TSUKUBA J. MATH.
Vol. 10, No. 2 (1986). 269-283

AUTOMORPHISMS OF FINITE ORDER OF THE AFFINE LIE ALGEBRA $A_l^{(1)}$

By

Zenji KOBAYASHI

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

0. Introduction

We will classify all automorphisms of prime order of the *affine Lie algebra* $A_l^{(1)}$ up to conjugacy in the group of all automorphisms of $A_l^{(1)}$. To do this, we will use non abelian *group cohomology* of some finite cyclic group acting on $PGL_{l+1}(C[t, t^{-1}])$.

The authors wishes to express his appreciation to Professor E. Abe, Professor K. Kato, Mr. Terasoma for their encouragement and help with the present work.

1. Preliminary

Let $\mathcal{G}$ be the *affine Lie algebra* over C of type $A_{n-1}^{(1)}(n\geqq 2)$, *i.e.* the Lie algebra over C generated by e_i, h_i, f_i $(1\leqq i\leqq n)$ with the following defining relations;
for $n>2$

$$[h_i, h_j]=0,\ [e_i, f_j]=\delta_{ij}h_i \quad \text{for all} \quad i, j.$$

$$[h_i, e_j]=\begin{cases} 2e_i & \text{if} \quad i=j, \\ -e_j & \text{if} \quad |i-j|=1 \quad \text{or} \quad n-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$[h_i, f_j]=\begin{cases} -2f_i & \text{if} \quad i=j, \\ f_j & \text{if} \quad |i-j|=1 \quad \text{or} \quad n-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$[e_i, [e_i, e_j]]=[f_i, [f_i, f_j]]=0 \quad \text{if} \quad |i-j|=1 \quad \text{or} \quad n-1,$$

$$[e_i, e_j]=[f_i, f_j]=0 \quad \text{if} \quad |i-j|\neq 1 \quad \text{and} \quad n-1,$$

and for $n=2$

$$[h_i, h_j]=0,\ [e_i, f_j]=\delta_{ij}h_i \quad \text{for all} \quad i, j,$$

$$[h_i, e_j]=\begin{cases} 2e_i & \text{if} \quad i=j, \\ -2e_j & \text{if} \quad i\neq j, \end{cases}$$

$$[h_i, f_j]=\begin{cases} -2f_i & \text{if} \quad i=j, \\ 2f_j & \text{if} \quad i\neq j, \end{cases}$$

$$[e_i, [e_i, [e_i, e_j]]]=[f_i, [f_i, [f_i, f_j]]]=0 \quad \text{if} \quad i\neq j.$$

Received September 25, 1985.

Let E_{ij} be the matrix unit with 1 in the i, j position and 0 elesewhere, and let $C[t, t^{-1}]$ be the ring of Laurent polynomials in t with the coefficients in C.

Then we have a universal central extension over C:

$$(1) \qquad 0 \longrightarrow Cz \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{sl}_n(C[t, t^{-1}]) \longrightarrow 0,$$

where $z=h_1+h_2+\cdots+h_n$, $\pi(e_i)=E_{i,i+1}$, $\pi(e_n)=tE_{n1}$, $\pi(h_i)=E_{ii}-E_{i+1,i+1}$, $\pi(h_n)=E_{11}-E_{nn}$, $\pi(f_i)=E_{i+1,}$ and $\pi(f_n)=t^{-1}E_{1n}$ for all $i=1, \cdots, n-1$.

By the universality of (1) and Theorem 2 [4],

$$(2) \qquad \mathrm{Aut}(\mathfrak{g}) \simeq \mathrm{Aut}_C(\mathfrak{sl}_n(C[t, t^{-1}])) \simeq \begin{cases} \mathrm{Aut}_{C\text{-}alg}(C[t, t^{-1}]) \ltimes PGL_2(C[t, t^{-1}]) & \text{when } n=2 \\ (\langle\tau\rangle \times \mathrm{Aut}_{C\text{-}alg}(C[t, t^{-1}])) \ltimes PGL_n(C[t, t^{-1}]) & \text{when } n \geqq 3, \end{cases}$$

where τ is the involutive automorphism induced by the Dynkin diagram automorphism of A_{n-1}. More precisely, τ is defined by $\tau(e_i)=-e_{n-i}$ and $\tau(f_i)=-f_{n-i}$ for $i=1,\cdots, n-1$, $\tau(e_n)=-e_n$ and $\tau(f_n)=-f_n$. $PGL_n(C[t, t^{-1}])$ acts on $\mathfrak{sl}_n(C[t, t^{-1}])$ by conjugation (cf. [3]).

To classify all the automorphisms of finite order of $\mathcal{G}$ up to conjugacy is equivalent to classifying the elements of finite order of $Aut_C(\mathfrak{sl}_n(C[t, t^{-1}]))$ up to the conjugacy. Put $G=Aut_C(\mathfrak{sl}_n(C[t, t^{-1}]))$, $G_1=\langle\tau\rangle \times Aut_{C\text{-alg}}(C[t, t^{-1}])$ and $G_2=PGL_n(C[t, t^{-1}])$. Then elements of order k ($k \geqq 2$) of $G=G_1 \quad G_2$ have the expression g_1g_2 ($g_i \in G_i$), where $g_1^k=1$ and $\Pi_{j=0}^{k-1} g_1^{-j} g_2 g_1^j=1$ and if g_1 is conjugate to g_1' in G then g_1g_2 is conjugate in G to an element which has the expression $g_1'g_2'$ for some $g_2' \in G_2$.

2. Group cohomology

From now on we will let $R=C[t, t^{-1}]$. Let θP be an element of $Aut_C(\mathfrak{sl}_n(R))$, where $\theta \in \langle\tau\rangle \times Aut_{C\text{-}alg}(R)$ and $P \in PGL_n(R)$. θP is of prime order k if and only if $\theta^k=1$ and $(\theta^{k-1}\cdot P)\cdots(\theta\cdot P)P=I_n$ (in $PGL_n(R)$), where $\theta\,\cdot$ notes the action of θ on $PGL_n(R)$. Let θP_1 and θP_2 be elements of order k. θP_1 is conjugate to θP_2 under $PGL_n(R)$ *i.e.* $\theta P_1=Q^{-1}(\theta P_2)Q$ for some $Q \in PGL_n(R)$, if and only if $P_1=(\theta\cdot Q^{-1})P_2Q$ for some $Q \in PGL_n(R)$. The condition $(\theta^{k-1}\cdot P)\cdots(\theta\cdot P)P=I_n$ (resp. $P_1=(\theta\cdot Q^{-1})P_2Q$) coincides with the *cocycle condition* (resp. the *coboundary condition*) of the *group cohomology* $H^1(\boldsymbol{Z}_k, PGL_n(R))$ under the action of θ (=a *generator* of $\boldsymbol{Z}_k$) on $PGL_n(R)$.

Let σ (resp. ε_2) be the automorphism of R induced by $t \longmapsto t^{-1}$ (resp. $t \longmapsto -t$), then the set $\{\sigma, \varepsilon_2, \tau, \tau\sigma, \tau\varepsilon_2\}$ is a set of representatives of the conjugacy classes of order 2 of $\langle\tau\rangle \times Aut_{C\text{-}alg}(R)$. Let ε_k be the automorphism of R induced by $t \longmapsto \zeta_k t$ (ζ_k*=fixed k-th primitive root of unity*), then the set $\{\varepsilon_k, (\varepsilon_k)^2, \cdots, (\varepsilon_k)^{(k-1)/2}\}$ is a set of representatives of the conjugacy classes of odd prime order k of $\langle\tau\rangle \times Aut_{C\text{-}alg}(R)$.

If the following section, we will determine some *cohomologies* $H^1(\boldsymbol{Z}_k, PGL_n(R))$ in the following situation:

(1) $k=2$

(a) trivial action

(b) σ-action
(c) ε_2-action
(d) τ-action
(e) $\tau\sigma$-action
(f) $\tau\varepsilon_2$-action.

(2) $k \geqq 3$
(a) trivial action
(b) ε_k-action

For the rest of this section, we will determine $H^1(\boldsymbol{Z}_k, GL_n(R))$ with the above actions. We begin with the case when $k=2$.

(1) $k=2$
(a) *trivial action.*

PROPOSITION 1.
Under the trivial action on $\boldsymbol{Z}_2$ *on* $GL_n(R)$,

$$H^1(\boldsymbol{Z}_2, GL_n(R)) = \left\{ I_{a,b} = \begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \middle| a+\mathrm{b}=\mathrm{n} \right\}$$

where I_k *is the* $k \times \mathrm{k}$ unit matrix.

PROOF.
In this case, the cocycle condition is $P^2 = I_n P \in GL_n(R)$ and the coboundary condition is $P_1 \sim P_2 \Leftrightarrow P_1 = Q^{-1} P_2 Q$ for some $Q \in GL_n(R)$. Let M be a free R module of rank n, and let ϕ be a R module endomorphism satisfying $\phi^2 = identity$. We define free R modules M_1 and M_2 as follows $M_1 = \{m \in M \mid \phi(m) = m\}$, $M_2 = \{m \in M \mid \phi(m) = -m\}$. Then $M = M_1 \oplus M_2$ (*direct sum*) because $(1/2) \in R$. q.e.d.

(b) *σ-action.*

PROPOSITION 2.
Under the σ-action of $\boldsymbol{Z}_2$ on $GL_n(R)$,

$$H^1(\boldsymbol{Z}_2, GL_n(R)) = \left\{ J_{a,b,c} = \begin{bmatrix} I_a & & \\ & tI_b & \\ & & -I_c \end{bmatrix}, J'_{a,b,c} = \begin{bmatrix} I_a & & \\ & -tI_b & \\ & & -I_c \end{bmatrix} \middle| a+b+c=n \right\}.$$

In this case, the cocycle condition is $(\sigma \cdot P)P = I_n$, $P \in GL_n(R)$ and the coboundary condition is $P_1 \sim P_2 \Leftrightarrow P_1 = (\sigma \cdot Q) P_2 Q^{-1}$ for some $Q \in GL_n(R)$. We first prove several lemmas.

LEMMA 1.
If P satisfies the cocycle condition and $P(1) = I_n$, $P(-1) = I_n$ *($P(a)$ is the specialization of*

$P \in \mathcal{L}_n(R)$ at $a \in C^{\times}$), then $P=(\sigma \cdot Q)Q^{-1}$ for some $Q \in GL_n(R)$.

PROOF.

Let R^{σ} be the invariant subring of σ. Then R is a finite projective *R^{σ}-algebra.* Using Grothendieck's "*theory of descent*", we can prove $P=(\sigma \cdot Q)Q^{-1}$ for some $Q \in GL_n(R)$ (cf. Theorem 4.3[1]).

LEMMA 2.

If P satisfies the cocycle condition, then there exists $Q \in GL_n(R)$ s.t.

(1) *Q satisfies the cocycle condition*

(2) *Q is cohomologous to P*

(3) $Q(1)=\begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}$, $Q(-1)=\begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}$

for some a, b, c, d.

PROOF.

Since $P(1)^2=I_n$, there exists $\alpha \in GL_n(C)$ s.t.

$$\alpha P(1)\alpha^{-1}=\begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}.$$

We may assume:

$$P(1)=\begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}.$$

Let $\beta \in SL_n(C)$ s.t.

$$\beta P(-1)\beta^{-1}=\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix},$$

then there exists $Z \in SL_n(R)$ s.t. $Z(1)=I_n$, $Z(-1)=\beta$. Set $Q=ZPZ^{-1}$, then

$$Q(1)=\begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \text{ and } Q(-1)=\begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}. \text{ q.e.d.}$$

We will call (a, b, c, d) the *invariant* of P. If P_1 is cohomologous to P_2, then P_1, P_2 have the same *invariant.*

LEMMA 3.

If P_1, P_2 satisfy the cocycle condition and have the same invariant (a, b, c, d), then P_1 is cohomologous to P_2.

PROOF.

We will only prove this when $a>$c. The cases $a=$c and $a<$c are similar.

$$I_{c,p,b}=\begin{bmatrix} I_c & & \\ & tI_p & \\ & & -I_b \end{bmatrix},$$

then the *invariant* of $I_{c,p,b}$ is (a, b, c, d). From the above lemma, We may assume

$$P_1(1)=\begin{bmatrix} I_a & \\ & -I_b \end{bmatrix} \quad \text{and} \quad P_1(-1)=\begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}.$$

We will prove P_1 is cohomologous to $I_{c,p,b}$.

Set

$$\Gamma=\left\{P\in GL_n(R)\,|\,P(1)=\begin{bmatrix} I_a & \\ & -I_b \end{bmatrix}, P(-1)=\begin{bmatrix} I_c & \\ & -I_d \end{bmatrix}\right\}$$

and

$$U=\begin{bmatrix} I_c & & \\ & \frac{1}{1+t}I_p & \\ & & \frac{1-t}{1+t}I_b \end{bmatrix}.$$

Then:

(1) $U^{-1}(\sigma\cdot U)=\begin{bmatrix} I_c & & \\ & tI_p & \\ & & -I_b \end{bmatrix}$

(2) $(\sigma\cdot U)\Gamma U^{-1}\in GL_n(R)$.

Let $P\in\Gamma$, then

$$((\sigma\cdot U)PU^{-1})\,(1)=I_n$$
$$((\sigma\cdot U)PU^{-1})\,(-1)=I_n.$$

Since P_1 satisfies the cocycle condition, $\sigma\cdot((\sigma\cdot U)P_1U^{-1})((\sigma\cdot U)P_1U^{-1})=I_n$.
From the above lemma, $(\sigma\cdot U)P_1U^{-1}=(\sigma\cdot Q)Q^{-1}$ for some $Q\in GL_n(R)$.
By the following lemma, we may assume $Q(1)=I_n$. and $Q(-1)=\lambda I_n$ for some $\lambda\in C^\times$.
Then $U^{-1}Q(\sigma\cdot U)\in GL_n(R)$ and

$$\begin{aligned}P_1&=(\sigma\cdot U)^{-1}(\sigma\cdot Q)Q^{-1}U\\ &=\sigma\cdot(U^{-1}Q(\sigma\cdot U))U^{-1}(\sigma\cdot U)(U^{-1}Q(\sigma\cdot U))^{-1}.\end{aligned}$$

Since

$$U^{-1}(\sigma\cdot U)=\begin{bmatrix} I_c & & \\ & tI_p & \\ & & -I_b \end{bmatrix},$$

P_1 is cohomologous to $I_{c,p,b}$.

LEMMA 4.

Let U, Q be as above. There exists $Q_1\in GL_n(R^\sigma)$ s.t. $QQ_1(1)=I_n$ *and* $QQ_1(-1)=\lambda I$ *for some* $\lambda\in C^\times$, *where* R^σ *is the σ-invariant subring of R. In particular,*

$$\begin{aligned}(\sigma\cdot(QQ_1))(QQ_1)^{-1}&=(\sigma\cdot Q)(\sigma\cdot Q_1)Q_1^{-1}Q^{-1}\\ &=(\sigma\cdot Q)Q^{-1}\\ &=(\sigma\cdot U)P_1U^{-1}.\end{aligned}$$

PROOF.

Set $\gamma=Q(1)^{-1}\in GL_n(C)$. Let $\lambda\in C^\times$ *s.t. det* $(Q(-1)\gamma)=\lambda^n$ and $A\in SL_n(R^\sigma)$ s.t. $A(-1)=Q(-1)\cdot\lambda^{-1}I_n\gamma$ and $A(1)=I_n$. Set $Q_1=\gamma A^{-1}$, then $Q(1)Q_1(1)=I_n$ and $Q(-1)Q_1(-1)=\lambda I_n$.

PROOF OF PROP. 2.

From the above lemmas, $\{J_{a,b,c}, J'_{a,b,c}\,|\,a+b+c=n\}$ is a set of representatives of $H^1(\boldsymbol{Z}_2, GL_n(R))$. q.e.d.

(c) *ε_2-action*

PROPOSITION 3.

Under the ε_2-action of $\boldsymbol{Z}_2$ on $GL_n(R)$, $H^1(\boldsymbol{Z}_2, GL_n(R))=\{I_n\}$.

PROOF.

Let R^{ε_2} be the invariant subring of ε_2. Then $R^{\varepsilon_2}\subset R$ is a galois extension (cf. P. 44 [1]). Using Grothendieck's "*theory of descent*", we can prove $H^1(\boldsymbol{Z}_2, GL_n(R))=\{I_n\}$ (cf. Theorem 5.1 [1]).

(d) *τ-action.*

PROPOSITION 4.

Under the τ-action of $\boldsymbol{Z}_2$ on $GL_n(R)$,

$$H^1(\boldsymbol{Z}_2, GL_n(R))=\left\{K_1=\begin{bmatrix} & & 1\\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix},\ K_2=\begin{bmatrix} & & 1\\ & 1\cdot^{\cdot^{\cdot}} & \\ t & & \end{bmatrix}\right\}.$$

Let $P=(p_{ij})\in GL_n(R)$, then $\tau\cdot P=(p_{n+1-j,n+1-i})^{-1}$. In this case, the cocycle condition is $(p_i)=(p_{n+1-j,n+1-i})$, $P=(p_{ij})\in GL_n(R)$ and the coboundary condition is $P_1\sim P_2 \Leftrightarrow P_1=(q_{n+1-j,n+1-i})P_2Q$ for some $Q=(q_{ij})\in GL_n(R)$. When we put $\tilde{P}=K_1P$, these conditions are equivalent to $\tilde{P}={}^t\tilde{P}$ and $\tilde{P}_1\sim\tilde{P}_2 \Leftrightarrow \tilde{P}_1={}^tQ\tilde{P}_2Q$ for some $Q\in GL_n(R)$, where ${}^tP$ is the transposed matrix of P.

We first prove several lemmas.

LEMMA 5.

Let $K=\boldsymbol{C}(t)$ and let $X\in \boldsymbol{GL}_n(K)$ s.t. ${}^tX=X$.

Let ψ_X be the bilinear form on K^n defined by $\psi_X(x, y)={}^txXy$ for $x, y\in K^n$. Then

(1) *When n is even, the dimension of a maximal anisotropic subspace is $n/2$ or $(n/2)-1$.*

(2) *When n is odd, the dimension of a maximal anisotropic subspace is $(n-1)/2$, where an anisotropic subspace is a subspace V of K^n s.t. $\psi_X(x, y)=0$ for all $x, y\in V$.*

PROOF.

Let V be a maximal anisotropic subspace, and let $V^\perp=\{x\in K^n \mid \psi_X(x, v)=0 \ \textit{for all}\ v\in V\}$. ψ_X defines a non-degenerate symmetric bilinear form $\tilde{\psi}_X$ on $V^\perp/V$. It is sufficient to prove $dim_K V^\perp/V\leqq 2$. Assume $dim_K V^\perp/V\geqq 3$ and take a three dimensional subspace W of $V^\perp/V$. Since K is a C_1*-field*, there exists an element $w\in W$ *s.t.* $\tilde{\psi}_X(w, w)=0$. Then $V\oplus wK$ is anisotropic, contradicting the maximality of V.

LEMMA 6.

Let $X\in GL_n(R)$ s.t. ${}^tX=X$, and let ψ_X satisfy one of the following conditions:

(1) *n=odd,*

(2) *n=even and the dimension of every maximal anisotropic subspace of K^n is $(n/2)-1$. Let V be a maximal anisotropic subspace of K^n, and let $W=V\cap R^n$. Put $W^\perp=V^\perp\cap R^n$. Then:*

(1) *The exact sequence $0\to W\to W^\perp\to W^\perp/W\to 0$ is split.*

(2) *Let U be the image of a spliting of (1). Then $R^n=U\oplus U^\perp$ and the dimension of a maximal anisotropic subspace of $U\otimes_R K$ is $(n/2)-1$ (resp. $(n-1)/2$) when n is even (resp. odd).*

PROOF.

(1) R^n/W is embedded in K^n/V and is torsion free. Then $W^\perp/W$ is torsion free and free since R is a principal ideal domain.

(2) $W^\perp=U\oplus W$ and $R^n\supset U$. Let x be a maximal ideal of R, and let $k(x)$ be the residual field. Then $\tilde{\psi}_X|(W^\perp/W)\otimes_R k(x)$ is non-degenerate and $\psi_X|U\otimes_R k(x)$ is also non-degenerate. Then $R^n=U\oplus U^\perp$ (cf. P. 5 [2]).

LEMMA 7.

Let $X\in \mathrm{GL}_n(R)$ s.t. ${}^tX=X$, and let n be even. Assume there exists a maximal anisotropic subspace V of K^n whose dimension is $n/2$. Then there exists a free basis $\{e_1,\cdots, e_n\}$ *of R^n s.t.*

$$(\psi_X(e_i, e_j))=\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

PROOF.

Put $W=V\cap R^n$, $W^{\perp}=V^{\perp}\cap R^n$. Let $x\in W$ and let H be a submodule of W *s.t.* $W=xR\oplus H$. Then $H^{\perp}\supset W^{\perp}$ and $H^{\perp}/W^{\perp}$ is torsion free. Let s be a *spliting* of the exact sequence $0\to W^{\perp}\to H^{\perp}\to H^{\perp}/W^{\perp}\to 0$, *and let* $y\in s(H^{\perp}/W^{\perp})$ *s.t.* $yR\oplus W^{\perp}=H^{\perp}$.

We define the map $f\ R\to R$ by $f\ (r)=\psi_X(x, ry)$. For every maximal ideal $\mathfrak{p}$ of R, there exists an element $r\in R$ *s.t.* $\psi_X(x, ry)\neq 0$ modulo $\mathfrak{p}$. Since the image of f is an ideal of R, f is surjective.

Then, there exists $z\in s(H^{\perp}/H)$ *s.t.* $\psi_X(x, z)=1$, $\psi_X(x, x)=0$. There exists $r\in R$ *s.t.* $\psi_X(x, rz)=\psi_X(z, z)$. Let $u=z-rx/2$, then $\psi_X(u, u)=0$, $\psi_H(u, x)=1$ and $H^{\perp}/H=xR\oplus uR$. Since $\tilde{\psi}_X|(H_{\perp}/H)$ is non-degenerate, $R^n=(H^{\perp}/H)\oplus(H^{\perp}/H)^{\perp}$ and $(H^{\perp}/H)^{\perp}\supset H$. By induction on n, we can prove this lemma.

LEMMA 8.

Let $X\in \mathrm{GL}_2(R)$ *s.t.* ${}^tX=X$.

(1) *If* $\sqrt{\det X}\in R$, *then there exists a basis* $\{e_1, e_2\}$ *of* R^2 *s.t.*

$$(\psi_X(e_i, e_j))=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(2) *If* $\sqrt{\det X}\notin R$, *then there exists a basis* $\{e_1, e_2\}$ *of* R^2 s.t.

$$(\psi_X(e_i, e_j))=\begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}.$$

PROOF.

(1) If $\sqrt{\det X}\in R$, then there exists an element $v\in K^2$ *s.t.* $\psi_X(v, v)=0$. Then there exists an element $u\in R^2$ *s.t.* $\psi_X(u, u)=0$. We can prove (1) by Lemma 7.

(2) Let $\tilde{R}=C[u, u^{-1}]$ *s.t.* $u^2=t$. Then $\tilde{R}\supset R$ and $\sqrt{\det X}\in\tilde{R}$. There exists $v\in\tilde{R}^2$ *s.t.* $\psi_X(v, v)=0$. Let δ be the automorphism of $\tilde{R}$ defined by $\delta(u)=-u$, then $\tilde{R}^{\delta}=R$. Let $e_1=v+\delta(v)$ and $e_2=uv-u\delta(v)$, then $e_i\in R^2$ and $\psi_X(e_1, e_2)=0$. q.e.d.

PROOF OF PROP. 4.

From the above lemmas, under the transposing action of $\boldsymbol{Z}_2$ on $GL_n(R)$,

$$H^1(\boldsymbol{Z}_2, GL_n(R))=\left\{I_n,\ \begin{bmatrix} t & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\right\}.$$

Then $\{K_1, K_2\}$ is a representative set of $H^1(\boldsymbol{Z}_2, GL_n(R))$ under the *τ-action* of $\boldsymbol{Z}_2$. q.e.d.

(e) *τσ-action.*

PROPOSITION 5.

Under the τσ-action of $\boldsymbol{Z}_2$ *on* $GL_n(R)$, $H^1(\boldsymbol{Z}_2, GL_n(R))=\{K_1\}$.

(f) *τε₂-action.*

PROPOSITION 6.

Under the $\tau\varepsilon_2$-action of $\boldsymbol{Z}_2$ on $GL_n(R)$, $H^1(\boldsymbol{Z}_2, GL_n(R))=\{K_1\}$.

Proofs of Prop. 5 and Prop. 6 are analogous to that of Prop. 4.

(2) $k\geqq 3$

(a) *trivial action.*

PROPOSITION 7.

Under the trivial action of $\boldsymbol{Z}_k$ on $GL_n(R)$,

$$H^1(\boldsymbol{Z}_k, GL_n(R))=\left\{ I_{a_0},\cdots,{}_{a_{k-1}}=\begin{bmatrix} I_{a_0} & & & \\ & \zeta_k\cdot I_{a_1} & & \\ & & \ddots & \\ & & & \zeta_k^{k-1}\cdot I_{a_{k-1}} \end{bmatrix} \;\middle|\; \sum_{m=0}^{k-1} a_m=n \right\}.$$

PROOF.

The proof is analogous to that of Prop. 1.

(b) *ε_k-action.*

PROPOSITION 8.

Under the ε_k-action of $\boldsymbol{Z}_k$ on $GL_n(R)$, $H^1(\boldsymbol{Z}_k, GL_n(R))=\{I_n\}$.

PROOF.

Proof is analogous to that of Prop. 3.

3. Determination of $H^1(\boldsymbol{Z}_k, PGL_n(R))$.

In this section, we will determine $H^1(\boldsymbol{Z}_k, PGL_n(R))$ with the previously mentioned actions.

THEOREM 1.

$H^1(\boldsymbol{Z}_k, PGL_n(R))$ is:

(1) $k=2$

(a) *trivial action.*

(i) $\{I_{a,b}\,|\,a+b=n,\ a\geqq b\}$ $\quad(n=odd)$.

(ii) $$\left\{ I_{a,b}(a\geqq b),\ \begin{bmatrix} 0 & t^{-1} & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & t^{-1} \\ & & & 1 & 0 \end{bmatrix} \right\} \quad (n=even).$$

(b) *σ-action.* $\quad\{J_{a,b,c}, J'_{d,e,f}\,|\,a\geqq b+c,\ d\geqq e+f,\ d\neq 0\}$

(c) *ε_2-action.* $\{I_n\}$.

(d) *τ-action.*

(i) $\{K_1, K_2\}$ $(n=odd)$.

$$\text{(ii)}\quad \left\{K_1, K_2, K_3=\begin{bmatrix} & & & -1\,0 \\ & & & 0\,1 \\ & & \cdot^{\cdot^{\cdot}} & \\ -1\,0 & & & \\ 0\,1 & & & \end{bmatrix}\right\} \quad (n=even).$$

(e) *$\tau\sigma$-action.*

(i) $\{K_1\}$ $(n=odd)$.

$$\text{(ii)}\quad \left\{K_1, K_3, K_4=\begin{bmatrix} & & & 1\,0 \\ & & & 0\,t^{-1} \\ & & \cdot^{\cdot^{\cdot}} & \\ 1\,0 & & & \\ 0\,t^{-1} & & & \end{bmatrix},\ K_5=\begin{bmatrix} & & & -1\,0 \\ & & & 0\,t^{-1} \\ & & \cdot^{\cdot^{\cdot}} & \\ -1\,0 & & & \\ 0\,t^{-1} & & & \end{bmatrix}\right\} \quad (n=even).$$

(f) *$\tau\varepsilon_2$-action.* $\{K_1\}$.

(2) $k\geqq 3$

(a) *trivial action.*

(i) *When k is not a divisor of n, $\{I_{a_0}, \cdots, a_{k-1} \mid \sum_{m=0}^{k-1} a_m=n$, $(a_0, \cdots, a_{k-1})$ runs a set of representatives of the equivalence relation generated by $(a_0, \cdots, a_{k-1})\sim(a'_0, \cdots, a'_{k-1})$ $\Leftrightarrow a'_0=a_1, \cdots, a'_{k-2}=a_{k-1}, a'_{k-1}=a_0.\}$.*

(ii) *When k is a divisor of n, $\left\{I_{a_0}, \cdots, a_{k-1}, \begin{bmatrix} L & & \\ & \ddots & \\ & & L\end{bmatrix} \;\middle|\; (a_0, \cdots, a_{k-1})\right.$: same condition as*

$$\text{(i)},\ L=\begin{bmatrix} & & & t^{-j} \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}:\ \ k\times k\text{-matrix and } j=1, \cdots, (k-1)/2.\}.$$

(b) *ε_k-action.* $\{I_n\}$.

PROOF.

(1) (a) and (b) are trivial.

(c) Since the sequence

$$1 \to R^{\times} \to GL_n(R) \to PGL_n(R) \to 1$$

is exact, the sequence

$$H^1(\mathbf{Z}_2, GL_n(R)) \to H^1(\mathbf{Z}_2, PGL_n(R)) \to H^2(\mathbf{Z}_2, R^\times)$$

is eact (cf. P. 125 [5]). Therefore $H^1(\mathbf{Z}_2, PGL_n(R))=\{I_n\}$ follows from $H^2(\mathbf{Z}_2, R^\times)=\{1\}$.

(d) In this case, the cocycle condition is

$$(p_{ij})=\pm(p_{n+1-j,n+1-i}), \quad P=(p_{ij})\in GL_n(R).$$

When we put $\tilde{P}=K_1P$, then this condition is equivalent to ${}^t\tilde{P}=\pm\tilde{P}$. If ${}^t\tilde{P}=-\tilde{P}$, then $\psi_{\tilde{P}}$ is a skew-symmetric bilinear form. Since R is a principal ideal domain, $\psi_{\tilde{P}}$ has a basis $\{e_1,\cdots, e_n\}$ *s.t.*

$$(\psi_{\tilde{P}}(e_i, e_j))=\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \quad \text{(cf. p.7 [2]).}$$

(e) This follows from the exact sequence:

$$H^1(\mathbf{Z}_2, GL_n(R)) \to H^1(\mathbf{Z}_2, PGL_n(R)) \to H^2(\mathbf{Z}_2, R^\times)$$

and

$$|H^2(\mathbf{Z}_2, R^\times)|=\begin{cases} 1 & n=odd \\ 4 & n=even \end{cases}.$$

(f) This follows from the above exact sequence and $H^2(\mathbf{Z}_2, R^\times)=\{1\}$.

(2) (a), (b) Proofs are analogous to those of (1) (a), (c).

4. Automorphisms of finite order of $A_{n-1}^{(1)}$

In this section, we will describe automorphisms of prime order of $A_{n-1}^{(1)}$ using the generator e_i, f_i $(1\leqq i\leqq n)$. Proofs follow from the results of the preceeding section. Put

$$E_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_j}=[\cdots[e_i, e_{i+1}], e_{i+2}]\cdots], e_j],$$
$$F_{\alpha_i+\alpha_{i+1}+\cdots+\alpha_j}=[\cdots[f_j, f_{j-1}], f_{j-2}]\cdots], f_i] \quad (i<j),$$
$$E_{\alpha_0}=E_{\alpha_1+\cdots+\alpha_{n-1}} \text{ and } F_{\alpha_0}=F_{\alpha_1+\cdots+\alpha_{n-1}}.$$

THEOREM 2.

When $n\geqq 3$, a complete set of representatives of automorphisms of prime order k up to conjugacy in $Aut(\mathcal{G})$ is the following.

$k=2$

(a) *For* $a=[\frac{n+1}{2}], \cdots, n-1$,

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i\neq a, n,$$
$$e_a \longmapsto -e_a, \qquad f_a \longmapsto -f_a,$$
$$e_n \longmapsto -e_n, \qquad f_n \longmapsto -f_n.$$

(a′) *When n is even,*

$$e_1 \longmapsto [[e_n, E_{\alpha_0}], f_1],$$
$$f_1 \longmapsto [[f_n, F_{\alpha_0}], e_1],$$
$$e_2 \longmapsto [[f_n, F_{\alpha_0}], E_{\alpha_1+\alpha_2+\alpha_3}],$$
$$f_2 \longmapsto [[e_n, E_{\alpha_0}], F_{\alpha_1+\alpha_2+\alpha_3}],$$

. . . . .

(b) *For* $a=[\frac{n+1}{2}], \cdots, n-1, a\geqq b\geqq 1$ *and* $a+b\leqq n-1$,

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i\neq a, a+b, n,$$
$$e_a \longmapsto [[[f_n, F_{\alpha_1+\cdots+\alpha_a}], F_{\alpha_1+\cdots+\alpha_{n-1}}], e_a],$$
$$f_a \longmapsto -[[[e_n, E_{\alpha_1+\cdots+\alpha_a}], E_{\alpha_{a+1}\cdots+\alpha_{n-1}}], f_a],$$
$$e_{a+b} \longmapsto -[[[e_n, E_{\alpha_1+\cdots+\alpha_{a+b}}], E_{\alpha_{a+b+1}+\cdots+\alpha_{n-1}}], e_{a+b}]$$
$$f_{a+b} \longmapsto [[[f_n, F_{\alpha_1+\cdots+\alpha_{a+b}}], F_{\alpha_{a+b+1}+\cdots+\alpha_{n-1}}], f_{a+b}],$$
$$e_n \longmapsto \frac{1}{2}[[f_n, F_{\alpha_0}], F_{\alpha_0}],$$
$$f_n \longmapsto \frac{1}{2}[[e_n, E_{\alpha_0}], E_{\alpha_0}].$$

(b′) *For* $a=[\frac{n+1}{2}], \cdots, n-1$,

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i\neq a, n,$$
$$e_a \longmapsto [[[f_n, F_{\alpha_1+\cdots+\alpha_a}], F_{\alpha_{a+1}+\cdots+\alpha_{n-1}}], e_a],$$
$$f_a \longmapsto -[[[e_n, E_{\alpha_1+\cdots+\alpha_a}], E_{\alpha_{a+1}+\cdots+\alpha_{n-1}}], f_a],$$
$$e_n \longmapsto F_{\alpha_0}, \qquad f_n \longmapsto F_{\alpha_0}.$$

(b″) *For* $a=[\frac{n+1}{2}], \cdots, n-1$,

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i\neq a, n,$$
$$e_a \longmapsto -e_a, \qquad f_a \longmapsto -f_a,$$
$$e_n \longmapsto \frac{1}{2}[[f_n, F_{\alpha_0}], F_{\alpha_0}],$$
$$f_n \longmapsto \frac{1}{2}[[e_n, E_{\alpha_0}], E_{\alpha_0}].$$

(b‴)

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i\neq n,$$
$$e_n \longmapsto -\frac{1}{2}[[f_n, F_{\alpha_0}], F_{\alpha_0}],$$

$$f_n \longmapsto -\frac{1}{2}[[e_n, E_{\alpha_0}], E_{\alpha_0}].$$

(c)

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i \neq n,$$
$$e_n \longmapsto -e_n, \qquad f_n \longmapsto -f_n.$$

(d)

$$e_i \longmapsto -f_i, \qquad f_i \longmapsto -e_i \quad i \neq n,$$
$$e_n \longmapsto \frac{1}{2}[[e_n, E_{\alpha_0}], E_{\alpha_0}],$$
$$f_n \longmapsto \frac{1}{2}[[f_n, F_{\alpha_0}], F_{\alpha_0}].$$

(d′)

$$e_i \longmapsto -f_i, \qquad f_i \longmapsto -e_i \quad i \neq n-1, n,$$
$$e_{n-1} \longmapsto -[[e_n, E_{\alpha_0}], f_{n-1}],$$
$$f_{n-1} \longmapsto -[[f_n, F_{\alpha_0}], e_{n-1}],$$
$$e_n \longmapsto -E_{\alpha_0}, \qquad f_n \longmapsto -F_{\alpha_0}.$$

(d″) *When n is even.*

$$e_1 \longmapsto e_1, \qquad f_1 \longmapsto f_1,$$
$$e_2 \longmapsto F_{\alpha_1+\alpha_2+\alpha_3}, \qquad f_2 \longmapsto E_{\alpha_1+\alpha_2+\alpha_3},$$

.....

$$e_n \longmapsto -[[e_n, E_{\alpha_1+\cdots+\alpha_{n-1}}], E_{\alpha_2+\cdots+\alpha_{n-1}}],$$
$$f_n \longmapsto -[[f_n, F_{\alpha_1+\cdots+\alpha_{n-1}}], F_{\alpha_2+\cdots+\alpha_{n-1}}].$$

(e)

$$e_i \longmapsto -f_i, \qquad f_i \longmapsto -e_i.$$

(e′) *When n is even,*

$$e_1 \longmapsto e_1, \qquad f_1 \longmapsto f_1,$$
$$e_2 \longmapsto F_{\alpha_1+\alpha_2+\alpha_3}, \qquad f_2 \longmapsto E_{\alpha_1+\alpha_2+\alpha_3},$$

.....

$$e_n \longmapsto -[[f_n, f_1], f_{n-1}],$$
$$f_n \longmapsto -[[e_n, e_1], e_{n-1}].$$

(e″) *When n is even,*

$$e_1 \longmapsto [[e_n, E_{\alpha_0}], e_1],$$
$$f_1 \longmapsto [[f_n, F_{\alpha_0}], f_1],$$
$$e_2 \longmapsto [[f_n, F_{\alpha_0}], [[f_1, f_2], f_3]],$$
$$f_2 \longmapsto [[e_n, E_{\alpha_0}], [[e_1, e_2], e_3]],$$

.....

$$e_n \longmapsto \frac{1}{2}[[[[f_n, E_{\alpha_0}], f_n], f_1], f_{n-1}],$$
$$f_n \longmapsto \frac{1}{2}[[[[e_n, F_{\alpha_0}], e_n], e_1], e_{n-1}].$$

(f)

$$e_i \longmapsto -f_i, \qquad f_i \longmapsto -e_i \quad i \neq n,$$

$$e_n \longmapsto -\frac{1}{2}[[e_n, E_{\alpha_0}], E_{\alpha_0}],$$

$$f_n \longmapsto -\frac{1}{2}[[f_n, F_{\alpha_0}], F_{\alpha_0}].$$

(2) $k \geqq 3$ *Put* $\zeta = \zeta_k$,

(a) *For* $a_0, \cdots, a_{k-1}$ *s.t.* $\sum_{m=0}^{k-1} a_m = n$, $(a_0, \cdots, a_{k-1})$ *runs a set of representatives of the equivalence relation generated by* $(a_0, \cdots, a_{k-1}) \sim (a'_0, \cdots, a'_{k-1}) \Leftrightarrow a'_0 = a_1, \cdots, a'_{k-2} = a_{k-1}$, $a'_{k-1} = a_0$,

$$e_i \longmapsto e_i, \qquad f_i \longmapsto f_i \quad i \neq a_m, n,$$

$$e_{a_m} \longmapsto \zeta e_{a_m}, \qquad f_{a_m} \longmapsto \zeta^{-1} f_{a_m},$$

$$e_n \longmapsto \zeta^{a_{m_0}} \cdot e_n, \qquad f_n \longmapsto \zeta^{-a_{m_0}} \cdot f_n,$$

where $m_0 = \max\{m \mid a_m \neq 0\}$.

(a′) *When k is a divisor of n, for* $j = 1, \cdots, \frac{k-1}{2}$,

$$e_1 \longmapsto ad([e_n, E_{\alpha_0}])]^j F_{\alpha_1 + \cdots + \alpha_{k-1}},$$

$$f_1 \longmapsto ad([f_n, F_{\alpha_0}])^j E_{\alpha_1 + \cdots + \alpha_{k-1}},$$

$$e_2 \longmapsto e_1, \qquad f_2 \longmapsto f_1, \qquad e_3 \longmapsto e_2, \qquad f_3 \longmapsto f_2,$$

$\cdots\cdots$

(b) *For* $j = 1, \cdots, \frac{k-1}{2}$,

$$e_i \longmapsto e_i, \qquad f_1 \longmapsto f_i \quad i \neq n,$$

$$e_n \longmapsto \zeta^j e_n, \qquad f_n \longmapsto \zeta^{-j} f_n.$$

THEOREM 3.

When $n = 2$, *a complete set of representatives of automorphisms of prime order k up to conjugacy in* $Aut(\mathcal{G})$ *is the following.*

(1) $k = 2$

(a)

$$e_1 \longmapsto -e_1, \quad f_1 \longmapsto -f_1, \quad e_2 \longmapsto -e_2, \quad f_2 \longmapsto -f_2.$$

(a′)

$$e_1 \longmapsto e_2, \quad f_1 \longmapsto f_2, \quad e_2 \longmapsto e_1, \quad f_2 \longmapsto f_1.$$

(b)

$$e_1 \longmapsto e_1, \quad f_1 \longmapsto f_1, \quad e_2 \longmapsto -\frac{1}{2}[[f_2, f_1], f_1], \quad f_2 \longmapsto -\frac{1}{2}[[e_2, e_1], e_1].$$

(b′)

$$e_1 \longmapsto -e_1, \quad f_1 \longmapsto -f_1, \quad e_2 \longmapsto \frac{1}{2}[[f_2, f_1], f_1], \quad f_2 \longmapsto \frac{1}{2}[[e_2, e_1], e_1].$$

(b″)

$$e_1 \longmapsto f_2, \quad f_1 \longmapsto e_2, \quad e_2 \longmapsto f_1, \quad f_2 \longmapsto e_1.$$

(c)

$$e_1 \longmapsto e_1, \quad f_1 \longmapsto f_1, \quad e_2 \longmapsto -e_2, \quad f_2 \longmapsto -f_2.$$

(2) $k \geqq 3$ *Put* $\zeta = \zeta_k$

(a) For $a = 1, \cdots, \frac{k-1}{2}$,

$$e_1 \longmapsto \zeta^a e_1, \quad f_1 \longmapsto \zeta^{-a} f_1, \quad e_2 \longmapsto \zeta^a e_2, \quad f_2 \longmapsto \zeta^{-a} f_2.$$

(b) *For* $b = 1, \cdots, \frac{k-1}{2}$.

$$e_1 \longmapsto e_1, \quad f_1 \longmapsto f_1, \quad e_2 \longmapsto \zeta^b e_2, \quad f_2 \longmapsto \zeta^{-b} f_2.$$

References

[1] Knus, M.-A. et Ojanguren, M., Théorie de la Descente et Algébres d'Azumaya. Springer Lecture Notes, **389**, 1974.

[2] Millnor, J. and Husemollor, D., Symmetric bilinear forms. Springer-Verlag, 1973.

[3] Morita, J., Conjugate classes of three dimensional simple Lie subalgebra of the affine Lie algebra $A_l^{(1)}$. Algebraic and Topological Theories. Kinokuniya, 1985.

[4] Peterson, D.H. and Kac, V.G., Infinite flag varieties and conjugacy theorems. Proc. Natl. Acad. Sci. USA **80** (1983), 1778–1782.

[5] Serre, J.-P., Local fields. GTM **67** Springer-Verlag, 1979.

Institute of Mathematics
University of Tsukuba
Sakura-mura, Niihari-gun
Ibaraki, 305 Japan