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CYCLIC-PARALLEL REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

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Abstract. We classify cyclic-parallel real hypersurfaces of quaternionic projective space.

1. Introduction.

Let M be a connected real hypersurface of a quaternionic projective space QP^m, $m \geqq 2$, endowed with the metric of constant quaternionic sectional curvature 4. If ζ denotes the unit local normal vector field and $\{J_1, J_2, J_3\}$ a local basis of the quaternionic structure of QP^m (see [2]), then $U_i = -J_i\zeta$, $i=1, 2, 3$, are vector fields tangent to M.

It is known, [4], that there do not exist parallel real hypersurfaces of QP^m, $m \geqq 2$. A real hypersurface of QP^m is called cyclic-parallel if it satisfies

$$\sigma(g((\nabla_X A)Y, Z))=0 \tag{1.1}$$

for any X, Y, Z tangent to M, where A denotes the Weingarten endomorphism of M and σ the cyclic sum.

Our purpose is to classify such real hypersurfaces by mean of the following

THEOREM. *A real hypersurface M of QP^m, $m \geqq 2$, is cyclic-parallel if and only if it is congruent to an open subset of a tube of radius r, $0<r<\Pi/2$, over QP^k, $k \in \{0, \cdots, m-1\}$.*

In the Theorem, QP^k is considered canonically and totally geodesically embedded in QP^m.

2. Preliminaries.

Let X be a vector field tangent to M. We write $J_iX = \phi_iX + f_i(X)\zeta$, $i=1, 2, 3$, where ϕ_iX denotes the tangent component of J_iX and $f_i(X) = g(X, U_i)$. From this, see [3], we have

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(2.1) $$g(\phi_i X, Y)+g(X, \phi_i Y)=0, \qquad \phi_i U_i=0, \qquad \phi_j U_k=-\phi_k U_j=U_t$$

for any X, Y tangent to M, $i=1, 2, 3$, (j, k, t) being a circular permutation of $(1, 2, 3)$.

From the expression of the curvature tensor of QP^m (see [2]) the equation of Codazzi is given by

(2.2) $$(\nabla_X A)Y-(\nabla_Y A)X=\sum_{i=1}^{3}\{f_i(X)\phi_i Y-f_i(Y)\phi_i X+2g(X, \phi_i Y)U_i\}$$

for any X and Y tangent to M. Moreover

(2.3) $$\nabla_X U_i=q_k(X)U_j-q_j(X)U_k+\phi_i AX$$

for any X tangent to M, (i, j, k) being a circular permutation of $(1, 2, 3)$ and q_i, $i=1, 2, 3$, certain local 1-forms on M (see [2]).

In the following we shall denote by $\mathfrak{D}^{\perp}=\mathrm{Span}\{U_1, U_2, U_3\}$ and by $\mathfrak{D}$ its orthogonal complement in TM.

3. Proof of Theorem.

From (1.1) and applying twice (2.2) we find

(3.1) $$(\nabla_X A)Y=-\sum_{i=1}^{3}\{f_i(Y)\phi_i X+g(Y, \phi_i X)U_i\}.$$

From (3.1), (2.1) and (2.3) we have for any X, Y, Z tangent to M

(3.2) $$\begin{aligned}(\nabla_Z(\nabla_X A))Y-(\nabla_{\nabla_Z X}A)Y=-\sum_{i=1}^{3}\{&g(\phi_i X, Y)\phi_i AZ\\ &+g(Y, \phi_i AZ)\phi_i X+f_i(X)g(AZ, Y)U_i\\ &-2f_i(Y)g(AX, Z)U_i+f_i(Y)f_i(X)AZ\}.\end{aligned}$$

If we exchange Z and X, from (3.2) we obtain

(3.3) $$\begin{aligned}(R(Z, X)A)Y=-\sum_{i=1}^{3}\{&f_i(X)g(AZ, Y)U_i-f_i(Z)g(AX, Y)U_i\\ &+g(\phi_i X, Y)\phi_i AZ-g(\phi_i Z, Y)\phi_i AX+g(\phi_i AZ, Y)\phi_i X\\ &-g(\phi_i AX, Y)\phi_i Z+f_i(Y)f_i(X)AZ-f_i(Y)f_i(Z)AX\}.\end{aligned}$$

Let now $\{E_j, j=1, \cdots, 4m-1\}$ be an orthonormal frame on M. From (3.3) we have

(3.4) $$\begin{aligned}\sum_{j=1}^{4m-1}g((R(E_j, X)A)Y, E_j)=-\sum_{i=1}^{3}\{&f_i(X)f_i(AY)-g(\phi_i X, Y)\,\mathrm{trace}(A\phi_i)\\ &-2f_i(Y)f_i(AX)-g(A\phi_i Y, \phi_i X)\\ &+\mathrm{trace}(A)f_i(Y)f_i(X)\}.\end{aligned}$$

But the left hand side of (3.4) is symmetric respect to X and Y. This gives

(3.5) $\quad -3\sum_{i=1}^{3} f_i(X)f_i(AY) = -3\sum_{i=1}^{3} f_i(Y)f_i(AX) + 2\sum_{i=1}^{3} g(\phi_i Y, X)\,\mathrm{trace}(A\phi_i)$.

From (2.1) it is easy to see that $\mathrm{trace}(A\phi_i)=0$. This implies

(3.6) $$\sum_{i=1}^{3} f_i(X)f_i(AY) = \sum_{i=1}^{3} f_i(Y)f_i(AX)$$

for any X, Y tangent to M. From (3.6) we obtain that $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = \{0\}$. Thus, [1], M is congruent to an open subset of either a tube of radius r, $0<r<\Pi/2$, over QP^k $k\in\{0, \cdots, m-1\}$ or a tube of radius r, $0<r<\Pi/4$ over CP^m.

If we consider the second case, the principal curvatures of M are, [3], $\cot(r)$, $-\tan(r)$, $2\cot(2r)$, $-2\tan(2r)$ with respective multiplicities $2(m-1)$, $2(m-1)$, 1 and 2. If X is a unit vector field such that $AX=-\tan(r)X$, from (2.3) the $\mathfrak{D}$-component of $(\nabla_X A)U_2$ is $-\tan(r)(-2\tan(2r)-\cot(r))\phi_2 X$. But if M is cyclic-parallel from (3.1) $(\nabla_X A)U_2$ must be equal to $-\phi_2 X$. This implies that $\cot^2(r)=-1$ which is impossible.

On the other hand it is easy to prove that the tubes over QP^k are cyclic-parallel and this finishes the proof.

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