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SOME RESULTS ON PSEUDO-VALUATION DOMAINS

By

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Introduction. In [7], Hedstrom and Houston defined a pseudo-valuation domain (for short, a *PVD*) to be an integral domain in which every prime ideal P has the property that whenever x and y are elements of the quotient field with $xy\in P$, then either $x\in P$ or $y\in P$. As the terminology suggests, these domains are closely related to valuation domains. In [7, Prop. 1.1], they showed that every valuation domain is a pseudo-valuation domain. They also showed, in [7, Theorem 2.10], that a *PVD* which is not a valuation domain is characterized as a quasilocal domain (D, M) with the property that $M^{-1}=D:_{K}M$ is a valuation overring with maximal ideal M, where K is the quotient field of D.

If I is an ideal of an integral domain R with quotient field K, then $I:_{K}I=\{x\in K|xI\subseteqq I\}$ is an overring of R. We shall call $I:_{K}I$ the "*conductor overring*" of R with respect to I. In [12], we investigated conductor overrings of a valuation domain. In that paper, we introduced the notion of "*recurrent closure*": If I is an ideal of an integral domain R with quotient field K, then the ideal $R:_{R}(I:_{K}I)$ is called the "*recurrent closure*" of I and is denoted by I_{r}. In [12, Theorem 13], we proved that if I is an ideal of a valuation domain V with quotient field K such that $I:_{K}I\neq V$, then I_{r} is a prime ideal of V and $I:_{K}I=V_{I_{r}}$. An ideal I of an integral domain R is said to be "*recurrent*" in case $I=I_{r}$. We also showed, in [12, Theorem 13], that every nonmaximal prime ideal P of a valuation domain V is recurrent. The main purpose of this paper is to study conductor overrings of a pseudo-valuation domain and to extend some results obtained in [12] to a pseudo-valuation domain.

Throughout this paper, D will be a pseudo-valuation domain with maximal ideal M, and K will denote its quotient field. Any unexplained terminology is standard, as in [5] and [10].

Let R be an integral domain with quotient field K and let $P\subset I$ be ideals of R with P prime. Then we cannot in general expect that P is also prime in $I:_{K}I$, as showed in [12, Example 15]. But we showed in [12, Corollary 16]

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that if $P \subset I$ are ideals of a valuation domain V with P prime, then P is also prime in $I:_K I$, where K is the quotient field of V. We show here that this result is also valid for a PVD.

THEOREM 1. *Let $P \subset I$ be ideals of D. If P is prime in D, then P is also prime in $I:_K I$.*

PROOF. By [11, Corollary 1.5], it suffices to prove that $P = P:_K I$. Since $P \subseteqq P:_K I$ is clear, we need only show that $P:_K I \subseteqq P$. To see this, let $x \in P:_K I$. If we choose an element $t \in I \backslash P$, then we have $xt \in P$. Then, since P is strongly prime (cf. [7, Definition, p. 138]), $xt \in P$ and $t \notin P$ implies that $x \in P$, which shows that $P:_K I \subseteqq P$.

COROLLARY 2. *Let I be an ideal of D and let P be a prime ideal of $I:_K I$. If $P \cap D \subset I$, then P is also a prime ideal of D.*

PROOF. If we set $Q = P \cap D$, then, by hypothesis, Q is properly contained in I and so, by [11, Proposition 1.3 (3)], we have $P = Q:_K I$. But then, by Theorem 1, $Q = Q:_K I$ and consequently $P = Q$, which implies that P is also a prime ideal of D as required.

In [7, Theorem 2.10], Hedstrom and Houston showed that $M^{-1} = D:_K M$ is a valuation overring with maximal ideal M. Since $M^{-1} = M:_K M$ by [9, Proposition 2.3], it then follows that M is the unique maximal ideal of $M:_K M$. In this paper it will be shown that if P is a prime ideal of D, then P is the unique maximal ideal of $P:_K P$.

We first establish the following lemma.

LEMMA 3. *Let P be a prime ideal of D. Then*

(1) *P is also a prime ideal of $P:_K P$.*

(2) *Any proper ideal I of $P:_K P$ is also an ideal of D.*

PROOF. (1) First, it is well known that P is an ideal of $P:_K P$. Then it is easily seen that P is also a prime ideal of $P:_K P$, since P is strongly prime.

(2) Let I be any proper ideal of $P:_K P$. It then suffices to show that $I \subseteqq D$. Assume the converse and choose an element $x \in I \backslash D$. Then, by [7, Proposition 1.2], $x^{-1} \in P:_K P$. Hence $1 = xx^{-1} \in I(P:_K P) = I$, which implies that $I = P:_K P$. But this contradicts our assumption, and consequently $I \subseteqq D$ as we wanted.

THEOREM 4. *If P is a prime ideal of D, then P is the unique maximal ideal of $P:_K P$.*

PROOF. Let I be any proper ideal of $P:_K P$. Then it is sufficient to show that I is contained in P. First, by Lemma 3, I is contained in D. Suppose that $I \not\subseteq P$ and choose an element $s \in I \backslash P$. Then $s/p \in K \backslash D$ for each nonzero $p \in P$. Therefore, by [7, Proposition 1.2], $p/s \in P:_K P$. Then, since P is strongly prime, $s(p/s) \in P$ and $s \notin P$ implies $p/s \in P$ and therefore $p \in sP$. Thus we have $P \subseteq sP \subseteq P$, and consequently $P = sP$. But then, by [12, Lemma 18], s is a unit of $P:_K P$ and so $I = P:_K P$, a contradiction. This completes the proof.

In [12, Theorem 13], we showed that every nonmaximal prime ideal P of a valuation domain V is a recurrent ideal, as stated in Introduction. We can now prove, as an easy consequence of Theorem 4, that this result is also valid for any nonmaximal prime ideal of a PVD.

COROLLARY 5. *If P is a nonmaximal prime ideal of D, then P is a recurrent ideal.*

PROOF. First, by [11, Lemma 1.1], $P_r = D:_D(P:_K P)$ is an ideal of $P:_K P$. Then, by Theorem 4, P_r is contained in P. But, by definition, the converse inclusion $P \subseteq P_r$ is always valid and thus $P = P_r$ as we wanted.

In [12, Theorem 1], we showed that if P is a proper prime ideal of a valuation domain V, then $P:_K P = V_P$ where K is the quotient field of V. We shall next show that this fact is also true for any nonmaximal prime ideal P of a PVD.

We begin by proving the following lemma.

LEMMA 6. *Let R be an integral domain with quotient field K. If P is a prime ideal of R such that R_P is a valuation domain and $PR_P = P$, then we have $P:_K P = R_P$.*

PROOF. First, if we take any element $x \in R \backslash P$, then $p/x \in PR_P = P$ for any $p \in P$, and consequently $p \in xP$. Thus $P \subseteq xP \subseteq P$, and therefore $P = xP$. Then, by [12, Lemma 18], x is a unit of $P:_K P$. Hence $x^{-1} \in P:_K P$ for any $x \in R \backslash P$. Now take any element r/s of R_P with $r \in R$ and $s \in R \backslash P$. Then, by the result shown above, $s^{-1} \in P:_K P$ and accordingly $r/s \in P:_K P$. Therefore we have $R_P \subseteq P:_K P$. Next, we shall show that $P:_K P \subseteq R_P$. Suppose not. Then we can choose an element $t \in P:_K P \backslash R_P$. Since R_P is a valuation domain, $t \notin R_P$ implies that $t^{-1} \in PR_P = P$. Then we get $1 = tt^{-1} \in (P:_K P)P \subseteq P$, a contradiction, whence we must have $P:_K P \subseteq R_P$. Thus our proof is complete.

THEOREM 7. *If P is a nonmaximal prime ideal of D, then $P:_K P = D_P$.*

PROOF. By [7, Proposition 2.6], D_P is a valuation domain. Next, any PVD is a divided ring, as noted in [3, p. 560], and consequently $PD_P=P$. Thus any nonmaximal prime ideal P of D satisfies the two conditions descrived in Lemma 6, and therefore our assertion follows from Lemma 6.

REMARK 8. Following [6], a prime ideal P of an integral domain R is called an "*F-ideal*" if R_P is a valuation domain and $PR_P=P$. Using this terminology, Lemma 6 says that if P is an F-ideal of an integral domain R with quotient field K, then $P:_K P=R_P$. Furthermore, the proof of Theorem 7 is based on the fact that any nonmaximal prime ideal P of a PVD is an F-ideal.

In [11, Corollary 2.5], we showed that if P is a prime ideal of an integral domain R with quotient field K, then $\dim(P:_K P)\geqq \operatorname{rank} P$. The following corollary is an immediate consequence of Theorem 7.

COROLLARY 9. *If P is a nonmaximal prime ideal of D, then we have* $\dim(P:_K P)=\operatorname{rank} P$.

It is well known that if I is an ideal of a valuation domain V, then $\bigcap_{n=1}^{\infty} I^n$ is a prime ideal of V (cf. [5, Theorem (17.1) (3)]) and furthermore if P is a prime ideal of V properly contained in I, then $P\subseteqq \bigcap_{n=1}^{\infty} I^n$ (cf. [5, Theorem (17.1) (4)]). In [7, Proposition 2.4], Hedstrom and Houston showed that if I is an ideal of a PVD, then $\bigcap_{n=1}^{\infty} I^n$ is a prime ideal. By virtue of [7, Theorem 1.4], it is easily proved that [5, Theorem (17.1) (4)] is also valid for a PVD.

PROPOSITION 10. *Let I be a proper ideal of D. If a prime ideal P of D is properly contained in I, then $P\subseteqq \bigcap_{n=1}^{\infty} I^n$.*

PROOF. If not, then $P\not\subseteqq I^m$ for some integer $m>0$. Then, by [7, Theorem 1.4], $MI^m\subseteqq P$. Now, since $P\subset I\subseteqq M$, there is an element $t\in M\backslash P$. Then $tI^m\subseteqq P$ and $t\notin P$ implies that $I^m\subseteqq P$, and accordingly $I\subseteqq P$, a contradiction. This completes our proof.

In [11, Lemma 1.1 (5)], we showed that if I is an ideal of an integral domain R and R' is a proper overring of R, then $I:_R R'$ is an ideal of R and is contained in I. It is natural to ask that if P is a prime ideal of R, does this imply that $P:_R R'$ is a prime ideal of R? In general, $P:_R R'$ need not be a prime ideal of R (Example 12), but in the case R is a PVD, the answer is yes.

THEOREM 11. *Let D' be a proper overring of D and let P be a prime ideal* *of* D. *Then*

(1) $P:_D D'$ *is also a prime ideal of* D *and is contained in* P.

(2) *If* $D' \subseteqq P:_K P$, *then we have* $P:_D D'=P$.

(3) *If* $P:_K P$ *is properly contained in* D', *then* $P:_D D'$ *is properly contained in* P. *Moreover*, $D' \to P:_D D'$ *gives a one-one correspondence between the set of all prime ideals* P' *properly contained in* P *and the set of all overrings* D' *of* D *properly containing* $P:_K P$.

PROOF. (1) By [11, Lemma 1.1 (5)], $P:_D D'$ is an ideal of D and is contained in P. Hence we need only show that $P:_D D'$ is a prime ideal of D. Suppose that $rs \in P:_D D'$, $s \notin P:_D D'$ with $r, s \in D$. Since $s \notin P:_D D'$, $st \notin P$ for some $t \in D'$. But then, we have $(rs)(tD') \subseteqq rsD' \subseteqq P$, since $tD' \subseteqq D'$. Then $(st)(rD') \subseteqq P$ and $st \notin P$ implies that $rD' \subseteqq P$, whence $r \in P:_D D'$. Thus $P:_D D'$ is a prime ideal of D, and our proof is over.

(2) By [11, Lemma 1.1 (6)], we always have $P=P:_D(P:_K P)$. Hence, if $D' \subseteqq P:_K P$, then $P=P:_D(P:_K P) \subseteqq P:_D D' \subseteqq P$, whence $P=P:_D D'$.

(3) If $P:_K P \subset D'$, then there exists an element $x \in D' \setminus P:_K P$. Since $x \notin P:_K P$, we can find an element $p \in P$ such that $xp \notin P$. Then $xp \notin P$ and $x \in D'$ implies that $p \notin P:_D D'$, whence $p \in P \setminus P:_D D'$. Thus $P:_D D' \neq P$ as we wanted. Next, we shall show that if D' is any overring of D properly containing $P:_K P$, then D' is of the form $P':_K P'$ with some prime ideal P' properly contained in P. First, we note that $P:_K P$ is a valuation domain by [7, Proposition 1.2]. Moreover, by Theorem 7, we have $P:_K P=D_P$. Hence, we get $D'=(D_P)_{P'D_P}=D_{P'}$ with some prime ideal P' properly contained in P. Using Theorem 7 again, we have $D'=D_{P'}=P':_K P'$, as we required. Next, we shall show that if $D'=P':_K P'$ with $P' \subset P$, then $P:_D D'=P'$. By [11, Lemma 1.1 (6)], $P'=P':_D(P':_K P')$ and moreover, by Corollary 5, $D:_D(P':_K P')=P'$. Hence it follows that $P'=P':_D(P':_K P')=P':_D D' \subseteqq P:_D D' \subseteqq D:_D(P':_K P')=P'$, whence $P:_D D'=P'$. Conversely, if P' is a prime ideal of D properly contained in P, then, by Theorem 7, $P':_K P'=D_{P'}$ is an overring of D properly containing $P:_K P=D_P$, and furthermore we have $P'=P:_D(P':_K P')$, as shown above. This completes our proof.

EXAMPLE 12. Let $R=k[X^3, X^4] \subset R'=k[X^2, X^3]$, where k is a field and X is an indeterminate over k. Then the quotient field of R is the field $k(X)$ and so R' is an overring of R. Set $P=RX^3+RX^4$, and note that P is a prime ideal of R since $R/P=k$. We claim that $P:_R R'$ is not a prime ideal of R. To see this, first observe that $X^3 \notin P:_R R'$. In fact, $X^3 X^2=X^5 \notin P$. But $X^6 \in P:_R R'$ since $X^6 X^2=(X^4)^2 \in P$ and $X^6 X^3=(X^3)^3 \in P$. Thus we have $X^3 \notin P:_R R'$ and $(X^3)^2 \in$

$P:{}_R R'$, and our claim is established.

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