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# HYPOELLIPTICITY OF SYSTEMS OF PSEUDO DIFFERENTIAL OPERATORS WITH DOUBLE CHARACTERISTICS

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## 0. Introduction.

In this paper, we are concerned with the hypoellipticity of a system of pseudodifferential operators on $\Omega$, an open subset of $R^N$, of the form

$$P(x, D)=I_d p(x, D)+Q(x, D), \tag{0.1}$$

where $I_d$ is the identity $d\times d$ matrix, $p(x, D)$ a scalar pseudodifferential operator of degree $m$, and $Q(x, D)$ a $d\times d$ system of pseudodifferential operators of degree at most $m-1$. We shall assume that its principal symbol $p(x, \xi)$ is non-negative on $T^*\Omega$, the cotangent bundle of $\Omega$ and that it vanishes exactly of order 2 on its characteristic set $\Sigma$, which is assumed to be a symplectic smooth submanifold of $T^*\Omega$. In [3], L. Boutet de Monvel and F. Treves have obtained a necessary and sufficient condition for $P(x, D)$ such as above (in fact, a little more general ones) to be hypoelliptic with loss of one derivative, which is the best hypoellipticity for $P(x, D)$ to have. In case of a scalar situation (i.e. $d=1$), hypoellipticity (and local solvability) of $P(x, D)$ was studied in [6], assuming, in addition, that the codimension of $\Sigma$ in $T^*\Omega$ is 2, in which case the analysis is much simpler than the present case. In this work, we obtain sufficient conditions for $P(x, D)$ to be hypoelliptic, which extend the results in [6] to the vector situation with no restriction on codimension of $\Sigma$. As in [6], we rely heavily on the method of concatenations which was initiated by F. Treves in [8] and turned out to be quite useful in some cases (cf. [2, 3, 5, 6, 8]). After reducing the operator under study into a canonical form near a characteristic point in section 1, we construct, in section 2, a series of operators, called concatenations, by which we can reformulate the condition under which $P(x, D)$ is hypoelliptic with loss of one derivative. Then, in section 3, we state and prove the main results of this paper.

We use $(x, \xi)=(x_1, \cdots, x_N, \xi^1, \cdots, \xi^N)$ for the variable point in $T^*\Omega$ and

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$\omega=\sum_{j=1}^{N} d\xi^j \Lambda dx_j$ will represent the fundamental symplectic two-form on $T^*\Omega$. All pseudodifferential operators in this paper will be classical ones, in the sense that their total symbols are asymptotic sum of homogeneous terms whose homogeneous degrees drop by integers. For a pseudodifferential operator $A$ of degree $\leqq k$, $\sigma(A)$ stands for its principal symbol and $\sigma_k(A)$ its principal symbol as an operator of degree $k$ (hence, if $A$ happens to be of degree$<k$, then $\sigma_k(A)\equiv 0$ while $\sigma(A)$ need not be identically 0). For two pseudodifferential operators $A$ and $B$, $A\sim B$ means that $A-B$ is regularizing and $[A, B]=AB-BA$ their commutation bracket. Other notations used but not stated here will be standard ones of distribution theory and of pseudodifferential operator theory. Functions and distributions here have their values in a large array of finite dimensional vector spaces over the complex numbers, which we do not specify, hoping that it will be clear from the context.

## 1. Canonical form of the operator near a characteristic point.

Let $\Omega$ be an open subset of $R^N$, $N\geqq 1$ and $P(x, D)$ be an operator of the form (0. 1). For $\sigma(P)=p(x, \xi)I_d$, we shall always assume the followings:

(1.1) $\quad p(x, \xi)\geqq 0 \quad$ on $T^*\Omega$;

(1.2) $\quad p(x, \xi)$ vanishes exactly of order 2 on its characteristic set $\Sigma=p^{-1}(0)$;

(1.3) $\quad \Sigma$ is a smooth submanifold of $\dot{T}^*\Omega=T^*\Omega\setminus 0$ and is symplectic, that is, the restriction of $\omega$ to its tangent space is nondegenerate everywhere.

Condition (1.3) requires that the dimension of $\Sigma$ be even and hence so be its codimension in $T^*\Omega$: we shall set

(1.4) $$\operatorname{codim}\Sigma=2n, \quad n\geqq 1 \text{ integer}.$$

For any characteristic point $\rho$, since $\omega$ is non-degenerate on $T_\rho(\Sigma)$, the tangent space of $\Sigma$ at $\rho$, $T_\rho(T^*\Omega)$ can be decomposed as

(1.5) $$T_\rho(T^*\Omega)=T_\rho(\Sigma)\oplus N_\rho(\Sigma),$$

where $N_\rho(\Sigma)$ denotes the orthogonal complement of $T_\rho(\Sigma)$ in $T_\rho(T^*\Omega)$ with respect to $\omega$.

By the assumption (1.2), we can intrinsically define a symmetric bilinear form $q_p(\rho)$ on $T_\rho(T^*\Omega)$ (cf. [7]) by

$$q_p(\rho)(v_1, v_2)=\frac{1}{2}X_1(X_2(p))(\rho), \qquad v_1, v_2\in T_\rho(T^*\Omega),$$

where $X_j$ are some vector fields defined in a neighborhood of $\rho$ with $X_j(\rho)=v_j$, $j=1, 2$.

Then, it is clearly well-defined and positive-definite on $N_\rho(\Sigma)$. Moreover, since $\omega$ is non-degenerate, it induces an endomorphism $A_p(\rho)$ of $T_\rho(T^*\Omega)$, called the Hamilton map of $q_p(\rho)$, defined by

$$q_p(\rho)(v_1, v_2)=\omega(v_1, A_p(\rho)(v_2)), \qquad v_1, v_2\in T_\rho(T^*\Omega). \tag{1.6}$$

If we set $Q_p(\rho)$ to be the quadratic form associated to $q_p(\rho)$, then it is nothing but the quadratic form, which begins the Taylor expansion of $p(x, \xi)$ at $\rho$.

The followings are special cases of the results in [3, Section 3] (cf. also [5, 7]).

Proposition 1.1. *Non-0 eigenvalues of $A_\rho(\rho)$ are $\pm i\lambda_j$, $1\leqq j\leqq n$, $\lambda_j>0$. Hence, when $\rho$ varies over $\Sigma$, $\lambda_j$ are positive real-valued functions on $\Sigma$, which are invariants (i.e. coordinate-free) associated to $P(x, D)$.*

Proposition 1.2. *In a small conic neighborhood $\Gamma$ of any point in $\Sigma$, there exist functions $z_j$, homogeneous of degree 1 and $d_j>0$, homogeneous of degree $m-2$ such that:*

$$p(x, \xi)=\sum_j d_jz_j\bar{z}_j \qquad \text{in } \Gamma; \tag{1.7}$$

$$d_j=\lambda_j \qquad \text{on } \Sigma\cap\Gamma,\ 1\leqq j\leqq n; \tag{1.8}$$

$$\{z_j, z_k\}=\{z_j, \bar{z}_k\}-i\delta_{jk}=0 \qquad \text{in } \Gamma,\ 1\leqq j,\ k\leqq n. \tag{1.9}$$

By a standard method of successive approximations, we can construct $n$ pseudodifferential operators $Z_j$ with their principal symbols $\sigma(Z_j)=z_j$, $1\leqq j\leqq n$, so that they satisfy the commutation relations in $\Gamma$

$$[Z_j, Z_k]\sim 0\sim[Z_j, Z_k^*]-I\delta_{jk}, \qquad 1\leqq j,\ k\leqq n \tag{1.10}$$

where $I$ is the identity operator.

Let $D_j$ be pseudodifferential operators with $d_j$ as their symbols and set

$$Z_0=P(x, D)-I_d(\sum_j D_jZ_j^*Z_j) \qquad \text{in } \Gamma. \tag{1.11}$$

Then $Z_0$ is a $d\times d$ matrix of pseudodifferential operators of degree at most $m-1$ in $\Gamma$. We shall denote by $\sigma_0=\sigma_0(x, \xi)$ the restriction to $\Sigma\cap\Gamma$ of $\sigma_{m-1}(Z_0)$ the principal symbol of $Z_0$ as an operator of degree $m-1$. By a standard symbolic calculus of pseudodifferential operators, we have

$$\sigma_0=\sigma_{sub}(P)+I_d\Big(\frac{1}{2}\sum\lambda_j\Big), \tag{1.12}$$

where $\sigma_{sub}(P)$ is the subprincipal symbol of $P(x, D)$, which is well defined on $\Sigma$, independent of the choice of local coordinates.

## 2. Concatenations and Hypoelliptcity wiith loss of one dervatice.

Throughout this section, we restrict our attention to the conic open set in which we can write $P(x, D)$ as

$$P(x, D)=I_d(\sum_j D_j Z_j^* Z_j)+Z_0 . \tag{2.1}$$

If we set $W_j=D_j Z_j^*$, $1\leqq j\leqq n$, we have in $\Gamma$

$$P(x, D)=I_d(\sum_j W_j \boldsymbol{Z}_j)+\boldsymbol{Z}_0 . \tag{2.2}$$

We shall construct a sequence of operators, of which any two consequtive ones are related by certain relations, called concatenations (cf. [8]), through which hypoellipticity can be transmitted backward and then give the connection between the concatenations and the hypoellipticity with loss of one derivative. The latter means that for any open set $U$ in $\Omega$, any real number $s$, and any distribution $u$ in $U$.

$$P(x, D)u\in H^s_{\text{loc}}(U) \text{ implies } u\in H^{s+m-1}_{\text{loc}}(U) . \tag{2.3}$$

PROPOSITION 2.1. *There are pseudodifferential operators $A_j$ of degree* 0 *and $Q_{jk}$ of degree $m-1$ (both are $d\times d$ systems), $1\leqq j$, $k\leqq n$, such that*

$$(\boldsymbol{Z}_j I_d-A_j)P=P(\boldsymbol{Z}_j I_d-A_j)+\sum_{k=1}^n Q_{jk}(\boldsymbol{Z}_k I_d-A_k) \tag{2.4}$$

*and*

$$\sigma(Q_{jk})=\sigma([\boldsymbol{Z}_j, W_k])I_d . \tag{2.5}$$

PROOF. Here, we temporarily assume that $P$ is a scalar operator, i.e., $d=1$. But, the proof when $P$ is a system remains unchanged except notational complexity. Write $P=\sum_{k=1}^n \{W_k(\boldsymbol{Z}_k-A_k)+W_k A_k\}+\boldsymbol{Z}_0$ and set $M_{jk}=[\boldsymbol{Z}_j-A_j, W_k]-Q_{jk}$. Then, (2.4) is equivalent to

$$\sum\{W_k[\boldsymbol{Z}_j-A_j, \boldsymbol{Z}_k-A_k]+[\boldsymbol{Z}_j-A_j, W_k A_k]+M_{jk}(\boldsymbol{Z}_k-A_k)\} +[\boldsymbol{Z}_j-A_j, \boldsymbol{Z}_0]=0 . \tag{2.6}$$

Note that degree of $M_{jk}$ is at most $m-1$ and we have (2.5) if $M_{jk}$ is of degree

$\leqq m-2$.

We are going to show that asymptotic expansions of $A_j$ and $M_{jk}$ can be determined successively so that they satisfy the required conditions. Let $A_j \sim \sum_{\alpha\geqq 0} A_j^\alpha$ and $M_{jk} \sim \sum_{\alpha\geqq 0} M_{jk}^\alpha$ and for any $\alpha\geqq 0$, $A_j^{(\alpha)}=\sum_{\beta<\alpha} A_j^\beta$ and $M_{jk}^{(\alpha)}=\sum_{\beta<\alpha} M_{jk}^\beta$, where the degree of $A_j^\alpha$ is $-\alpha$ and that of $M_{jk}^\alpha$ is $m-2-\alpha$. If we set

$$R_j^{(\alpha)}=\sum\{W_k[Z_j-A_j^{(\alpha)}, Z_k-A_k^{(\alpha)}]+[Z_j-A_j^{(\alpha)}, W_kA_k^{(\alpha)}]$$
$$+M_{jk}^{(\alpha)}(Z_k-A_k^{(\alpha)})\}+[Z_j-A_j^{(\alpha)}, Z_0],$$

we need to show that $R_j^{(\alpha)}$ is of degree $\leqq m-1-\alpha$, which inductively proves our assertion.

When $\alpha=0$, $R_j^{(0)}=[Z_j, Z_0]$ since $[Z_j, Z_k]=0$ for all $j$, $k$ and hence is of degree $m-1$.

Assume that $R_j^{(\alpha)}$ is of degree$\leqq m-1-\alpha$, for some $\alpha\geqq 0$. Counting degrees of each term in $R_j^{(\alpha+1)}$, we get

$$R_j^{(\alpha+1)}=R_j^{(\alpha)}+\sum[Z_j, W_k]A_k^\alpha-\sum W_k[A_j^\alpha, Z_k]+\sum M_{jk}^\alpha Z_k,$$

modulo operators of degree$\leqq m-2-\alpha$. Thus, we need to have

$$\sigma_{m-1-\alpha}(R_j^{(\alpha)}+\sum[Z_j, W_k]A_k^\alpha-\sum W_k[A_j^\alpha, Z_k]+\sum M_{jk}^\alpha Z_k) \tag{2.10}$$
$$=r_j^{(\alpha)}-i\sum\{z_j, w_k\}a_k^\alpha+i\sum w_k\{a_j^\alpha, z_k\}+\sum m_{jk}^\alpha z_k=0,$$

where we use the corresponding small letters for the principal symbols of operators.

Therefore, the problem is reduced to find smooth functions $a_j^\alpha$, $1\leqq j\leqq n$, which are homogeneous of degree $-\alpha$ so that $r_j^{(\alpha)}-i\sum\{z_j, w_k\}a_k^\alpha+i\sum w_k\{a_j^\alpha, z_k\}$ wanishes of order infinity on $\Sigma$. From now on, we shall omit the superscript $\alpha$ to simplify the notations. If $a_j$ vanishes of order $l$ on $\Sigma$, it can be written as

$$a_j=\sum_{|\alpha+\beta|=l} b_{\alpha,\beta}^j z^\alpha\bar{z}^\beta,$$

(here $\alpha$ and $\beta$ denote the multiindices)

with $b_{\alpha,\beta}^j$ having the suitable homogeneity. Then we have, via (1,9),

$$\sum w_k\{a_j, z_k\}-\sum\{z_j, w_k\}a_k \tag{2.11}$$

$$=\sum_{|\alpha+\beta|=l}\sum_{k=1}^n[w_k(-i\beta_k)b_{\alpha,\beta}^j z^\alpha\bar{z}^{\beta-\langle k\rangle}-ib_{\alpha,\beta}^k\{z_i, w_k\}z^\alpha\bar{z}^\beta]$$

$$=\sum_{|\alpha+\beta|=l}\sum_{k=1}^n[(-i\beta_k)b_{\alpha,\beta}^j d_k-ib_{\alpha,\beta}^k d_k\delta_{jk}]z^\alpha\bar{z}^\beta$$

modulo functions vanishing of order at least $l+1$ on $\Sigma$. Here, $\langle k\rangle$ is the $k$-th unit vector in $R^n$. From the right hand side of the equation (2.11), let us pick up the coefficients of $z^\alpha\bar{z}^\beta$ for any fixed multiindices $\alpha$, $\beta$ with $|\alpha+\beta|=l$. It is

$$(-i)\sum_{k=1}^{n}(\beta_k b^j_{\alpha,\beta}d_k+b^k_{\alpha,\beta}d_k\delta_{jk}).$$

Therefore, the problem is whether we can find $b^k_{\alpha,\beta}$ to satisfy

$$\Big(\sum_{k=1}^{n}\beta_k d_k+d_j\Big)b^j_{\alpha,\beta}=-r^j_{\alpha,\beta}\,, \qquad 1\leqq j\leqq n, \tag{2.12}$$

where $r^j_{\alpha,\beta}$ is the coefficient of $z^\alpha\bar{z}^\beta$ in the Taylor series expansion of $r_j$ on $\Sigma$.

The equation (2.12) is trivially solvable for $b^j_{\alpha,\beta}$ for any given $r^j_{\alpha,\beta}$ since the coefficient $\sum_k\beta_k d_k+d_j$ is positive for $j=1,\cdots,n$. If we set $P_{jk}=\delta_{jk}P+Q_{jk}$, $P^{(1)}=(P_{jk})$ to be an $nd\times nd$ system of operators and $Z^{(0)}$ to be a column vector with $n$ entries, each one of which is $Z_jI_d-A_j$, then (2.4) reads as

$$Z^{(0)}P^{(0)}=P^{(1)}Z^{(0)}\,, \qquad P^{(0)}=P. \tag{2.13}$$

On the other hand, we may write

$$P^{(1)}=I_{nd}(\sum_k W_kZ_k)+Z^{(1)}_0\,, \tag{2.14}$$

where $Z^{(1)}_0$ is an $nd\times nd$ system of operators of degree at most $m-1$. With $P^{(1)}$ instead of $P^{(0)}=P$, we can repeat the same argument as the one in proposition 2.1 and by induction, we can get a sequence of operators $P^{(j)}$, $j\geqq 0$, satisfying

$$Z^{(j)}P^{(j)}=P^{(j+1)}Z^{(j)} \qquad \text{(called (left) concatenations)}, \tag{2.15}$$

$$P^{(j)}=I_{njd}(\sum_k W_kZ_k)+Z^{(j)}_0=\sum_k W_k(Z_k-A^{(j)}_k)+\tilde{Z}^{(j)}_0\,, \tag{2.16}$$

where $Z^{(0)}_0=Z_0$, $A^{(0)}_k=A_k$, and

$$\tilde{Z}^{(j)}_0=Z^{(j)}_0+\sum_k W_kA^{(j)}_k\,. \tag{2.17}$$

Now, let $\sigma^{(j)}_0=\sigma_{m-1}(\tilde{Z}^{(j)}_0)|_{\Sigma\cap\Gamma}=\sigma_{m-1}(Z^{(j)}_0)|_{\Sigma\cap\Gamma}$ to be the restriction to $\Sigma\cap\Gamma$ of the principal symbol of $\tilde{Z}^{(j)}_0$ regarded as an operator of degree $m-1$. From the definition of $P^{(1)}$, it follows immediately that $\sigma^{(1)}_0=\sigma^{(0)}_0\otimes I_n+I_d\otimes A$, where $A=(\lambda_j\delta_{jk})$ and $\otimes$ denotes the direct product of matrices. Similarly, by induction, we have

$$\sigma^{(j)}_0=\sigma^{(j-1)}_0\otimes I_n\times I_{nj-1d}\otimes A\,, \qquad j\geqq 1, \tag{2.18}$$

of which any eigenvalue is a sum of an eigenvalue of $\sigma^{(j-1)}_0$ and an eigenvalue of $A$. Therefore, again by induction on $j$ and using (1.12), we get

$$\text{spec}\,\sigma_0^{(j)}=\text{spec}\,\sigma_{sub}(P)+\frac{1}{2}\sum_{k=1}^{n}\lambda_k+\underbrace{\text{spec}\,A+\cdots+\text{spec}\,A}_{j\text{-time}}, \quad j\geqq 0, \tag{2.19}$$

where the plus signs on the right stand for the addition of complex numbers. Let us consider the following condition on $P(x, D)$:

(2.20) all $\tilde{Z}_0^{(j)}$ (equivalently $Z_0^{(j)}$) are elliptic of order $m-1$ on $\Sigma$ (hence in $\Gamma$ if it is small enough).

In view of (2.19), condition (2.20) is equivalent to:

(2.21) for any point $\rho$ in $\Sigma$, any $r$-tuple of nonnegative integers $r=(r_1, \cdots, r_n)$, and any eigenvalue $\mu$ of $\sigma_{sub}(P)(\rho)$,

$$\mu+\sum_{j=1}^{n}\left(\frac{1}{2}+r_j\right)\lambda_j\neq 0\,.$$

As is now well known (cf. [1, 3]), condition (2.21) is necessary and sufficient for $P(x, D)$ to be hypoelliptic with loss of one derivative (cf. (2.3)), which is sufficient but not necessary for $P(x, D)$ to be hypoelliptic.

## 3. Hypoellipticity.

Although it is not clear how to weaken the condition (2.21) itself to get just hypoellipticity of $P(x, D)$, we can replace it by the equivalent condition (2.20), which can be modified to give some sufficient conditions for $P(x, D)$ to be hypoelliptic. In this section, as in section 2, we restrict ourselves to the sufficiently small conic open subset $\Gamma$ of arbitrary characteristic point of $P(x, D)$, in which we can write $P$ as (2.2) and construct the concatenations (2.15). This is possible since hypoellipticity is purely a local (or rather microlocal) property and out of $\Sigma$, $P(x, D)$ is elliptic. Let us consider the following weakened form of the condition (2.20):

$$\text{all } \tilde{Z}_0^{(j)} \text{ are hypoelliptic.} \tag{3.1}$$

Returning to the equation (2.19), we can see that $\tilde{Z}_0^{(j)}$ will be elliptic of degree $m-1$ for large enough $j$ since any eigenvalue $\lambda_k$ of the matrix $A$ is strictly positive. Therefore, the condition (3.1) involves, in fact, only a finitely many $\tilde{Z}_0^{(j)}$.

LEMMA 3.1. *Suppose $P(x, D)$ satisfies the condition* (3.1). *If, for some $j>0$, $P^{(j+1)}$ is hypoelliptic, then so is $P$.*

PROOF. Let $u$ be a distribution such that $P^{(j)}u$ is $C^\infty$. Then, from (2.6), so is $P^{(j+1)}Z^{(j)}u=Z^{(j)}P^{(j)}u$. Since $P^{(j+1)}$ is hypoelliptic, $Z^{(j)}u$ and so equivalently $(Z_k-A_k^{(j)})u$, $1\leqq k\leqq n$, are $C^\infty$. Then $\tilde{Z}_0^{(j)}u=P^{(j)}u-\sum W_k(Z_k-A_k^{(j)})u$ and so $u$ is $C^\infty$ since $\tilde{Z}_0^{(j)}$ is hypoelliptic by the assumption, which means that $P^{(j)}$ is also hypoelliptic. By repeating the same argument $j$-times, we reach to the conclusion that $P$ is hypoelliptic.

Lemma 3.1 means that under the condition (3.1), hypoellipticity of $P^{(j)}$ can be transmitted backward along the concatenation (2.15). Now, we can give the main results of this paper as follows.

THEOREM 3.1. *If the condition* (3.1) *is satisfied, then* $P(x, D)$ *is hypoelliptic.*

PROOF. By the Lemma 3.1, it suffices to have hypoellipticity of $P^{(j)}$ for some $j\geqq 0$. Reminding that $\tilde{Z}_0^{(j)}$ is elliptic of order $m-1$ for large enough $j$, say, for $j\geqq J$ and that each $P^{(j)}$ has the same principal part as $P$, $P^{(J)}$ satisfies the condition (2.20) and so is hypoelliptic (with loss of one derivative).

REMARK 3.1. Of course, the condition (3.1) is satisfied provided that

(3.2) all $\tilde{Z}_0^{(j)}$ are elliptic of degree$\leqq m-1$ in $\Gamma$.

in which case the construction of concatenations is much simpler than ours (cf. [3, 5]). In that case, (analytic) hypoellipticity of $P(x, D)$ was already treated in [5].

The following special case of the theorem (3.1) is worth to mention since it involves only $\tilde{Z}_0^{(0)}$.

COLOLLARY 3.1. *If* $\tilde{Z}_0^{(0)}$ *is hypoelliptic and is of degree strictly less than* $m-1$, *then* $P(x, D)$ *is hypoelliptic.*

PROOF. By the assumption, $\sigma_0^{(0)}$, a principal symbol of $\tilde{Z}_0^{(0)}$ as an operator of degree $m-1$, must vanish identically on $\Sigma$. Therefore due to (1.12) and (2.19), any eigenvalue of $\sigma_0^{(j)}$, $j\geqq 1$, is of the form $\sum r_i\lambda_j$, where $r=(r_1, \cdots, r_n)\in \boldsymbol{Z}_+^n$, $|r|=r_1+\cdots+r_n=j$ and $\lambda_j$ are positive numbers introduced in proposition 1.1. Hence, 0 can not be an eigenvalue of $\tilde{Z}_0^{(j)}$, $j\geqq 1$, and so they are elliptic of degree $m-1$ on $\Sigma$. Therefore, the condition (3.1) is satisfied.

Careful inspection of the proof of theorem 3.1 reveals that the condition (3.1) is sufficient but not necessary for $P(x, D)$ to be hypoelliptic as the following result shows.

THEOREM 3.2. *$P(x, D)$ is hypoelliptic provided that the characteristic set of* *each* $Z_0^{(j)}$, $j \geqq 0$, *is disjoint from* $\Sigma$.

PROOF. It suffices to have an analog of lemma 3.1 under the given assumption. So, assume that, form some $j \geqq 0$, $P^{(j+1)}$ is hypoelliptic and that $P^{(j)}u$ is $C^\infty$ for a distribution $u$. As in the proof of lemma 3.1, $Z_0^{(j)}u$ is also $C^\infty$. Hence, $u$ must be $C^\infty$ also since its wave-front set is a subset of the intersection of char $\tilde{Z}_0^{(j)}$ and char $P^{(j)}=\Sigma$, which is empty by the assumption. The rest of the proof is the same as that of theorem 3.1.

From the theorem 3.2, it seems to us that that the hypoellipticity of $P(x, D)$ may be determined by the behavior of the principal symbol of $\tilde{Z}_0^{(j)}$ on $\Sigma$. In [6], we have already pushed in this direction when the codimension of $\Sigma$ in $T^*\Omega$ is 2 (i.e. $n=1$) by introuducing boundary operators which live only on $\Sigma$ (cf. section 3 in [6]). However, at the moment, it is not clear how to define the boundary operators in the present case.

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