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MINIMAL IMMERSION OF PSEUDO-RIEMANNIAN MANIFOLDS

By

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1. Preliminares.

Let E_q^n be the n-dimensional Pseudo-Euclidean space with metric tensor given by

$$g=-\sum_{i=1}^{q}(dx_i)^2+\sum_{j=q+1}^{n}(dx_j)^2$$

where $(x_1, x_2, \cdots, x_n)$ is a rectangular coordinate system of E_q^n. (E_q^n, g) is a flat Pseudo-Riemannian manifold of signature $(q, n-q)$.

Let c be a point in E_q^{n+1} (or E_{q+1}^{n+1}) and $r>0$. We put

$$S_q^n(c, r)=\{x\in E_q^{n+1} : g(x-c, x-c)=r^2\}$$
$$H_q^n(c, r)=\{x\in E_{q+1}^{n+1} : g(x-c, x-c)=-r^2\}.$$

It is known that $S_q^n(c, r)$ and $H_q^n(c, r)$ are complete Pseudo-Riemannian manifolds of signature $(q, n-q)$ and respective constant sectional curvatures r^{-2} and $-r^{-2}$. $S_q^n(c, r)$ and $H_q^n(c, r)$ are called the Pseudo-Riemannian sphere and the Pseudo-hyperbolic space, respectively. The point c is called the center of $S_q^n(c, r)$ and $H_q^n(c, r)$. In the following, $S_q^n(0, r)$ and $H_q^n(0, r)$ are simply denoted by $S_q^n(r)$ and $H_q^n(r)$, respectively. N_p^n denotes the Pseudo-Riemannian manifold with metric tensor of signature $(p, n-p)$. The Pseudo-Riemannian manifold, the Pseudo-Euclidean space, the Pseudo-Riemannian sphere and the Pseudo-hyperbolic space are simply denoted by the $P-R$ manifold, the $P-E$ space, the $P-R$ sphere and the $P-h$ space. The $P-R$ manifold N_1^n is called the Lorentz manifold and the $P-E$ space E_1^p is called the Minkowski space.

Let $f: M_p^m\to N_q^n$ be an isometric immersion of a $P-R$ manifold M_p^m in another $P-R$ manifold N_q^n. That is $f*\bar{g}=g$, where g and $\bar{g}$ are the indefinite metric tensors of M_p^m and N_q^n, respectively. $T(M_p^m)$ and $T^\perp(M_p^m)$ denote the tangent bundle and the normal bundle of M_p^m. ∇, $\bar{\nabla}$ and $\nabla^\perp$ denote the Riemannian connections and the normal connection on M_p^m, N_q^n and $T^\perp(M_p^m)$, respectively. Then for any vector fields $X, Y\in T(M_p^m)$, $v\in T^\perp(M_p^m)$, we have the Gauss formula

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y),$$

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the Weingarten formula

$$\bar{\nabla}_X v = -A^v(X) + \nabla_X^{\perp} v,$$

where B is the second fundamental form of the immersion, A^v is the Weingarten map with respect to v, and

$$g(A^v(X), Y) = \bar{g}(B(X, Y), v).$$

Let N_q^n be a $P-R$ manifold with the metric tensor $\bar{g}$. A tangent vector x to N_q^n is said to be space-like, time-like or light-like (null) if $\bar{g}(x, x)>0$ (or $x=0$), $\bar{g}(x, x)<0$ or $\bar{g}(x, x)=0$ (and $x\neq 0$), respectively.

Let M_p^m be a submanifold of N_q^n. If the Pseudo-Riemannian metric tensor $\bar{g}$ of N_q^n induces a Pseudo-Riemannian metric tensor, a Riemannian metric tensor or a degenerate metric tensor on M_p^m, then M_p^m is called a $P-R$ submanifold, a Riemannian submanifold or a degenerate submanifold, respectively. For the nondegenerate submanifold, we have the direct sum decomposion

$$T(N_q^n) = T(M_p^m) \oplus T^{\perp}(M_p^m)$$

and $T^{\perp}(M_p^m)$ (the normal bundle) is also nondegenerate. In the following, we assume that the submanifold is nondegenerate.

A normal vector field $v \in T^{\perp}(M_p^m)$ is said to be parallel if $\nabla_X^{\perp} v=0$ for any vector $X \in T(M_p^m)$.

Let M_p^m be a nondegenerate submanifold in N_q^n and $e_1, e_2, \cdots, e_m$ be an orthonormal local basis on M_p^m. The mean curvature vector H of M_p^m in N_q^n is defined by

$$H = \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i B(e_i, e_i), \qquad \varepsilon_i = g(e_i, e_i) = \pm 1.$$

The nondegenerate submanifold M_p^m of N_q^n is said to be minimal if the mean curvature vector H of M_p^m in N_q^n vanishes identically.

For any real function f on M_p^m, the Laplacian Δf of f is defined by

$$\Delta f = -g^{ji} \nabla_j \nabla_i f = -\sum_{i=1}^{m} \varepsilon_i (e_i e_i f - \nabla_{e_i} e_i f)$$

(cf. [2]).

LEMMA 1. ([3], [4]) *An isometric immersion x of a $P-R$ manifold M_p^m in a $P-E$ space E_q^n satisfies*

$$\Delta x = -mH$$

where H is the mean curvature vector of the immersion and Δ is the Laplacian of M_p^m.

LEMMA 2. ([3], [4]) *Let M_p^m be isometrically immersed in a $P-R$ sphere*

$S_q^{m+k-1}(c, r)$ or a $P-h$ space $H_{q-1}^{m+k-1}(c, r)$ of the $P-E$ space E_q^{m+k}. Then the mean curvature vector H of M_p^m in E_q^{m+k} and the mean curvature vector H_0 of M_p^m in S_q^{m+k-1} or H_{q-1}^{m+k-1} satisfy

$$H=H_0-\varepsilon(x-c)/r^2.$$

Where x is the immersion of M_p^m (as the vector field in E_q^{m+k}) and $\varepsilon=\pm 1$, if $x: M_p^m \to S_q^{m+k-1}(c, r)$, then $\varepsilon=1$, if $x: M_p^m \to H_{q-1}^{m+k-1}(c, r)$, then $\varepsilon=-1$.

2. The minimal immersion in $S_q^{m+k-1}(r)$ or $H_{q-1}^{m+k-1}(r)$.

LEMMA 3. *Let M_p^m $(m \geqq 2)$ be a nondegenerate submanifold of a $P-E$ space E_q^n and H be the mean curvature vector of M_p^m in E_q^n. x denotes the position vector field of M_p^m in E_q^n. If $x=aH$ for some $a \neq 0$ on M_p^m, then $\bar{g}(H, H) \neq 0$ on M_p^m, where $\bar{g}$ is the metric tensor of E_q^n.*

PROOF. Suppose $\bar{g}(H, H)=0$ and $x=aH$ for some $a \neq 0$ on M_p^m. Then $\bar{g}(x, x)=a^2\bar{g}(H, H)=0$. Since $\Delta x=-mH$, so

$$\begin{aligned}0=\Delta\bar{g}(x, x)&=2\bar{g}(\Delta x, x)-2\bar{g}(\nabla x, \nabla x)\\&=-2m\bar{g}(H, x)-2\bar{g}(\nabla x, \nabla x)\\&=-2\bar{g}(\nabla x, \nabla x),\end{aligned}$$

that is $\bar{g}(\nabla x, \nabla x)=0$. It is impossible because M_p^m $(m \geqq 2)$ is nondegenerate.

Q. E. D.

THEOREM 1. *If an isometric immersion $x: M_p^m \to E_q^{m+k}$ of a $P-R$ manifold M_p^m $(m \geqq 2)$ in a $P-E$ space E_q^{m+k} satisfies $\Delta x=bx$ for some constant $b \neq 0$*

(1) *when $b>0$, then x realizes a minimal immersion in a $P-R$ sphere $S_q^{m+k-1}(\sqrt{m/b})$ of the sectional curvature b/m in E_q^{m+k}; conversely if x realizes a minimal immersion in a $P-R$ sphere of the sectional curvature $r^{-2}(r>0)$ in E_q^{m+k}, then x satisfies $\Delta x=bx$ up to a parallel displacement in the $P-E$ space E_q^{m+k} and $b=m/r^2$.*

(2) *when $b<0$, then x realizes a minimal immersion in a $P-h$ space $H_{q+1}^{m+k-1}(\sqrt{m/-b})$ of the sectional curvature b/m in E_q^{m+k}; conversely if x realizes a minimal immersion in a $P-h$ space of the sectional curvature $-r^2(r>0)$ in E_q^{m+k}, then x satisfies $\Delta x=bx$ up to a parallel displacement in the $P-E$ space E_q^{m+k} and $b=-m/r^2$.*

PROOF. Let $\Delta x=bx$, $b \neq 0$, then we have $bx=-mH$ by Lemma 1. Since $X\bar{g}(x, x)=2\bar{g}(X, x)=0$, where $\bar{g}$ is the metric of E_q^{m+k}, it yields that $\bar{g}(x, x)=$

constant$\neq 0$ by Lemma 3. So x realizes an immersion in $S_q^{m+k-1}(c, r)$ or $H_{q-1}^{m+k-1}(c, r)$. And by Lemma 2 and $bx=-mH$, we have $H_0=0$. Thus x realizes a minimal immersion in $S_q^{m+k-1}(c, r)$ or $H_{q-1}^{m+k-1}(c, r)$ in E_q^{m+k} and $r=\sqrt{m/\varepsilon b}$ ($\varepsilon=\pm 1$).

Conversely, if x realizes a minimal immersion in $S_q^{m+k-1}(c, r)$ or $H_{q-1}^{m+k-1}(c, r)$ in E_q^{m+k}, then by Lemma 2, we have

$$H=-\varepsilon(x-c)/r^2 \qquad (\varepsilon=\pm 1)$$

and $\Delta x=-mH$. Thus, we obtain

$$\Delta(x-c)=-m(-\varepsilon(x-c)/r^2)=\varepsilon m(x-c)/r^2$$

$$b=\varepsilon m/r^2 \qquad (\varepsilon=\pm 1). \qquad \text{Q.E.D.}$$

COROLLARY 1. *An isometric immersion* $x: M_p^m \to E_q^{m+k}$ *of a* $P-R$ *manifold* M_p^m *in a* $P-E$ *space* E_q^{m+k} *is minimal if and only if* $\Delta x=0$.

COROLLARY 2. *If an isometric immersion* $x: M_p^m \to E_p^{m+k}$ *of a* $P-R$ *manifold* M_p^m *in a* $P-E$ *space* E_p^{m+k} *satisfies* $\Delta x=bx$ *for some constant* $b\neq 0$, *then* b *is necessarily positive and* x *realizes a minimal immersion of a* $P-R$ *manifold* M_p^m *in a* $P-R$ *sphere* $S_p^{m+k-1}(\sqrt{m/b})$ *in the* $P-E$ *space* E_p^{m+k}.

PROOF. For any isometric immersion $x: M_p^m \to E_p^{m+k}$, the vectors of the normal space of M_p^m in E_p^{m+k} are space-like. Then by Lemma 2, $\varepsilon=+1$.

Q.E.D.

COROLLARY 3. *If an isometric immersion* $x: M_p^m \to E_{p+k}^{m+k}$ *of a* $P-R$ *manifold* M_p^m *in a* $P-E$ *space* E_{p+k}^{m+k} *satisfies* $\Delta x=bx$ *for some constant* $b\neq 0$, *then* b *is necessarily negative and* x *realizes a minimal immersion of a* $P-R$ *manifold* M_p^m *in a* $P-h$ *space* $H_{p+k-1}^{m+k-1}(\sqrt{m/-b})$ *in the* $P-E$ *space* E_{p+k}^{m+k}.

PROOF. By the condition, we know the vectors of the normal space of M_p^m in E_{p+k}^{m+k} are time-like. So in Lemma 2, $\varepsilon=-1$. Q.E.D.

3. The spectrum of $S_p^m(r)$ and $H_{p-1}^m(r)$.

In this section we consider the Laplacians Δ of $S_p^m(r)$ and $H_{p-1}^m(r)$ acting on functions. We obtain the constant b that satisfies $\Delta f=bf$, $f\not\equiv 0$, where Δ is the Laplacian of $S_p^m(r)$ or $H_{p-1}^m(r)$.

Let M_p^m be a $P-R$ manifold. The Laplacian of M_p^m has various expressions

$$\begin{aligned}\Delta f &= -g^{ji}\nabla_j\nabla_i f\\ &= -\text{trace }(\nabla d f)\\ &= -\text{trace (Hess } f),\end{aligned}$$

where Hess f denotes the Hessian of the function f. Let $e_1, e_2, \cdots, e_m$ be an orthonormal local basis on M_p^m, then

$$\Delta f = -\sum_{i=1}^{m}\varepsilon_i \text{ Hess } f(e_i, e_i) \qquad (\varepsilon_i = g(e_i, e_i) = \pm 1).$$

For each point $y \in M_p^m$, pick an orthonormal set of geodesics (v_i) parameterized by arc length and passing through $y \in M_p^m$ at $s=0$ and satisfying $v_i'(0)=e_i$. Then we have

$$\Delta f(y) = -\sum_{i=0}^{m}\varepsilon_i \frac{d^2}{ds^2}(f\circ v_i)(0)$$

(cf. [2] P. 33, P. 86).

For the $P-R$ sphere $S_p^m(1)$ and the $P-h$ space $H_{p-1}^m(1)$ in the $P-E$ space E_p^{m+1}, let $y \in S_p^m(1)$ or $H_{p-1}^m(1)$ be a point. Then y determines a unit vector e_1 in E_p^{m+1}. For $S_p^m(1)$ e_1 is a space-like vector and for $H_{p-1}^m(1)$ e_1 is a time-like vector. Let $e_2, e_3, \cdots, e_{m+1}$ be an orthonormal basis of $T_y(S_p^m(1))$ or $T_y(H_{p-1}^m(1))$. Then $e_1, e_2, \cdots, e_m, e_{m+1}$ form an orthonormal basis of $T_y(E_p^{m+1})$.

If $\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)=1$ $(i \geqq 2)$ on $S_p^m(1)$ or $H_{p-1}^m(1)$, the geodesic v_i $(i \geqq 2)$ through y with velocity vector e_i at y is given by

$$v_i(s) = (\cos s)e_1 + (\sin s)e_i \qquad i=2, 3, \cdots, (m+1)$$

where s is arc length parameter.

If $\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)=-1$ $(i \geqq 2)$ on $S_p^m(1)$ or $H_{p-1}^m(1)$, the geodesic v_i $(i \geqq 2)$ through y with velocity vector e_i at y is given by

$$v_i(s) = (\cosh s)e_1 + (\sinh s)e_i \qquad i=2, 3, \cdots, (m+1).$$

Let f be a function on E_p^{m+1} and $x^1, x^2, \cdots, x^{m+1}$ be the Euclidean coordinates associated with $e_1, e_2, \cdots, e_{m+1}$. Consider the functions $(f\circ v_i)(s) = f(v_i(s))$. By using the chain rule, we have

$$\frac{d(f\circ v_i)}{ds} = -(\sin s)\frac{\partial f}{\partial x^1} + (\cos s)\frac{\partial f}{\partial x^i}$$

if $\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)=1$ $(i \geqq 2)$;

$$\frac{d(f\circ v_i)}{ds} = (\sinh s)\frac{\partial f}{\partial x^1} + (\cosh s)\frac{\partial f}{\partial x^i}$$

if $\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)=-1$ $(i \geqq 2)$.

Therefore, for $y=v_i(0)$, we have

$$\frac{d^2(f\circ v_i)}{ds^2}(0)=-\frac{\partial f}{\partial x^1}(y)+\frac{\partial^2 f}{(\partial x^i)^2}(y)$$

if $\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)=1$ $(i\geqq 2)$;

$$\frac{d^2(f\circ v_i)}{ds^2}(0)=\frac{\partial f}{\partial x^1}(y)+\frac{\partial^2 f}{(\partial x^i)^2}(y)$$

if $\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)=-1$ $(i\geqq 2)$.

Let $\varepsilon=-\bar{g}(e_1, e_1)\bar{g}(e_i, e_i)$ $(i\geqq 2)$. Then

$$\begin{aligned}
\Delta^{S_p^m(1)}(f/S_p^m(1))(y)&=-\sum_{i=2}^{m+1}\varepsilon_i\frac{d^2(f\circ v_i)}{ds^2}(0)\\
&=-\sum_{i=2}^{m+1}\varepsilon_i\left(\varepsilon\frac{\partial f}{\partial x^1}(y)+\frac{\partial^2 f}{(\partial x^i)^2}(y)\right)\\
&=-\sum_{i=2}^{m+1}\varepsilon_i\frac{\partial^2 f}{(\partial x^i)^2}(y)-\sum_{i=2}^{m+1}\varepsilon_i\varepsilon\frac{\partial f}{\partial x^1}(y)\\
&=-\sum_{i=2}^{m+1}\varepsilon_i\frac{\partial^2 f}{(\partial x^i)^2}(y)+m\frac{\partial f}{\partial x^1}(y),\\
\Delta^{H_{p-1}^m(1)}(f/H_{p-1}^m(1))(y)&=-\sum_{i=2}^{m+1}\varepsilon_i\frac{d^2(f\circ v_i)}{ds^2}(0)\\
&=-\sum_{i=2}^{m+1}\varepsilon_i\left(\varepsilon\frac{\partial f}{\partial x^1}(y)+\frac{\partial^2 f}{(\partial x^i)^2}(y)\right)\\
&=-\sum_{i=2}^{m+1}\varepsilon_i\frac{\partial^2 f}{(\partial x^i)^2}(y)-m\frac{\partial f}{\partial x^1}(y)
\end{aligned}$$

But

$$(\Delta^{E_p^{m+1}}f)(y)=-\sum_{i=2}^{m+1}\varepsilon_i\frac{\partial^2 f}{(\partial x^i)^2}(y)-\varepsilon_1\frac{\partial^2 f}{(\partial x^1)^2}(y).$$

If we denote by r the "distance" function from a point in E_p^{m+1} to the origin, then we obtain

$$(*)\quad\begin{aligned}
(\Delta^{E_p^{m+1}}f)/S_p^m(1)&=\Delta^{S_p^m(1)}(f/S_p^m(1))-\frac{\partial^2 f}{\partial r^2}\Big/S_p^m(1)-m\frac{\partial f}{\partial r}\Big/S_p^m(1),\\
(\Delta^{E_p^{m+1}}f)/H_{p-1}^m(1)&=\Delta^{H_{p-1}^m(1)}(f/H_{p-1}^m(1))+\frac{\partial^2 f}{\partial r^2}\Big/H_{p-1}^m(1)+m\frac{\partial f}{\partial r}\Big/H_{p-1}^m(1).
\end{aligned}$$

Consider a homogeneous polynomial $\bar{Q}$ of degree $k\geqq 0$ on E_p^{m+1}. Let $Q=\bar{Q}/S_p^m(1)$ or $H_{p-1}^m(1)$. Then $\bar{Q}=r^kQ$. Thus we find

$$\frac{\partial\bar{Q}}{\partial r}=kr^{k-1}Q,\qquad \frac{\partial^2\bar{Q}}{\partial r^2}=k(k-1)r^{k-2}Q.$$

Therefore,

$$
\begin{aligned}
(\Delta^{E_p^{m+1}}\bar{Q})/S_p^m(1) &= \Delta^{S_p^m(1)}Q-k(k-1)Q-mkQ \\
&= \Delta^{S_p^m(1)}Q-k(k+m-1)Q \\
(\Delta^{E_p^{m+1}}\bar{Q})/H_{p-1}^m(1) &= \Delta^{H_{p-1}^m(1)}Q+k(k-1)Q+mkQ \\
&= \Delta^{H_{p-1}^m(1)}Q+k(k+m-1)Q\,.
\end{aligned}
$$

If $\bar{Q}$ satisfies $\Delta^{E_p^{m+1}}\bar{Q}=0$, $\bar{Q}$ is called the harmonic-like homogeneous polynormial. So we have

$$
\begin{aligned}
&\Delta^{S_p^m(1)}Q=k(m+k-1)Q\,, \\
&\Delta^{H_{p-1}^m(1)}Q=-k(m+k-1)Q\,.
\end{aligned}
$$

Let $\mathcal{H}_k$ be the vector space of harmonic-like homogeneous polynomials of degree k on E_p^{m+1}. With the same method of [5] P. 238-P. 240, we can prove

$$
\dim \mathcal{H}_k=\binom{m+k}{k}-\binom{m+k-2}{k-2}.
$$

Here, we give out another proof about $\dim \mathcal{H}_k$.

Assume

$$
\Delta=-\sum_{t=1}^{m+1}\frac{\partial^2}{(\partial x^t)^2} \qquad \Delta'=\sum_{t=1}^{p}\frac{\partial^2}{(\partial x^t)^2}-\sum_{t=p+1}^{m+1}\frac{\partial^2}{(\partial x^t)^2}\,,
$$

they are the Laplacians of E^{m+1} and E_p^{m+1}, respectively. $\mathcal{A}$ denotes the vector space of complex coefficient harmonic homogeneous polynomials of degree k about Δ. $\bar{\mathcal{A}}$ denotes the vector space of complex coefficient harmonic-like homogeneous polynomials of degree k about Δ'. Let $F(x^1, x^2, \cdots, x^{m+1})\in\mathcal{A}$. We have

$$
\begin{aligned}
0&=\Delta F(x^1, x^2, \cdots, x^{m+1}) \\
&=\sum_{t=1}^{m+1}\frac{\partial^2 F}{(\partial x^t)^2} \\
&=\sum_{t=1}^{p}\frac{\partial^2 F}{(\partial x^t)^2}+\sum_{t=p+1}^{m+1}\frac{\partial^2 F}{(\partial x^t)^2} \\
&=\sum_{t=1}^{p}\frac{-\partial^2 F}{(i\partial x^t)^2}+\sum_{t=p+1}^{m+2}\frac{\partial^2 F}{(\partial x^t)^2} \qquad (i^2=-1) \\
&=\Big(-\sum_{t=1}^{p}\frac{\partial^2}{(\partial x^t)^2}+\sum_{t=p+1}^{m+1}\frac{\partial^2}{(\partial x^t)^2}\Big)F(-ix^1, -ix^2, \cdots, -ix^p, \\
&\qquad\qquad x^{p+1}, x^{p+2}, \cdots, x^{m+1}) \\
&=\Delta' F(-ix^1, -ix^2, \cdots, -ix^p, x^{p+1}, \cdots, x^{m+1}).
\end{aligned}
$$

Therefore, $F(-ix^1, -ix^2, \cdots, -xi^p, x^{p+1}, \cdots, x^{m+1})\in\bar{\mathcal{A}}$ and

$$\left|\frac{\partial(y^1, y^2, \cdots, y^{m+1})}{\partial(x^1, x^2, \cdots, x^{m+1})}\right| = i^p \neq 0$$

where, $y^1 = ix^1$, $y^2 = ix^2$, $\cdots$, $y^p = ix^p$, $y^{p+1} = x^{p+1}$, $\cdots$, $y^{m+1} = x^{m+1}$. Thus we obtain $\dim \mathcal{A} = \dim \tilde{\mathcal{A}}$. But $\dim \mathcal{A} = \binom{m+k}{k} - \binom{m+k-2}{k-2}$. So

$$\dim \tilde{\mathcal{A}} = \binom{m+k}{k} - \binom{m+k-2}{k-2}.$$

THEOREM 2. *The spectrum of the Laplacians of the $P-R$ sphere $S_p^m(1)$ and the $P-h$ space $H_{p-1}^m(1)$ in the $P-E$ space E_{p+}^{m+1} is given by*

$$b_k = k(m+k-1) \qquad (k \geqq 0)$$

and

$$b_k = -k(m+k-1) \qquad (k \geqq 0)$$

respectively. And the multiplicity $j(b_k)$ of b_k is given by

$$j(b_0) = 1, \qquad j(b_1) = m+1,$$

$$j(b_k) = \binom{m+k}{k} - \binom{m+k-2}{k-2}$$

$$= \frac{(m+k-2)(m+k-3)\cdots(m+1)m}{k!}(m+2k-1). \qquad (k \geqq 2).$$

Since $S_p^m(r)$ with $S_p^m(1)$ and $H_{p-1}^m(r)$ with $H_{p-1}^m(1)$ are homothetic, we have

THEOREM 3. *The syectrum of the Laplacians of the $P-R$ sphere $S_p^m(r)$ and the $P-h$ space $H_{p-1}^m(r)$ in the $P-E$ space E_p^{m+1} is given by*

$$b_k = r^{-2}k(m+k-1)$$

and

$$b_k = -r^{-2}k(m+k-1) \qquad (k \geqq 0,\ r > 0),$$

respectively. And the multiplicity $j(b_k)$ of b_k is given by

$$j(b_0) = 1, \qquad j(b_1) = m+1,$$

$$j(b_k) = \binom{m+k}{k} - \binom{m+k-2}{k-2} \qquad (k \geqq 2).$$

4. The minimal immersions of the $P-R$ sphere and $P-$h space.

THEOREM 4. *Let $M = S_p^m(r)$ or $H_{p-1}^m(r)$. M is isometrically minimally immersed in $S_q^n(1)$ or $H_{q-1}^n(1)$. Then for $k = 0, 1, 2, \cdots$, we have*

$$r^{-2}=\frac{m}{k(m+k-1)}, \qquad n\leqq(m+k2-1)\frac{(m+k-2)!}{k!(m-1)!}.$$

PROOF. By Theorem 1, for the immersion f,

$$\Delta f=bf, \quad b>0;\ f: M \longrightarrow S_q^n(\sqrt{m/b})=S_q^n(1)$$

$$b<0;\ f: M \longrightarrow H_{q-1}^n(\sqrt{m/-b})=H_{q-1}^n(1).$$

Then, $b=m$ or $b=-m$ for $S_q^n(1)$ or $H_{q-1}^n(1)$, respectively. With Theorem 3, we have

$$b_k=k(m+k-1)r^{-2} \quad \text{for } S_p^m(r)$$

$$b_k=-k(m+k-1)r^{-2} \quad \text{for } H_{p-1}^m(r).$$

So

$$m=b=k(m+k-1)r^{-2} \quad \text{or} \quad m=-b=-(-k)(m+k-1)r^{-2}.$$

Therefore

$$r^{-2}=\frac{m}{k(m+k-1)}, \qquad n\leqq(m+2k-1)\frac{(m+k-2)!}{k!(m-1)!}. \qquad \text{Q.E.D.}$$

REMARK. By Theorem 1 and Theorem 3, we have

(1) $M_p^m(r)$ ($r<0$ is a constant) can not be isometrically minimally immersed in $S_q^n(1)$.

(2) The Riemannian manifold $M^m(r)$ with the constant sectional curvature $r<0$ can not be isometrically minimally immersed in the Riemannian sphere $S^n(1)$.

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