown\cdots{}^\frown\bar{x}^{j_n}:\bar{y})&: \phi \in L(T),\\ \bar{x}^{i_1}, \cdots, \bar{x}^{i_n} \in X\ (i_1<\cdots<i_n)&,\\ \bar{x}^{j_1}, \cdots, \bar{x}^{j_n} \in X\ (j_1<\cdots<j_n)&,\ \bar{y} \in \cup Y\}, \end{aligned}$$

where $\bar{y} \in \cup Y$ means that every element in the tuple $\bar{y}$ belongs to the set $\cup Y$. We shall show the consistency of

$$q_\beta(X_\alpha)_{\alpha<\kappa}=\bigcup_{F \in {}^\kappa\mathcal{F}} p(X_\alpha^{F(\alpha)})_{\alpha<\kappa} \cup \bigcup_{\gamma<\beta} r(X_\gamma, \bigcup_{\delta\neq\gamma} X_\delta),$$

by induction on β. If $\beta=0$, then the consistency follows from a). If β is a limit ordinal, the consistency is clear by compactness. Let $\beta=\gamma+1$ and suppose that $(B_\alpha)_{\alpha<\kappa}$ realizes $q_\gamma(X_\alpha)_{\alpha<\kappa}$. For given formulas (*) $\phi_i(\bar{x}_1\wedge\cdots\wedge\bar{x}_n:\bar{y}_i)\in L(T)$ $(i=1,\cdots,m)$ and elements $\bar{b}_i\in\bigcup_{\delta\in\gamma}\cup B_\delta$ $(i=1,\cdots,m)$, we can choose a function $f\in\mathcal{F}$ such that

$$\mathfrak{C}\models\phi_i(\bar{b}_\gamma^{f(i_1)}{}^\frown\cdots{}^\frown\bar{b}_\gamma^{f(i_n)}:\bar{b}_i)\longleftrightarrow\phi_i(\bar{b}_\gamma^{f(j_1)}{}^\frown\cdots{}^\frown\bar{b}_\gamma^{f(j_n)}:\bar{b}_i)\qquad(i=1,\cdots,m),$$

whenever $i_1<\cdots<i_n$ and $j_1<\cdots<j_n$, by using Ramsey's theorem. Then the sequence $(B_\delta)_{\delta<\gamma}\wedge(B_\gamma^f)\wedge(B_\delta)_{\delta<\gamma}$ realizes the following type:

$$q_\gamma(X_\alpha)_{\alpha<\kappa}\cup\bigcup_{i=1}^{m}\{\phi_i(\bar{x}_\gamma^{i_1}{}^\frown\cdots{}^\frown\bar{x}_\gamma^{i_n}:\bar{b}_i)\longleftrightarrow\phi_i(\bar{x}_\gamma^{j_1}{}^\frown\cdots{}^\frown\bar{x}_\gamma^{j_n}:\bar{b}_i):\\ i_1<\cdots<i_n,\ j_1<\cdots<j_n\}.$$

This shows the consistency of $q_\beta(X_\alpha)_{\alpha<\kappa}$. This means that b) holds.

c)$\Rightarrow$d): The proof of this case is similar to that of a)$\Rightarrow$b). For each $F=(f_\alpha)_{\alpha<\beta}\in{}^{\kappa>}\mathcal{F}$, we prepare new variables $X_F=(\bar{x}_F^i)_{i<\omega}$. We shall show the contency of

$$q'_\beta(X_F)_{F\in{}^{\kappa>}\mathcal{F}}=\bigcup_{F\in{}^\kappa\mathcal{F}}p(X_{F\restriction\alpha}^{F(\alpha)})\cup\bigcup_{\gamma<\beta}\ \bigcup_{F\in{}^\gamma\mathcal{F}}r(X_F,\bigcup_{\delta<\gamma}X_{F\restriction\delta}^{F(\delta)}),$$

by induction on β. As in a)$\Rightarrow$b), we can assume that β is a successor and that $\beta=\gamma+1$. Let $(B_F)_{F\in{}^{\kappa>}\mathcal{F}}$ realize $q'_\gamma(X_F)_{F\in{}^{\kappa>}\mathcal{F}}$. For given $\phi_{i,j}(\bar{x}_1{}^\frown\cdots{}^\frown\bar{x}_n:\bar{y})\in L(T)$ $(i=1,\cdots,l;\ j=1,\cdots,m)$, $F_i\in{}^\gamma\mathcal{F}$ $(i=1,\cdots,l)$, and $b_{i,j}\in\bigcup_{\delta<\gamma}\cup B_{F_i\restriction\delta}^{F_i(\delta)}$ $(i=1,\cdots,l;\ j=1,\cdots,m)$, we can choose a function $f\in\mathcal{F}$ such that, for each i, j,

$$\mathfrak{C}\models\phi_{i,j}(\bar{b}_{F_i}^{f(i_1)}{}^\frown\cdots{}^\frown\bar{b}_{F_i}^{f(i_n)}:\bar{b}_{i,j})\longleftrightarrow\phi_{i,j}(\bar{b}_{F_i}^{f(j_1)}{}^\frown\cdots{}^\frown\bar{b}_{F_i}^{f(j_n)}:\bar{b}_{i,j}),$$

whenever $i_1<\cdots<i_n$ and $j_1<\cdots<j_n$, by using Ramsey's theorem. Then define $(A_F)_{F\in{}^{\kappa>}\mathcal{F}}$ by

$$\begin{aligned}
&A_F=B_F &&\text{if}\quad lh(F)<\gamma,\\
&A_F=B_F^f &&\text{if}\quad lh(F)=\gamma,\\
&A_{F^\frown(g)^\frown H}=B_{F^\frown(g\circ h)^\frown H} &&\text{if}\quad lh(F)=\gamma.
\end{aligned}$$

It is a routine to check that $(A_F)_{F\in{}^\kappa\mathcal{F}}$ realizes the following type:

$$q'_\gamma(X_F)_{F\in{}^\kappa\mathcal{F}}\cup\bigcup_{\substack{1\le i\le l\\1\le j\le n}}\{\phi_{i,j}(\bar{x}_{F_i}^{i_1}{}^\frown\cdots{}^\frown\bar{x}_{F_i}^{i_n}:\bar{b}_{i,j})\\ \longleftrightarrow\phi_{i,j}(\bar{x}_{F_i}^{j_1}{}^\frown\cdots{}^\frown\bar{x}_{F_i}^{j_n}:\bar{b}_{i,j}):i_1<\cdots<i_n,\ j_1<\cdots<j_n\}.$$

The above argument shows the consistency of $q'_\beta(X_F)_{F\in{}^\kappa\mathcal{F}}$. Let $(A_F)_{F\in{}^{\kappa>}\mathcal{F}}$ realize $q'_\kappa(X_F)_{F\in{}^{\kappa>}\mathcal{F}}$ and define $(A_\alpha)_{\alpha<\kappa}$ by

(*) Exactly speaking, n depends on i, but we can assume that n does not depend on i without loss of generality.

$$A_\alpha = A_{(id)_{\beta<\alpha}} = \underbrace{A_{(id,\, id, \ldots)}}_{\alpha \text{ times}}.$$

$(A_\alpha)_{\alpha<\kappa}$ is the desired sequence which satisfies the conditions i) and ii) of d).

d)⇒c): For each $\alpha<\kappa$ and each $f\in\mathcal{F}$, let $E_{\alpha,f}$ be an elementary map such that

1) $\mathrm{dom}(A_{\alpha,f})=\bigcup_{\beta\leq\alpha+1}\cup A_\beta$, $E_{\alpha,id}$=the identity map,

2) $E_{\alpha,f}\restriction(\bigcup_{\beta<\alpha}\cup A_\beta)$=the identity map,

3) $E_{\alpha,f}(\bar{a}^i_\alpha)=\bar{a}^{f(i)}_\alpha$, for each $i<\omega$.

Using these $E_{\alpha,f}$ ($\alpha<\kappa$, $f\in\mathcal{F}$), let's define elementary maps I_F ($F\in{}^{\kappa>}\mathcal{F}$) such that

4) $\mathrm{dom}(I_F)=\bigcup_{\alpha\leq lh(F)}\cup A_\alpha$,

5) $I_F\restriction(\bigcup_{\alpha<lh(F)}\cup A_\alpha)\subseteq I_G\restriction(\bigcup_{\alpha<lh(G)}\cup A_\alpha)$, for all F, $G\in{}^{\kappa\geq}\mathcal{F}$ such that $F\subseteq G$,

6) $I_{F^\frown(f)}=I_{F^\frown(id)}\circ E_{lh(F),f}$.

Suppose that we have already constructed I_F ($F\in{}^{\alpha>}\mathcal{F}$). Our construction splits into the following two cases:

CASE 1. $\alpha=\beta+1$. For each $F\in{}^\beta\mathcal{F}$, let J be an arbitrary elementary map such that $J\supseteq I_F$ and $\mathrm{dom}(J)=\bigcup_{\gamma\leq\alpha}\cup A_\gamma$. Then put $I_{F\wedge(f)}=J\circ E_{\beta,f}$.

CASE 2. α is a limit ordinal. For each $F\in{}^\alpha\mathcal{F}$, let $I^*_F=\bigcup_{\beta<\alpha}(I_{F\restriction\beta}\restriction(\bigcup_{\gamma<\beta}A_\gamma))$. By 5), I^*_F is an elementary map. We define I_F as an elementary map such that $I_F\supseteq I^*_F$ and $\mathrm{dom}(I_F)=\bigcup_{\beta<\alpha}\cup A_\beta$. If we put $A_F=(I_F(\bar{a}^i_{lh(F)}))_{i<\omega}$, then $(A_F)_{F\in{}^{\kappa>}\mathcal{F}}$ guarantees the almost strong consistency of $p(X_\alpha)_{\alpha<\kappa}$, i.e., it realizes the type $\bigcup_{F\in{}^\kappa\mathcal{F}} p(Y^{F(\alpha)}_{F\restriction\alpha})_{\alpha<\kappa}$. For this we must show that $(A^{F(\alpha)}_{F\restriction\alpha})_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$ for each $F\in{}^\kappa\mathcal{F}$. But this is clear, since the followings hold in turn:

$(A_\alpha)_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$;
$((I_F(\bar{a}^i_\alpha))_{i<\omega})_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$;
$((I_{F\restriction\alpha+1}(\bar{a}^i_\alpha))_{i<\omega})_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$;
$((I_{F\restriction\alpha\wedge(id)}\circ E_{\alpha,F(\alpha)}(\bar{a}^i_\alpha))_{i<\omega})_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$;
$(((I_{F\restriction\alpha}(\bar{a}^i_\alpha))_{i<\omega})^{F(\alpha)})_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$;
$(A^{F(\alpha)}_{F\restriction\alpha})_{\alpha<\kappa}$ realizes $p(X_\alpha)_{\alpha<\kappa}$.

REMARK. If T is stable, any indiscernible sequence becomes an indiscernible set. Hence, in such cases, we can require in b) and d) that A_α is an indiscernible set. We are inspired by Chapter III of [2]. In fact, the construction of I^*_F in the proof of d)⇒c) is very similar to that of F_η in Theorem 3.7 of [**2**]. We use Theorems A and B freely in our forthcoming paper [**3**].

Added in proof.

In Theorems A and B, each sequence X_α ($\alpha<\kappa$) is assumed to be an ω-sequence of finite tuples of variables (to avoid unnecessary complexity). But the restriction to ω-sequence is not necessary. By using compactness, we can prove Theorems A and B for a type $p(X_\alpha)_{\alpha<\kappa}$ with $lh(X_\alpha)=\lambda$, where λ is an arbitrary infinite cardinal. Moreover, if $p(X_\alpha)_{\alpha<\kappa}$ is a strongly consistent type of a stable theory T, its realization $(A_\alpha)_{\alpha<\kappa}$ can be assumed to be independent over some A with $|A|=\kappa$. (Precisely speaking, $\bigcup_{\alpha<\kappa} A_\alpha$ is independent over A.) To prove this, we must note that for each $A_\alpha=(\bar{a}_\alpha^i)_{i<\lambda}$, the average type $Av(A_\alpha/A_\alpha)$ is a unique non-forking extension of $Av(A_\alpha/(\bar{a}_\alpha^i)_{i<\omega})$ to the domain A_α. A hint for its proof will be found in [3].

References

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