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SCATTERING THEORY FOR WAVE EQUATIONS WITH LONG-RANGE PERTURBATIONS

By

Hirokazu IWASHITA

Introduction.

In this paper we are concerned with the existence and completeness of modified wave operators for the wave equation with long-range perturbations:

$$\partial_t^2 u+Lu=0,\ L=-\sum_{j,k=1}^{n}\partial_{x_j}a^{jk}(x)\partial_{x_k}+V(x)\ \text{ in } \boldsymbol{R}^n(n\geq 3).$$

Scattering theory for the Schrödinger operators $-\Delta+V$ with long-range perturbations has been extensively investigated and already reached a satisfactory stage, while few have been known about long-range scattering for classical wave equations. It is well known (cf., e.g., Reed-Simon [17] and Mochizuki [14]) that the Schrödinger and classical wave equations are related by the invariance principle of Kato and Birman theory in short-range scattering and it has been expected that also in long-range scattering the invariance principle allows us to treat classical wave equations.

In the present paper we first prove the invariance principle for modified wave operators intertwining L and $-\Delta$ which is applicable to the wave equation. As for the invariance principle in long-range scattering, several authors have studied it for modified wave operators intertwining $-\Delta+V$ and $-\Delta$ which are known to exist (cf., e.g., Matveev [11], Chandler-Gibson [2] and Kitada [9]). Our approach is quite different from those of the above authors, however similar to that of Mochizuki [14]. We employ a spectral representation theory to justify the invariance principle directly, which means, with no knowledge of the existence of time dependent modified wave operators for the Schrödinger operator L. This method is influenced by Ikebe-Isozaki [4]. However an L^2-estimate of an integral operator plays a crucial role in place of the stationary phase method (see Proposition 4.4). The invariance principle assures the existence and completeness of modified wave operators for the wave equation in the square integrable space, from which we construct modified wave operators in the energy spaces by modifying the results of Reed-Simon [17], based upon two-Hilbert-space scattering theory of Kato [8].

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§ 1. Assumptions and statement of the main results.

Consider the following Schrödinger operators in $\boldsymbol{R}^n (n \geq 2)$:

$$L_0 = -\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2,$$

$$L_1 = -\Delta_A + V(x) = -\sum_{j,k=1}^{n} \partial_{x_j}(a^{jk}(x)\partial_{x_k}\cdot) + V(x),$$

where $\partial_{x_j} = \partial/\partial_{x_j}$ and the coefficients $a^{jk}(x)$ and $V(x)$ are supposed to satisfy the following

ASSUMPTION 1.1. (A) The real symmetric matrix $A(x) = (a^{jk}(x))$ is a C^∞ and everywhere positive function of x over $\boldsymbol{R}^n$ such that for some positive constant $\delta < 1$,

$$|\partial_x^\alpha(a^{jk}(x) - \delta^{jk})| \leq C_\alpha(1+|x|)^{-|\alpha|-\delta} \tag{1.1}$$

for any non-negative multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $1 \leq j,\ k \leq n$, where δ^{jk} is Kronecker's delta and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

(V) $V(x)$ is a non-negative $C^\infty(\boldsymbol{R}^n)$-function such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha(1+|x|)^{-|\alpha|-\delta}$$

for any non-negative multi-index α, where δ is the same constant as in (A).

Under Assumption 1.1 the formal differential operators $L_j (j=0,\ 1)$ have the unique selfadjoint realizations in $L^2(\boldsymbol{R}^n)$, which will be denoted by L_j again. Then note that L_j have no point spectrum.

Let $\phi(\lambda)$ be a real $C^\infty(\boldsymbol{R}_+)(\boldsymbol{R}_+ = (0,\ \infty))$ function such that $\phi'(\lambda) = (d\phi/d\lambda)(\lambda) > 0$ for $\lambda \in \boldsymbol{R}_+$. In order to formulate our main results we require time dependent modifiers $X_{\phi,\pm}(\xi,\ t)$ associated with L_1 and $\phi(\lambda)$, which are real $C^\infty((\boldsymbol{R}^n \backslash \{0\}) \times \boldsymbol{R}_+)$-functions possessing the following properties: Given any compact set B in $\boldsymbol{R}^n \backslash \{0\}$, we can find a positive constant T such that if $\xi \in B$, $t > T$ and $|\alpha| \geq 0$, then

$$|\partial_\xi^\alpha X_{\phi,\pm}(\xi,\ \mathrm{t})| \leq C_\alpha(1+t)^{1-\delta}, \tag{1.2}$$

where δ is given as in Assumption 1.1 and the positive constant C_α is independent of $\xi \in B$, and the functions $W_{\phi,\pm}(\xi,\ t) = \pm t\phi(|\xi|^2) + X_{\phi,\pm}(\xi,\ t)$ solve the equations

$$\partial_t W_{\phi,\pm}(\xi,\ t) = \pm\phi(\xi \cdot A(\nabla_\xi W_{\phi,\pm}(\xi,\ t))\xi + V(\nabla_\xi W_{\phi,\pm}(\xi,\ t)))$$

for $\xi \in B$ and $t > T$, respectively, where $\nabla_\xi = {}^t(\partial_{\xi_1}, \cdots, \partial_{\xi_n})$.

DEFINITION 1.2. For time dependent modifiers $X_{\phi,\pm}(\xi,\ t)$, define

$$e^{-iX_{\phi,\pm}(t)}u = F_0^{-1}[e^{-iX_{\phi,\pm}(\cdot,t)}F_0 u] \text{ for } u \in L^2(\boldsymbol{R}^n),$$

where F_0 is the Fourier transform:

$$(F_0u)(\xi)=\hat{u}(\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n} e^{-ix\cdot\xi}u(x)dx.$$

THEOREM 1.3 (*Invariance principle*). *Let $\phi(\lambda)$ be a real-valued $C^\infty(\boldsymbol{R}_+)$-function such that $\phi'(\lambda)>0$ for any $\lambda\in\boldsymbol{R}_+$. Under Assumption* 1.1 *there exist time dependent modifiers $X_{\phi,\pm}(\xi, t)$ such that the modified wave operators*

$$\Omega_\pm(\phi(L_1), \phi(L_0))=\text{s-}\lim_{t\to\infty} e^{\pm it\phi(L_1)}e^{\mp it\phi(L_0)-iX_{\phi,\pm}(t)} \text{ in } L^2(\boldsymbol{R}^n) \tag{1.3}$$

exist and are unitary from $L^2(\boldsymbol{R}^n)$ onto $L^2(\boldsymbol{R}^n)$ with the intertwining property. Furthermore $\Omega_\pm(\phi(L_1), \phi(L_0))$ are independent of ϕ, that is

$$\Omega_\pm(\phi(L_1), \phi(L_0))=\Omega_\pm(L_1, L_0),$$

where $\Omega_\pm(L_1, L_0)$ are obtained from (1.3) *when $\phi(\lambda)=\lambda$.*

REMARK. The condition on $\phi(\lambda)$ in Theorem 1.3 is weaker than that of Matveev [11] or Kitada [9] where it is assumed in addition to ours that $\phi''(\lambda)\neq 0$ on $\boldsymbol{R}_+$.

We now consider the following wave equations in $\boldsymbol{R}^n (n\geq 3)$: For $j=0$, 1,

$$\partial_t^2u+H_j^2u=0, \tag{1.4$_j$}$$

where each H_j is the positive square root of L_j: $H_j=\sqrt{L_j}$. Since $\phi(\lambda)=\sqrt{\lambda}$ satisfies the condition in Theorem 1.3, we have

THEOREM 1.4. *Under Assumption* 1.1 *there exist time dependent modifiers $X_\pm(\xi, t)$ such that the modified wave operators intertwining H_1 and H_0,*

$$\Omega_\pm=\text{s-}\lim_{t\to\infty} e^{\pm itH_1}e^{\mp itH_0-iX_\pm(t)} \text{ in } L^2(\boldsymbol{R}^n)$$

exist and are complete.

On the basis of Theorem 1.4 we consider the wave equations in the energy spaces along the ideas of Reed-Simon [17]. Let $j=0$ or 1. Let $[\mathcal{D}(H_j)]$ be the closure of $\mathcal{D}(H_j)$, the domain of H_j in the norm $\|H_j\cdot\|$, where $\|\cdot\|$ denotes the norm in $L^2(\boldsymbol{R}^n)$. Let $\mathcal{H}_j$ be the Hilbert space defined by

$$\mathcal{H}_j=[\mathcal{D}(H_j)]\oplus L^2(\boldsymbol{R}^n)$$

equipped with norm

$$\left\|\begin{pmatrix}u\\v\end{pmatrix}\right\|_{\mathcal{H}_j}^2=\|H_ju\|^2+\|v\|^2,$$

and define

$$A_j=i\begin{pmatrix}0 & I\\ -H_j^2 & 0\end{pmatrix}, \quad \mathcal{D}(A_j)=\mathcal{D}(H_j^2)\oplus\mathcal{D}(H_j),$$

where

$$\mathcal{D}(H_j^2)=\{u\in[\mathcal{D}(H_j)]\,;\ H_j u\in\mathcal{D}(H_j)\}$$

and we are denoting both H_j and its extension to $[\mathcal{D}(H_j)]$ by H_j. Then $(1.4)_j$ is written in the vector form:

$$(1.5)_j \qquad \partial_t f=-iA_j f.$$

The operator A_j is selfadjoint in $\mathcal{H}_j$ and generates a unitary group of the solution operator $U_j(t)$:

$$U_j(t)=e^{-itA_j}=\begin{pmatrix}\cos(H_j t) & H_j^{-1}\sin(H_j t)\\ -H_j\sin(H_j t) & \cos(H_j t)\end{pmatrix}.$$

Define

$$T_j=\frac{1}{\sqrt{2}}\begin{pmatrix}H_j & i\\ H_j & -i\end{pmatrix}.$$

Then T_j is a unitary operator from $\mathcal{H}_j$ to $L^2(\boldsymbol{R}^n)\oplus L^2(\boldsymbol{R}^n)$ and satisfies

$$T_j A_j T_j^{-1}=\begin{pmatrix}H_j & 0\\ 0 & -H_j\end{pmatrix}$$

and thus

$$T_j U_j(t) T_j^{-1}=\begin{pmatrix}e^{-itH_j} & 0\\ 0 & e^{itH_j}\end{pmatrix}.$$

Let J be the identification operator between $\mathcal{H}_0$ and $\mathcal{H}_1$ defined by

$$J=T_1^{-1}T_0.$$

Let $J_j^{\pm}(t)$ $(t>0)$ be the modified identification operators between $\mathcal{H}_0$ and $\mathcal{H}_j$ defined by

$$J_j^{\pm}(t)=T_j^{-1}\begin{pmatrix}e^{-iX_{\pm}(t)} & 0\\ 0 & e^{-iX_{\mp}(t)}\end{pmatrix}T_0,$$

where $e^{-iX_{\pm}(t)}$ are given in Theorem 1.4. Then $J_0^{\pm}(t)$ and $J_1^{\pm}(t)$ are unitary and related by

$$J_1^{\pm}(t)=JJ_0^{\pm}(t).$$

Further it follows directly from the definitions that $J_0^{\pm}(t)$ commute with $U_0(t)$. We are now in a position to state

THEOREM 1.5. *Suppose that Assumption* 1.1 *is fulfilled. Let* $U_j(t)$ $(j=0, 1)$ *and* $J_1^{\pm}(t)$ *be as above. Then the modified generalized wave operators*

$$W_{\pm}(J)=\text{s-}\lim_{t\to\infty} U_1(\mp t)J_1^{\pm}(t)U_0(\pm t)$$

exist on $\mathcal{H}_0$ *to* $\mathcal{H}_1$ *and are complete. Furthermore* $W_{\pm}(J)$ *are isometries intertwining* $U_0(t)$ *and* $U_1(t)$.

PROOF. Since

$$\begin{aligned} &U_1(\mp t)J_1^{\pm}(t)U_0(\pm t)\\ &=T_1^{-1}(T_1U_1(\mp t)T_1^{-1})(T_1J_1^{\pm}(t)T_0^{-1})(T_0U_0(\pm t)T_0^{-1})T_0\\ &=T_1^{-1}\begin{pmatrix} e^{\pm itH_1}e^{\mp itH_0-iX_{\pm}(t)} & 0\\ 0 & e^{\mp itH_1}e^{\pm itH_0-iX_{\mp}(t)}\end{pmatrix}T_0\,, \end{aligned}$$

the assertion of the theorem follows from Theorem 1.4 and the unitarity of T_j $(j=0, 1)$. Q.E.D.

We now restrict our consideration to the perturbed equation $(1.5)_1$ with the coefficients on which we impose the following

ASSUMPTION 1.6. $a^{jk}(x)(1\leq j,\ k\leq n)$ satisfy (A) in Assumption 1.1 and $V(x)$ satisfies

$(V)'$ $V(x)$ is a non-negative $C^0(\boldsymbol{R}^n)$-function such that

$$V(x)=O(|x|^{-2}) \text{ as } |x|\to\infty\,. \tag{1.6}$$

We remark that the hypothesis $V(x)\in C^0(\boldsymbol{R}^n)$ is put for the sake of simplicity and $V(x)$ may have certain local singularities (see, e.g., Phillips [16]). It will be easily seen that the same results as in Theorems 1.4 and 1.5 hold with the time dependent modifiers $X_{\pm}(\xi, t)$ solving the equations

$$\partial_t X_{\pm}(\xi,\ t)=\mp\{|\xi|-\sqrt{\xi\cdot A(\pm t\xi/|\xi|+\nabla_\xi X_{\pm}(\xi,\ t))\xi}\}\,.$$

We use the same notation as before.

THEOREM 1.7. *Suppose that Assumption* 1.6 *is satisfied. Then* $\mathcal{H}_0$ *and* $\mathcal{H}_1$ *are setwise equal with equivalent norms, and the modified wave operators*

$$W_{\pm}=\text{s-}\lim_{t\to\infty} U_1(\mp t)J_0^{\pm}(t)U_0(\pm t)$$

exist and coincide with $W_{\pm}(J)$. *Hence* $W_{\pm}$ *are complete and intertwine* $U_0(t)$ *and* $U_1(t)$.

By the intertwining property and unitarity of $W_{\pm}$, we have the following result partially extending Phillips [16].

COROLLARY 1.8. *Let* $\mathcal{D}_-^0$ *and* $\mathcal{D}_+^0$ *be the incoming and outgoing subspaces relative to* $U_0(t)$ *on* $\mathcal{H}_0$ *defined by*

$$\mathcal{D}_\pm^0=\{f\in\mathcal{H}_0;\ U_0(t)f=0 \ for\ |x|<\pm t,\ \pm t>0\}$$

(see Lax-Phillips [10], p. 99). *Define*

$$\mathcal{D}_\pm^1=W_\pm\mathcal{D}_\pm^0 .$$

Then $\mathcal{D}_-^1$ *and* $\mathcal{D}_+^1$ *are the incoming and outgoing subspaces relative to* $U_1(t)$.

§ 2. Spectral representations for L_1.

In this section we shall establish the spectral representations for $L=L_1$ under Assumption 1.1. We start with

LEMMA 2.1. *There exists a real* $C^\infty(\boldsymbol{R}^n\times(\boldsymbol{R}\backslash\{0\}))$*-function* $K(x, \sigma)$ *satisfying the following requirements*:

(1) *For any compact set* Σ *in* $\boldsymbol{R}\backslash\{0\}$ *there exists a constant* $R>0$ *such that if* $\sigma\in\Sigma$ *and* $|x|\geq R$, *then*

$$(\nabla_x K)(x, \sigma)\cdot A(x)(\nabla_x K)(x, \sigma)+V(x)=\sigma^2 . \tag{2.1}$$

(2) $K(x, \sigma)=-K(x, -\sigma)$ *for* $\sigma<0$.

(3) $|\partial_x^\alpha\partial_\sigma^k(K(x, \sigma)-\sigma|x|)|\leq C_{\alpha k}(1+|x|)^{1-|\alpha|-\delta}(|x|\geq 1)$

for any non-negative multi-index α, *non-negative integer* k *and* $\sigma\in\Sigma$, *a compact set in* $\boldsymbol{R}\backslash\{0\}$, *where the constant* $C_{\alpha k}$ *is independent of* $\sigma\in\Sigma$.

This lemma can be proved in a similar method as in Theorem I. 16 of Isozaki [5] and so we may omit the proof.

As for the function $K(x, \sigma)$ introduced in Lemma 2.1, we shall find it convenient to rewrite $K(x, \sigma)$ as follows:

$$K(x, \sigma)=\sigma r+Y(x, \sigma) \text{ for } r=|x|\geq 1 . \tag{2.2}$$

Then Lemma 2.1 implies that

$$|\partial_x^\alpha\partial_\sigma^k Y(x, \sigma)|\leq C_{\alpha k}(1+|x|)^{1-|\alpha|-\delta} \tag{2.3}$$

for any $|\alpha|\geq 0$ and integer $k\geq 0$, where $C_{\alpha k}$ is independent of σ in a compact set in $\boldsymbol{R}\backslash\{0\}$.

DEFINITION 2.2. Let $Y(x, \sigma)$ be as above. Define a C^∞-function $\rho(x, \kappa)$ of $x\in\boldsymbol{R}^n$ and $\kappa=\sigma+i\tau$ with $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $\tau\in\boldsymbol{R}$ by

$$\rho(x, \kappa)=-i(\kappa r+Y(x, \sigma))+\frac{n-1}{2}\log r \text{ for } r=|x|\geq 1 .$$

PROPOSITION 2.3. *Let $\rho(x, \kappa)$ be as in Definition 2.2. Then for any $\kappa=\sigma+i\tau$ with $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $\tau\in[0, 1]$, we have*

$$\kappa^2-V(x)+(\nabla_x\rho)(x, \kappa)\cdot A(x)(\nabla_x\rho)(x, \kappa)-\Delta_A\rho(x, \kappa)$$
$$=O(|x|^{-2})+p(x, \kappa)+\tau q(x, \kappa)$$

as $|x|\to\infty$, where the functions $p(x, \kappa)$ and $q(x, \kappa)$ satisfy when $|x|\to\infty$ and $|\alpha|=0$, 1

$$\partial_x^\alpha p(x, \kappa)=O(|x|^{-1-|\alpha|-\delta}),$$
$$\partial_x^\alpha q(x, \kappa)=O(|x|^{-|\alpha|-\delta})$$

uniformly in $\sigma\in\Sigma$, a compact set in $\boldsymbol{R}\backslash\{0\}$ and τ.

PROOF. A straightforward calculation yields the following identities:

$$\kappa^2-V+(\nabla_x\rho)\cdot A(\nabla_x\rho)-\Delta_A\rho$$
$$=-\{(\nabla_xK)\cdot A(\nabla_xK)+V-\sigma^2\}+\frac{(n-1)(n+3)}{4r^2}\Phi-\frac{n-1}{2r^2}\mathrm{Trace}(A)+$$
$$-\frac{n-1}{2r}\sum\tilde{x}_k\partial_{x_j}a^{jk}+p(x, \kappa)+\tau q(x, \kappa);$$

$$p(x, \kappa)=i\kappa\frac{1}{r}(\mathrm{Trace}(A)-n\Phi)-i\sum(\partial_{x_j}a^{jk})\{\kappa\tilde{x}_k+\partial_{x_k}Y\}+$$
$$+i\sum a^{jk}\partial_{x_j}\partial_{x_k}Y,$$
$$q(x, \kappa)=\tau(\Phi-1)-2i\sigma(\Phi-1)-2i\tilde{x}\cdot A\nabla_xY,$$

where $\tilde{x}=(\tilde{x}_1, \cdots, \tilde{x}_n)=x/|x|$ and $\Phi=\Phi(x)=\tilde{x}\cdot A(x)\tilde{x}$. Hence the assertion is easily deduced from (2.1), (2.3) and the above identities. Q.E.D.

We introduce the weighted L^2-spaces to state the radiation condition. For a real number s, $L_s^2(G)$ denotes the Hilbert space of all measurable functions u such that

$$||u||_{s,G}^2=\int_G(1+|x|)^{2s}|u(x)|^2dx$$

is finite. If $s=0$ or $\boldsymbol{G}=\boldsymbol{R}^n$, we often omit the corresponding subscript.

DEFINITION 2.4. Let $\rho(x, \kappa)$ be as in Definition 2.2. A solution of the equation

$$(2.4)\qquad (L-\kappa^2)u=f$$

is said to satisfy the radiation condition if

$$u\in L^2_{(-1-\mu)/2} \text{ and } \tilde{x}\cdot A(x)(\nabla_x+(\nabla_x\rho)(x,\ \kappa))u\in L^2_{(-1+\nu)/2}(\Omega)\,,$$

where $\Omega=\{x\in \boldsymbol{R}^n;\ |x|\geq 1\}$ and μ, ν are positive constants satisfying $\mu\leq\nu$ and $\mu+\nu\leq 2$.

PROPOSITION 2.5. *Let $\tau>0$ and $0<\mu\leq\nu$, $\mu+\nu<2$. Let u be a solution of* (2.4) *with $f\in L^2_{(1+\nu)/2}$, satisfying the radiation condition. Then there exist positive constants C and R independent of κ, f and u such that*

$$\|(\nabla_x+(\nabla_x\rho)(\cdot\ ,\ \kappa))u\|^2_{(-1+\nu)/2,E(R)}\leq C\{\|u\|^2_{(-1-\mu)/2}+\|f\|^2_{(1+\nu)/2}\}\,,$$

where $E(R)=\{x\in\boldsymbol{R}^n;\ |x|\geq R\}$.

SKETCH OF THE PROOF. Putting $\theta=(\nabla_x+(\nabla_x\rho)(x,\ \kappa))u$ and $\eta=\mathrm{Im}\ \nabla_x\rho(x,\ \kappa)/\mathrm{Im}\ \partial_r\rho(x,\ \kappa)$ with $\partial_r=\tilde{x}\cdot\nabla_x$, we have (2.3) of [6] with $-\Delta_A\rho$ and β replaced by $-\Delta_A\rho-V$ and ν, respectively, which we denote by (2.3)′. Similarly the identity (2.4) of [6] holds with $-\Delta_A\rho$, $p(x,\ \kappa)$ and β replaced by $-\Delta_A\rho-V$, $p(x,\ \kappa)+\tau q(x,\ \kappa)$ given in Proposition 2.3 and ν, respectively. We denote this identity by (2.4)′. The integrals containing $p(x,\ \kappa)$ or its derivatives in (2.4)′ can be estimated in the same way as in [6]. The integrals containing $\tau q(x,\ \kappa)$ or its derivatives except for the term $\tau q(x,\ \kappa)\theta\cdot A\bar{\theta}$ can be estimated by use of Lemma 2.6 mentioned below. As for the integral containing $\tau q(x,\ \kappa)\theta\cdot A\bar{\theta}$, shift it from the right-hand side of (2.3)′ to the left and estimate it together with the second integral term in the left-hand side of (2.3)′. Then we have the inequality by the same manipulation as in [6].

Q.E.D.

LEMMA 2.6. *Let $\tau>0$ and $f\in L^2_{\gamma/2}$ for some positive $\gamma\leq(\nu-\mu)/2$. Then the solution u of* (2.4) *satisfying the radiation condition belongs to $L^2_{\gamma/2}$ and satisfies for some constant C independent of τ, u and f*

$$\tau\|u\|_{\gamma/2}\leq C\{\|u\|_{(\gamma-2)/2}+\|f\|_{\gamma/2}\}\,. \tag{2.5}$$

PROOF. Multiply (2.4) by $r^\gamma\bar{u}(r=|x|)$ and integrate the result over $\{|x|>R\}$ $(R>0)$. Integration by parts gives

$$-\int_{|x|=R}(\tilde{x}\cdot A(x)\nabla_x u)r^\gamma\bar{u}\,dS+\int_{|x|<R}(\nabla_x u)\cdot A(x)\left\{r^\gamma\left(\frac{\gamma}{r}\tilde{x}+\nabla_x\right)\bar{u}\right\}dx+$$

$$+\int_{|x|<R}r^\gamma(V(x)|u|^2-\kappa^2|u|^2)dx=\int_{|x|<R}r^\gamma f\bar{u}\,dx\,.$$

Taking the imaginary part we obtain

$$2\sigma\tau\int_{|x|<R}r^\gamma|u|^2dx-\int_{|x|=R}r^\gamma\ \mathrm{Im}\,\{\tilde{x}\cdot A(x)(\nabla_x\rho)(x,\ \kappa)\}|u|^2dS$$

$$=-\int_{|x|=R} r^{\gamma}\,\mathrm{Im}\,[\tilde{x}\cdot A(x)\{(\nabla_x+(\nabla_x\rho)(x,\ \kappa))u\}\bar{u}]dS$$

$$+\int_{|x|<R} r^{\gamma}\left[\frac{\gamma}{r}\,\mathrm{Im}\,\{\tilde{x}\cdot A(x)(\nabla_x u)\bar{u}\}-\mathrm{Im}\,(f\bar{u})\right]dx\,.$$

Noting that

$$\sigma^{-1}\,\mathrm{Im}\,\{\tilde{x}\cdot A(x)(\nabla_x\rho)(x,\ \kappa)\}=\Phi(x)+O(|x|^{-\delta})\ \text{as}\ |x|\to\infty$$

and the right-hand side is non-negative for sufficiently large $|x|$, we have for sufficiently large R

$$\tau\int_{|x|<R} r^{\gamma}|u|^2dx\leq C\|u\|_{\gamma/2}\{\|u\|_{(\gamma-2)/2}+\|f\|_{\gamma/2}\}$$

$$+\frac{1}{2\sigma}\left\{\int_{|x|=R} r^{\nu}|\tilde{x}\cdot A(x)(\nabla_x+(\nabla_x\rho)(x,\ \kappa))u|^2dS\right\}^{1/2}$$

$$\times\left\{\int_{|x|=R} r^{-\mu}|u|^2dS\right\}^{1/2},$$

where we have used the inequality $\gamma\leq(\nu-\mu)/2$ and the elliptic estimate

$$\|\nabla_x u\|_s\leq C\{\|u\|_s+\|f\|_s\},\ s\in\boldsymbol{R}\,.$$

The radiation condition allows us to let $R\to\infty$. Dividing the both sides by $\|u\|_{\gamma/2}$, we are led to the estimate (2.5). Q.E.D.

With the aid of Proposition 2.5 we now have the limiting absorption principle:

THEOREM 2.7. *Let $\mathcal{R}_\kappa=(L-\kappa^2)^{-1}$ and let μ, ν be positive constants satisfying $\mu\leq\nu$ and $\mu+\nu<2$.*

(1) *For $\sigma\in\boldsymbol{R}\backslash\{0\}$ there exists a strong limit*

$$\operatorname*{s-lim}_{\tau\downarrow 0}\mathcal{R}_{\sigma+i\tau}=\mathcal{R}_\sigma$$

as a bounded operator from $L^2_{(1+\nu)/2}$ to $L^2_{(-1-\mu)/2}$. Further $\mathcal{R}_\sigma f$ for $f\in L^2_{(1+\nu)/2}$ is continuous in $\sigma\in\boldsymbol{R}\backslash\{0\}$ in the $L^2_{(-1-\mu)/2}$-topology.

(2) *For $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $f\in L^2_{(1+\nu)/2}$, $u=\mathcal{R}_\sigma f$ is the unique solution of*

$$(L-\sigma^2)u=f,$$

satisfying the radiation condition.

The following proposition is proved in a similar manner as in Propositions 2.3 and 2.4 of [6].

PROPOSITION 2.8. (1) *Let μ, ν satisfy $0<\mu\leq\nu$ and $\mu+\nu<2$. For any $\sigma\in\boldsymbol{R}\backslash\{0\}$ and $f\in L^2_{(1+\nu)/2}$ there exists a sequence $\{r_m\}$ tending to infinity such that*

$$\lim_{m\to\infty}\int_{|x|=r_m}(r^{-\mu}|u|^2+r^{\nu}|(\nabla_x+(\nabla_x\rho)(x,\sigma))u|^2)dS=0, \tag{2.6}$$

where $u=\mathcal{R}_\sigma f$ *and* $r=|x|$.

(2) *Let* μ, ν *satisfy* $0<\mu\leq\nu$, $\mu+\nu<2$, $\mu<\delta$ *and* $\mu+2(1-\delta)<\nu$. *Let* $\{r_m\}$ *be any sequence satisfying* (2.6) *with these* μ *and* ν. *Then*

$$[\mathcal{F}(\sigma, r_m)f](\tilde{x})=-ie^{\pm(n-1)\pi i/4}\sqrt{\frac{2}{\pi}}\,\sigma e^{\rho(r_m\tilde{x},\sigma)}(\mathcal{R}_\sigma f)(r_m\tilde{x}) \ \ if\ \ \pm\sigma>0$$

converges to $\mathcal{F}(\sigma)f$ *strongly in* $L^2(S^{n-1})$ *and*

$$\frac{\sigma}{\pi i}(\mathcal{R}_\sigma f-\mathcal{R}_{-\sigma}f,\ f)=\|\mathcal{F}(\sigma)f\|^2_{S^{n-1}}.$$

Furthermore $\mathcal{F}(\sigma)$ *is independent of the choice of the sequence* $\{r_m\}$ *specified by* (2.6).

(3) *Let* $\tilde{\mu}$, $\tilde{\nu}$ *satisfy* $0<\tilde{\mu}<\tilde{\nu}\leq1$ *and* $\tilde{\mu}+\tilde{\nu}\leq\min\{2\delta, 1\}$. *Then the operator* $\mathcal{F}(\sigma)$ *initially defined on* $L^2_{(1+\nu)/2}$ *can be extended to a bounded operator from* $L^2_{(1+\tilde{\nu})/2}$ *to* $L^2(S^{n-1})$ *which will be denoted by* $\mathcal{F}(\sigma)$ *again. For* $f\in L^2_{(1+\tilde{\nu})/2}$ *and* $\phi\in L^2(S^{n-1})$, *we have*

$$(\mathcal{F}(\sigma)f,\ \phi)_{L^2(S^{n-1})}=\lim_{m\to\infty}(\mathcal{F}(\sigma, r_m)f,\ \phi)_{L^2(S^{n-1})},$$

where $\{r_m\}$ *is the sequence specified by* (2.6) *when* $\mu=\tilde{\mu}$ *and* $\nu=\tilde{\nu}$.

Making use of Proposition 2.8, we now arrive at the goal of this section.

Theorem 2.9. (1) *Let* $0<\tilde{\nu}<\min\{2\delta, 1\}$. *For* $f\in L^2_{(1+\tilde{\nu})/2}$, *define*

$$(\mathcal{F}_\pm f)(\sigma, \tilde{x})=[\mathcal{F}(\sigma)f](\tilde{x}) \ \ if\ \ (\sigma, \tilde{x})\in\boldsymbol{R}_\pm\times S^{n-1}.$$

Then $\mathcal{F}_\pm$ *can be extended to unitary operators from* $L^2(\boldsymbol{R}^n)$ *to* $L^2(\boldsymbol{R}_\pm\times S^{n-1})$, *which will be denoted by* $\mathcal{F}_\pm$ *again.*

(2) *For any bounded Borel function* $\alpha(\lambda)$ *on* $\boldsymbol{R}$ *and* $f\in L^2(\boldsymbol{R}^n)$, *we have*

$$\begin{aligned}\alpha(L)f&=\mathcal{F}^*_\pm\alpha(\sigma^2)\mathcal{F}_\pm f\\ &=\text{s-}\lim_{N\to\infty}\int_{e_{\pm N}}\mathcal{F}(\sigma)^*\alpha(\sigma^2)(\mathcal{F}_\pm f)(\sigma,\ \cdot\ )d\sigma \ \ in\ \ L^2(\boldsymbol{R}^n),\end{aligned}$$

where $e_{+N}=(1/N,\ N)$, $e_{-N}=(-N,\ -1/N)$.

§ 3. Asymptotic behaviour of certain integrals.

This section is devoted to preliminaries for calculations carried out in later sections.

PROPOSITION 3.1. *Consider the integral*

$$I(t)=\int_{\Omega}e^{i\Psi(x,t)}a(x, t)\,dx \ (\Omega: \text{ a domain in } \boldsymbol{R}^n).$$

Assume that the phase function $\Psi(x, t)$ *satisfies the following*: $\Psi(x, t)$ *is a real* $C^\infty(\Omega\times\boldsymbol{R}_+)$*-function such that*

$$|\partial_x^\alpha\Psi(x, t)|\leq C_\alpha(1+t) \text{ for } |\alpha|\geq 0,\ x\in\Omega \text{ and } t>0. \tag{3.1}$$

Moreover, there exist constants $C>0$ *and* $T\geq 0$ *such that*

$$|\nabla_x\Psi(x, t)|\geq Ct \text{ for } x\in\Omega \text{ and } t>T. \tag{3.2}$$

Suppose that the amplitude function $a(x, t)$ *is in* $C^\infty(\Omega\times\boldsymbol{R}_+)$ *and there exists a subdomain* $\Omega_0\Subset\Omega$ *such that for* $t>0$, $a(x, t)=0$ *if* $x\notin\Omega_0$. *Then for any integer* $N>0$ *there exists a positive constant* C_N *such that*

$$|I(t)|\leq C_N\sup_{\substack{x\in\Omega_0\\|\alpha|\leq N}}|\partial_x^\alpha a(x, t)|(1+t)^{-N} \text{ for } t>T.$$

PROOF. Let L_x be the differential operator defined by

$$L_x=\sum_{j=1}^n(1+|\nabla_x\Psi(x, t)|^2)^{-1}\left(\frac{1}{n}-(\partial_{x_j}\Psi)(x, t)i\partial_{x_j}\right).$$

Then the formal adjoint L_x^* of L_x is given by

$$\begin{aligned}L_x^*&=\sum_{j=1}^n(1+|\nabla_x\Psi(x, t)|^2)^{-1}(\partial_{x_j}\Psi)(x, t)i\partial_{x_j}+C(x, t),\\ C(x, t)&=(1+|\nabla_x\Psi(x, t)|^2)^{-1}+i\sum_{j=1}^n\partial_{x_j}[(1+|\nabla_x\Psi(x, t)|^2)^{-1}(\partial_{x_j}\Psi)(x, t)].\end{aligned} \tag{3.3}$$

It follows from (3.1) and (3.2) that

$$|\partial_x^\alpha C(x, t)|\leq C_\alpha(1+t)^{-1} \text{ for } x\in\Omega \text{ and } t>T. \tag{3.4}$$

Since $e^{i\Psi(x,t)}=L_xe^{i\Psi(x,t)}$, integration by parts shows that for any integer $N>0$,

$$I(t)=\int_{\Omega}e^{i\Psi(x,t)}(L_x^*)^N a(x, t)\,dx. \tag{3.5}$$

The assertion of the proposition is easily deduced from (3.1)-(3.5). Q.E.D.

The following proposition is proved in Ikebe-Isozaki [4] and prepared just for Theorem 4.10.

PROPOSITION 3.2. *Consider the integral*

$$I(t)=\int_{\Omega}e^{itS(x,t)}a(x, t)dx(\Omega: \text{ a domain of } \boldsymbol{R}^n)$$

Assume that the phase function $S(x, t)$ *is divided into two parts* $S_0(x)$ *and* $S_1(x, t)$: $S(x, t)=S_0(x)+S_1(x, t)$ *and they satisfy the following conditions*:

(i) $S_0(x)$ *is a real* $C^\infty(\Omega)$*-function such that*

$$|\partial_x^\alpha S_0(x)|\leq C_\alpha \text{ for } |\alpha|\geq 0 \text{ and } x\in\Omega.$$

(ii) *There exists a unique critical point* $x_0\in\Omega$ *of* $S_0(x)$: $(\nabla_x S_0)(x_0)=0$.

(iii) *The Hessian matrix* A_0 *of* $S_0(x)$ *at* $x=x_0$ *is non-singular.*

(iv) $S_1(x, t)$ *is a real* $C^\infty(\Omega\times\boldsymbol{R}_+)$*-function such that for some* $\delta>0$,

$$|\partial_x^\alpha S_1(x, t)|\leq C_\alpha t^{-\delta} \text{ if } |\alpha|\geq 0,\ x\in\Omega \text{ and } t>0.$$

The amplitude function $a(x, t)$ *is assumed to satisfy the following*:

(v) *There exists a subdomain* $\Omega_0\Subset\Omega$ *such that for* $t>0$, $a(x, t)=0$ *if* $x\notin\Omega_0$.

(vi) $a(x, t)$ *is a* $C^\infty(\Omega\times\boldsymbol{R}_+)$*-function such that*

$$|\partial_x^\alpha a(x, t)|\leq C_\alpha \text{ for } |\alpha|\geq 0.$$

Under the conditions (i)-(vi) *we conclude the following*:

(1) *We can find constants* $T>0$ *and* $C>0$ *such that for any* $t>T$ *there exists a unique critical point* $x(t)\in\Omega$ *of* $S(x, t)$: $(\nabla_x S)(x(t), t)=0$ *and* $|x(t)-x_0|\leq Ct^{-\delta}$.

(2) *We have the following asymptotic expansion as* $t\to\infty$:

$$I(t)=(2\pi)^n|\det A_0|^{-1/2}t^{-n/2}e^{i\pi\sigma/4}e^{iS(x(t),t)}a(x_0, t)+R(t),$$

$$|R(t)|\leq Ct^{-n/2-\delta} \text{ for } t \text{ large},$$

where σ *is the signature of* A_0 *and* C *is a constant independent of* t *large.*

§4. Invariance principle; proof of Theorem 1.3.

In this section we shall prove Theorem 1.3. We verify the theorem only for $\Omega_+(\phi(L_1), \phi(L_0))$ since $\Omega_-(\phi(L_1), \phi(L_0))$ can be treated in a similar manner.

Let $\eta(r)$ be a $C^\infty(\boldsymbol{R}_+)$-function such that $0\leq\eta\leq 1$, $\eta(r)=0$ if $r\leq 1$ and $\eta(r)=1$ if $r\geq 2$. Set for any $\psi(\sigma, \tilde{x})\in C_0^\infty(\boldsymbol{R}_+\times S^{n-1})$

$$v_\psi(x, \sigma)=e^{-(n-3)\pi i/4}(2\pi)^{-1/2}\eta(r)e^{-\rho(x,\sigma)}\psi(\sigma, \tilde{x})\quad (r=|x|), \tag{4.1}$$

$$\begin{aligned} g_\psi(x, \sigma)&=(-\Delta_A+V(x)-\sigma^2)v_\psi(x, \sigma)\\ &=a_\psi(x, \sigma)e^{-\rho(x,\sigma)}, \end{aligned} \tag{4.2}$$

where $\rho(x, \sigma)$ is introduced in Definition 2.2.

LEMMA 4.1. *Let* $\psi(\sigma, \tilde{x})\in C_0^\infty(\boldsymbol{R}_+\times S^{n-1})$ *and* $v_\psi(x, \sigma)$, $a_\psi(x, \sigma)$ *be given by* (4.1), (4.2). *Then*

$$|\tilde{x}\cdot A(x)(\nabla_x+(\nabla_x\rho)(x, \sigma))v_\psi(x, \sigma)|\leq C(1+|x|)^{-1-\delta-(n-1)/2},$$

$$|\partial_\sigma^k a_\psi(x, \sigma)| \leq C_k(1+|x|)^{-1-\delta} \textit{ for any integer } k \geq 0.$$

PROOF. For $|x|$ large we have with $C_0 = e^{-(n-3)\pi i/4}(2\pi)^{-1/2}$

$$\tilde{x} \cdot A(\nabla_x + (\nabla_x \rho))v_\psi = C_0 e^{-\rho} \tilde{x} \cdot (A-I)\nabla_x \psi ,$$

$$a_\psi = C_0\{-\sigma^2 + V - (\nabla_x \rho) \cdot A(\nabla_x \rho) + \Delta_A \rho + 2(\nabla_x \rho) \cdot (A-I)(\nabla_x \psi)\} .$$

The assertion is easily deduced from Proposition 2.3 and the fact $\nabla_x \psi = O(|x|^{-1})$ as $|x| \to \infty$. Q.E.D.

PROPOSITION 4.2. *Let* $\psi(\sigma, \tilde{x}) \in C_0^\infty(\boldsymbol{R}_+ \times S^{n-1})$ *and* $v_\psi(x, \sigma)$, $g_\psi(x, \sigma)$ *be defined by* (4.1), (4.2). *Then*

$$\mathcal{F}(\sigma)^* \psi(\sigma, \cdot) = -iv_\psi(\cdot, \sigma) + i\mathcal{R}_{-\sigma} g_\psi(\cdot, \sigma).$$

PROOF. Let $\tilde{\mu}$, $\tilde{\nu}$ be given as in (3) of Proposition 2.8. Lemma 4.1 and Theorem 2.7 show that $g_\psi(x, \sigma) \in L^2_{(1+\tilde{\nu})/2}$ and

$$v_\psi(\cdot, \sigma) = \mathcal{R}_\sigma g_\psi(\cdot, \sigma).$$

For $f \in L^2_{(1+\tilde{\nu})/2}$ let $u = \mathcal{R}_\sigma f$ and $\{r_m\}$ be the sequence specified by (2.6) with these $\tilde{\mu}$, $\tilde{\nu}$, f and u. By Green's formula we have

$$\begin{aligned}
&\int_{|x|<r_m} (u\bar{g}_\psi - f\bar{v})\,dx \\
&= \int_{|x|=r_m} [\tilde{x} \cdot A\{(\nabla_x + (\nabla_x \rho))u\}\bar{v}_\psi - u\tilde{x} \cdot A(\nabla_x + (\overline{\nabla_x \rho}))\bar{v}_\psi \\
&\quad + O(|x|^{-\delta-(n-1)/2})u\bar{\psi}]dS + i(\mathcal{F}(\sigma, r_m)f, \psi)_{L^2(S^{n-1})} .
\end{aligned}$$

Letting $m \to \infty$ we obtain

$$i(\mathcal{F}(\sigma)f, \psi) = -(f, v_\psi) + (u, g_\psi) ,$$

where we have used (1) and (3) of Proposition 2.8. The assertion follows from the above identity. Q.E.D.

We now choose arbitrary numbers σ_1, σ_2 such that $0 < \sigma_1 < \sigma_2 < \infty$. Let Σ be an open interval defined by $\Sigma = (\sigma_1, \sigma_2)$.

LEMMA 4.3. *Let* $\phi(\lambda)$ *be the function given in Theorem* 1.3. *Let* $\psi(\sigma, \tilde{x}) \in C_0^\infty$ $(\Sigma \times S^{n-1})$ *and* $g_\psi(x, \sigma)$ *be defined by* (4.2). *Tnen*

$$\lim_{t \to \infty} \left\| \int_0^\infty e^{-i\phi(\sigma^2)t} \mathcal{R}_{-\sigma} g_\psi(\cdot, \sigma)d\sigma \right\| = 0.$$

Similarly as in Lemma 3.4 of Mochizuki [13], this lemma is derived as a corollary of the following

PROPOSITION 4.4. *Consider the integral*

$$I(x, s, t)=r^{-(n-1)/2}\int_0^{\infty} e^{-i\{\sigma^2 s+\phi(\sigma^2)t-K(x,\sigma)\}}a(\sigma, x)d\sigma \quad (r=|x|)$$

for $s, t\in \boldsymbol{R}_+$ *and* $x\in \boldsymbol{R}^n\backslash\{0\}$, *where* $K(x, \sigma)$ *is the function introduced in Lemma* 2.1 *and functions* $\phi(\lambda)$, $a(\sigma, x)$ *are assumed to possess the following properties with an interval* $\Sigma=(\sigma_1, \sigma_2)(0<\sigma_1<\sigma_2<\infty)$:

(i) $\phi(\lambda)$ *is a real* $C^\infty(\boldsymbol{R}_+)$*-function such that for some constant* $m>0$,

(4.3) $\phi'(\lambda)\geq m$ *if* $\sigma_1^2<\lambda<\sigma_2^2$.

(ii) $a(\sigma, x)$ *is a* C^∞*-function supported in* $\Sigma\times\{x\in \boldsymbol{R}^n;\ |x|\geq 1\}$ *and satisfies*

(4.4) $$|\partial_\sigma^k a(\sigma, x)|\leq C_k(1+|x|)^{-1-\delta} \text{ for any integer } k\geq 0,$$

where δ *is the same constant as in Assumption* 1.1.

Then there exist positive constants T *and* C *independent of* s, t *such that if* $t>T$, *then*

$$\|I(\cdot, s, t)\|\leq C(s+t)^{-1-\delta} \text{ for any } s>0.$$

PROOF. We set

(4.5) $$d=\min\{\sigma_1, m\sigma_1\}.$$

Choose a partition of unity $\{\chi_j\}_{j=1,2}$ on $\boldsymbol{R}_+$ such that $\chi_1(\lambda)=0$ for $\lambda\geq 5d/4$ and $\chi_2(\lambda)=0$ for $\lambda\leq 3d/4$. We set

$$I_j(x, s, t)=\chi_j(|x|/(s+t))I(x, s, t) \quad (j=1, 2).$$

1*st Step*. We shall estimate $I_1(x, s, t)$. Putting $x=(s+t)y$ we have

$$I_1((s+t)y, s, t)=\{(s+t)|y|\}^{-(n-1)/2}\int_0^{\infty} e^{-i\Psi(\sigma,y,s,t)}\chi_1(|y|)a(\sigma, (s+t)y)d\sigma;$$

$$\Psi(\sigma, y, s, t)=\Psi_0(\sigma, y, s, t)-Y((s+t)y, \sigma),$$

$$\Psi_0(\sigma, y, s, t)=\sigma^2 s+\phi(\sigma^2)t-\sigma(s+t)|y|,$$

where $Y(x, \sigma)$ has been introduced in (2.2). Then it follows from (4.3) and (4.5) that

$$|\partial_\sigma\Psi_0(\sigma, y, s, t)|=|2\sigma s+2\sigma\phi'(\sigma^2)-(s+t)|y||\geq\frac{3}{4}d(s+t)$$

for $(\sigma, y, s)\in\Gamma=\{(\sigma, y, s);\ \sigma\in\Sigma,\ \chi_1(|y|)\neq 0,\ s>0\}$. By (2.3) there exists a constant $T'>0$ such that if $t>T'$ and $(\sigma, y, s)\in\Gamma$, then

$$|\partial_\sigma Y((s+t)y, \sigma)|\leq\frac{1}{4}d(s+t).$$

Hence we obtain the inequality

$$|\partial_\sigma \Psi(\sigma, y, s, t)| \geq \frac{d}{2}(s+t)$$

for $(\sigma, y, s) \in \Gamma$ and $t > T'$. We also have for any integer $k \geq 0$

$$|\partial_\sigma^k \Psi(\sigma, y, s, t)| \leq C_k(1+s+t), \ |\partial_\sigma^k \{\chi_1(|y|)a(\sigma, (s+t)y)\}| \leq C_k$$

if $(\sigma, y, s) \in \Gamma$ and $t > T'$. Thus we can apply Proposition 3.1 to obtain for any $s > 0$, $t > T'$ and large integer $N > 0$

$$|I_1((s+t)y, s, t)| \leq C_N(s+t)^{-N}$$

uniformly for $|y| \leq 5d/4$, which gives

$$\|I_1(\cdot, s, t)\| \leq C_N(s+t)^{-N} \tag{4.6}$$

for $s > 0$, $t > T'$ and N large.

2nd Step. In order to estimate $I_2(x, s, t)$ we need the following

LEMMA 4.5. *For any $\psi(\sigma, \tilde{x}) \in C_0^\infty(\Sigma \times S^{n-1})$ let $A(s, t)$ be defined by*

$$[A(s, t)\psi](x) = r^{-(n-1)/2}\int_\Sigma e^{-i\{\sigma^2 s + \phi(\sigma^2)t - K(x, \sigma)\}}\chi_2(|x|/(s+t))a(\sigma, x)\psi(\sigma, \tilde{x})d\sigma .$$

Then there exist constants $C > 0$ and $T > T'$ such that for any $s > 0$ and $t > T$,

$$\|A(s, t)\psi\| \leq C(s+t)^{-1-\delta}\|\psi\|_{\Sigma \times S^{n-1}} . \tag{4.7}$$

Hence the operator $A(s, t)$ can be extended to an operator from $L^2(\Sigma \times S^{n-1})$ to $L^2(\boldsymbol{R}^n)$ with bound $C(s+t)^{-1-\delta}$.

The proof of the lemma will be given in Section 5. It now follows from the inequality (4.7) with $\psi(\sigma, \tilde{x}) \equiv 1$ that

$$\|I_2(\cdot, s, t)\| \leq C(s+t)^{-1-\delta}$$

for any $s > 0$ and $t > T$. Combining this with (4.6) we are led to the assertion of the lemma. Q.E.D.

The following result stating the asymptotic behaviour of $e^{-it\phi(L_1)}$ as $t \to \infty$ is derived from Theorem 2.9, Proposition 4.2 and Lemma 4.3.

THEOREM 4.6. *Let $\phi(\lambda)$ be as in Theorem 1.3. Let $u \in L^2(\boldsymbol{R}^n)$ such that $\mathcal{F}_+ u \in C_0^\infty(\boldsymbol{R}_+ \times S^{n-1})$, and define for $\psi(\sigma, \tilde{x}) \in C_0^\infty(\boldsymbol{R}_+ \times S^{n-1})$*

$$w_\phi^\infty(x, t; \psi) = -i\int_0^\infty e^{-it\phi(\sigma^2)}v_\psi(x, \sigma)\,d\sigma$$

$$= e^{-(n-1)\pi i/4}(2\pi)^{-1/2}\eta(r)r^{-(n-1)/2}\int_0^\infty e^{-i\{\phi(\sigma^2)t - K(x, \sigma)\}}\psi(\sigma, \tilde{x})\,d\sigma.$$

Then

$$\lim_{t\to\infty} \|e^{-it\phi(L_1)}u - w_\phi^\infty(\cdot, t; \mathcal{F}_+u)\| = 0.$$

REMRK. When $\phi(\lambda)=\sqrt{\lambda}$, $w_\phi^\infty(x, t; \mathcal{F}_+u)$ corresponds to the asymptotic wave functions constructed in Mochizuki [13] and the author [6], but in our setting where the potential $V(x)$ is in a long-range class, $K(x, \sigma)$ is not a linear function of σ and therefore w_ϕ^∞ cannot be described as a modified diverging spherical wave.

LEMMA 4.7. *Let $\phi(\lambda)$ and $K(x, \sigma)$ be given as in Theorem* 1.3 *and Lemma* 2.1, *respectively. Then we have $C^\infty(\boldsymbol{R}^n\backslash\{0\})\times \boldsymbol{R}_+)$-functions $x(\xi, t)$ and $\sigma(\xi, t)$ possessing the following properties: For any compact set B in $\boldsymbol{R}^n\backslash\{0\}$ there exists a positive constant T such that if $\xi\in B$ and $t>T$, then*

$$\xi=(\nabla_x K)(x(\xi, t)), \sigma(\xi, t)),$$

$$2\sigma(\xi, t)\phi'(\sigma^2(\xi, t))t=(\partial_\sigma K)(x(\xi, t), \sigma(\xi, t));$$

$$\left.\begin{array}{l} |\partial_\xi^\alpha(x(\xi, t)-2\xi\phi'(|\xi|^2)t)|\le C_\alpha(1+t)^{1-\delta}, \\ |\partial_\xi^\alpha(\sigma(\xi, t)-|\xi|)|\le C_\alpha(1+t)^{-\delta}, \end{array}\right\} \quad (|\alpha|\ge 0)$$

where the constant $C_\alpha>0$ is independent of $\xi\in B$.

This lemma is a consequence of the inverse function theorem and since the proof can be carried out similarly as for Lemma 6.1 of Ikebe-Isozaki [4] (see also Proposition 2.2 of Kitada [9] and Lemmas 4.1, 4.2 of Ikebe-Isozaki [3]), we may omit the proof.

DEFINITION 4.8. Let $x(\xi, t)$ and $\sigma(\xi, t)$ be as introduced in Lemma 4.7. Define

$$X_\phi(\xi, t)=-t\phi(|\xi|^2)+x(\xi, t)\cdot\xi+t\phi(\sigma^2(\xi, t))-K(x(\xi, t), \sigma(\xi, t)).$$

It follows from Lemmas 2.1 and 4.7 that $X_\phi(\xi, t)\in C^\infty(\boldsymbol{R}^n\backslash\{0\})\times\boldsymbol{R}_+)$ and

$$|\partial_\xi^\alpha X_\phi(\xi, t)|\le C_\alpha(1+t)^{1-\delta}$$

for $\xi\in B$, a compact set in $\boldsymbol{R}^n\backslash\{0\}$ and $|\alpha|\ge 0$. Put $W_\phi(\xi, t)=t\phi(|\xi|^2)+X_\phi(\xi, t)$. Then Lemma 4.7 implies that for any compact set B in $\boldsymbol{R}^n\backslash\{0\}$ there exists a positive constant T such that

$$\nabla_\xi W_\phi(\xi, t)=x(\xi, t), \ \partial_t W_\phi(\xi, t)=\phi(\sigma^2(\xi, t))$$

for $\xi\in B$ and $t>T$. Since $K(x, \sigma)$ satisfies

$$(\nabla_x K)(x, \sigma)\cdot A(x)(\nabla_x K)(x, \sigma)+V(x)=\sigma^2,$$

we replace x, $\nabla_x K(x, \sigma)$ and σ^2 by $\nabla_\xi W_\phi(\xi, t)$, ξ and $\phi^{-1}(\partial_t W_\phi(\xi, t))$, respectively in the above equation. Then we get the Hamilton-Jacobi equation

$$\partial_t W_\phi(\xi, t)=\phi(\xi\cdot A(\nabla_\xi W_\phi(\xi, t))\xi+V(\nabla_\xi W_\phi(\xi, t)))$$

for $\xi\in B$ and $t>T$. Thus we have obtained the time dependent modifier $X_\phi(\xi, t)$.

DEFINITION 4.9. Let F_+ be the operator from $L^2(\boldsymbol{R}^n)$ to $L^2(\boldsymbol{R}^n)$ defined by

$$(F_+u)(\xi)=|\xi|^{-(n-1)/2}(\mathcal{F}_+u)(|\xi|, \xi/|\xi|),\ u\in L^2(\boldsymbol{R}^n),$$

where $\mathcal{F}_+$ is the operator defined in Theorem 2.9.

Theorem 2.9 implies that F_+ is a unitary operator from $L^2(\boldsymbol{R}^n)$ onto $L^2(\boldsymbol{R}^n)$.

THEOREM 4.10. *For any* $u\in L^2(\boldsymbol{R}^n)$ *we have*

$$\|e^{-it\phi(L_1)}u-F_0^{-1}[e^{-iW_\phi(\cdot, t)}F_+u]\|\to 0 \ \textit{as}\ t\to\infty,$$

where $W_\phi(\xi, t)=t\phi(|\xi|^2)+X_\phi(\xi, t)$.

PROOF. Since the operators are unitary, we have only to verify the theorem when $\tilde{u}(\sigma, \tilde{x})=(\mathcal{F}_+u)(\sigma, \tilde{x})\in C_0^\infty(\Sigma\times S^{n-1})$ with $\Sigma=(\sigma_1, \sigma_2)$ $(0<\sigma_1<\sigma_2<\infty)$. The proof follows the same lines as in Lemma 6.3 of Ikebe-Isozaki [4]. Let $a=\inf\{2\sigma\phi'(\sigma^2);\ \sigma\in\Sigma\}$ and $b=\sup\{2\sigma\phi'(\sigma^2);\ \sigma\in\Sigma\}$. We cover $\boldsymbol{R}_+$ by three open subsets $\mathcal{U}_1=(0, a)$, $\mathcal{U}_2=(a-2\varepsilon, b+2\varepsilon)$ and $\mathcal{U}_3=(b, \infty)$, where ε is a sufficiently small positive constant. Let $\{\chi_j\}_{j=1,2,3}$ be a partition of unity subordinate to this covering such that $\chi_1(s)=0$ for $s\geq a-\varepsilon$, $\chi_2(s)=1$ for $s\in[a, b]$ and $\chi_3(s)=0$ for $s\leq b+\varepsilon$. We set

$$f_j(x, t)=\chi_j(|x|/t)w_\phi^\infty(x, t;\ \tilde{u}),\ j=1,2,3.$$

1*st Step.* We begin by estimating $f_j(x, t)$ for $j=1,3$. Putting $x=ty$ we have

$$f_j(ty, t)=e^{-(n-1)\pi i/4}(2\pi)^{-1/2}\chi_j(|y|)\int_0^\infty e^{i\Psi(\sigma, y, t)}h(\sigma, y, t)\,d\sigma;$$

$$\Psi(\sigma, y, t)=\Psi_0(\sigma, y, t)-Y(ty, \sigma)$$

$$\Psi_0(\sigma, y, t)=t(\sigma|y|-\phi(\sigma^2)),$$

$$h(\sigma, y, t)=\eta(t|y|)(t|y|)^{-(n-1)/2}\tilde{u}(\sigma, y/|y|)$$

with $Y(x, \sigma)$ introduced in (2.2). Put

$$\Omega_j=\{(\sigma, y);\ \sigma\in\Sigma,\ \chi_j(|y|)\neq 0\},\ j=1,\ 3.$$

Then we have with some constant $C>0$

$$|\partial_\sigma\Psi_0(\sigma, y, t)|\geq Ct(1+|y|) \tag{4.8}$$

for $(\sigma, y)\in\Omega_j$, $j=1, 3$. In fact, since $\partial_\sigma\Psi_0(\sigma, y, t)$ is calculated as

$$\partial_\sigma\Psi_0(\sigma, y, t)=t(|y|-2\sigma\phi'(\sigma^2)),$$

it follows that if $(\sigma, y)\in\Omega_1$, then

$$|\partial_\sigma \Psi_0(\sigma, y, t)| \geq \varepsilon[1+(a-\varepsilon)]^{-1}t(1+|y|).$$

Next, take a sufficiently large constant $R_0>0$ such that $R_0(1+R_0)^{-1}\geq 3/4$ and $b(1+R_0)^{-1}\leq 1/4$. When $(\sigma, y)\in\Omega_3$, we have

$$|\partial_\sigma \Psi_0(\sigma, y, t)| \geq \begin{cases} \varepsilon(1+R_0)^{-1}t(1+|y|) & \text{if } |y|\leq R_0, \\ t(1+|y|)/2 & \text{if } |y|>R_0. \end{cases}$$

Thus we have shown (4.8). On the other hand, it follows immediately from (2.3) that for t large,

$$|\partial_\sigma Y(ty, \sigma)| \leq C[t(1+|y|)]^{1-\delta}.$$

Hence we can find positive constants C and T such that

$$|\partial_\sigma \Psi(\sigma, y, t)| \geq Ct(1+|y|)$$

for $(\sigma, y)\in\Omega_j$, $j=1, 3$ and $t>T$. Noting that

$$\begin{aligned} &|\partial_\sigma^k \Psi(\sigma, y, t)| \leq C_k t(1+|y|), \\ &|\partial_\sigma^k h(\sigma, y, t)| \leq C_k[t(1+|y|)]^{-(n-1)/2} \end{aligned} \qquad (k\geq 0)$$

for $(\sigma, y)\in\Omega_j$, $j=1, 3$ and $t>T$, we can use Proposition 3.1 to obtain

$$|f_j(ty, t)| \leq C_N t^{-N}(1+|y|)^{-N}, \ j=1, 3$$

for any large $N>0$ and $t>T$. Therefore we have for $j=1, 3$

$$\|f_j(\cdot, t)\| \to 0 \text{ as } t\to\infty. \tag{4.9}$$

2nd Step. We now choose $\{\mathcal{V}_j\}_{j=1,2,3}$, a covering of $\boldsymbol{R}_+$ such that $\mathcal{V}_1=(0, \sigma_1)$, $\mathcal{V}_2=(\sigma_1-2\varepsilon, \sigma_2+2\varepsilon)$ and $\mathcal{V}_3=(\sigma_1, \infty)$. Take a partition of unity $\{\psi_j\}_{j=1,2,3}$ subordinate to this covering such that $\psi_1(s)=0$ for $s\geq\sigma_1-\varepsilon$, $\psi_2(s)=1$ for $s\in\Sigma$ and $\psi_3(s)=0$ for $s\leq\sigma_2+\varepsilon$. Noting that $f_2(\cdot, t)\in L^2(\boldsymbol{R}^n)$, we set for $j=1, 2, 3$

$$g_j(\xi, t)=(2\pi)^{-n/2}\psi_j(|\xi|)\int_{\boldsymbol{R}^n} e^{-ix\cdot\xi}f_2(x, t)\,dx,$$

$$\Gamma_j=\{(y, \sigma, \xi);\ \chi_2(|y|)\neq 0,\ \sigma\in\Sigma,\ \psi_j(|\xi|)\neq 0\}.$$

We shall estimate $g_j(\xi, t)$ for $j=1, 3$. Putting $x=ty$ we have

$$\begin{aligned} g_j(\xi, t)&=(2\pi)^{-(n+1)/2}e^{-(n-1)\pi i/4}t^{(n+1)/2}\psi_j(|\xi|) \\ &\quad\times\int_{\boldsymbol{R}^n}\int_0^\infty e^{-i\Psi(y,\sigma,\xi,t)}h(y, \sigma, t)\,d\sigma dy; \end{aligned}$$

$$\Psi(y, \sigma, \xi, t)=\Psi_0(y, \sigma, \xi, t)-Y(ty, \sigma),$$

$$\Psi_0(y, \sigma, \xi, t)=t(y\cdot\xi+\phi(\sigma^2)-\sigma|y|),$$

$$h(y, \sigma, t)=\chi_2(|y|)\eta(t|y|)|y|^{-(n-1)/2}\tilde{u}(\sigma, y/|y|).$$

Then it is easily seen as in 1st Step that for some constant $C>0$,

$$|\nabla_y \Psi_0(y, \sigma, \xi, t)| \geq Ct(1+|\xi|)$$

if $(y, \sigma, \xi)\in\Gamma_j$, $j=1, 3$. If we take account of the inequality

$$|\nabla_y Y(ty, \sigma)| \leq C[t(1+|\xi|)]^{1-\delta}$$

for $(y, \sigma, \xi)\in\Gamma_j$, $j=1, 3$ and t large, then we can find positive constants C and T such that

$$|\nabla_y \Psi(y, \sigma, \xi, t)| \geq Ct(1+|\xi|)$$

for $t>T$ and $(y, \sigma, \xi)\in\Gamma_j$, $j=1, 3$. We also see that

$$\left.\begin{aligned} &|\partial_y^\alpha \Psi(y, \sigma, \xi, t)| \leq C_\alpha t(1+|\xi|), \\ &|\partial_y^\alpha h(y, \sigma, t)| \leq C_\alpha \end{aligned}\right. \qquad (|\alpha|\geq 0)$$

for $(y, \sigma, \xi)\in\Gamma_j$, $j=1, 3$ and $t>T$. Thus we can apply Proposition 3.1 to the y-integral to obtain for any integer $N>0$ and $t>T$

$$|\psi_j(|\xi|)\int_{\boldsymbol{R}^n} e^{-i\Psi(y,\sigma,\xi,t)} h(y, \sigma, t)\,dy| \leq C_N t^{-N}(1+|\xi|)^{-N}, \ j=1, 3$$

uniformly in $\sigma\in\Sigma$. Hence we have

$$\|g_j(\cdot, t)\| \to 0 \text{ as } t\to\infty, \ j=1, 3. \tag{4.10}$$

3rd Step. Rewrite $g_2(\xi, t)$ as follows:

$$g_2(\xi, t) = (2\pi)^{-(n+1)/2} e^{-(n-1)\pi i/4} t^{(n+1)/2} \psi_2(|\xi|)$$
$$\times \int_{\boldsymbol{R}^n}\int_0^\infty e^{-itS(y,\sigma,\xi,t)} h(y, \sigma, t)\,d\sigma dy\,;$$
$$S(y, \sigma, \xi, t) = S_0(y, \sigma, \xi) + S_1(y, \sigma, t),$$
$$S_0(y, \sigma, \xi) = y\cdot\xi + \phi(\sigma^2) - \sigma|y|,$$
$$S_1(y, \sigma, t) = -t^{-1} Y(ty, \sigma),$$
$$h(y, \sigma, t) = \chi_2(|y|)\eta(t|y|)|y|^{-(n-1)/2}\tilde{u}(\sigma, y/|y|).$$

Then it is easily seen that the following inequalities hold:

$$|\partial_y^\alpha \partial_\sigma^k S_0(y, \sigma, \xi)| \leq C_{\alpha k}, \ |\partial_y^\alpha \partial_\sigma^k S_1(y, \sigma, t)| \leq C_{\alpha k} t^{-\delta},$$
$$|\partial_y^\alpha \partial_\sigma^k h(y, \sigma, t)| \leq C_{\alpha k} \ (|\alpha|\geq 0, \ k\geq 0)$$

for $(y, \sigma, \xi)\in\Gamma_2$ and t large. Thus we have checked the assumptions (i) and (iv)-(vi) of Proposition 3.2. Let $B=\{\xi\in\boldsymbol{R}^n\,;\ \psi_2(|\xi|)\neq 0\}$. For $\xi\in B$ there exists a unique critical point $(2\xi\phi'(|\xi|^2), |\xi|)$ of $S_0(y, \sigma, \xi)$. Let $A_0(\xi)$ be the Hessian matrix of $S_0(y,$

$\sigma, \xi)$ at $(y, \sigma)=(2\xi\phi'(|\xi|^2), |\xi|)$ and sign A_0 its signature. A direct calculation yields

$$\det A_0(\xi)=(-1)^n(2\phi'(|\xi|^2))^{-(n-1)},$$
$$\operatorname{sign} A_0=-(n-1)$$

for $\xi\in B$, hence $\det A_0(\xi)\neq 0$ since $\phi'(\lambda)>0$. Thus we have a unique critical point $(y_c(\xi, t), \sigma_c(\xi, t))$ of $S(y, \sigma, \xi, t)$ for t large and $\xi\in B$ if we apply Proposition 3.2, (1). Noting that

$$tS(y_c(\xi, t), \sigma_c(\xi, t), \xi, t)=W_\phi(\xi, t),$$
$$h(2\xi\phi'(|\xi|^2), |\xi|, t)=|2\xi\phi'(|\xi|^2)|^{-(n-1)/2}\tilde{u}(|\xi|, \xi/|\xi|)$$

when $\xi\in B$ and $t\to\infty$, we obtain by Proposition 3.2, (2)

$$g_2(\xi, t)-e^{-iW_\phi(\xi,t)}(F_+u)(\xi)=o(1)$$

uniformly for $\xi\in B$ as $t\to\infty$. This shows that

$$\|g_2(\cdot, t)-e^{-iW_\phi(\cdot,t)}F_+u\|\to 0 \text{ as } t\to\infty \tag{4.11}$$

since F_+u vanishes outside B. Hence the assertion follows from combining (4.9)-(4.11) and Theorem 4.6, and using the inverse Fourier transformation. Q.E.D.

PROOF OF THEOREM 1.3. We have already constructed the time dependent modifier $X_\phi(\xi, t)$. Theorem 4.10 implies that for $u\in L^2(\boldsymbol{R}^n)$,

$$\|e^{-it\phi(L_1)}u-e^{-it\phi(L_0)-iX_\phi(t)}F_0^*F_+u\|\to 0 \text{ as } t\to\infty.$$

This together with the unitarity of $e^{-it\phi(L_1)}$ and $F_0^*F_+$ in $L^2(\boldsymbol{R}^n)$ shows that

$$\|e^{it\phi(L_1)}e^{-it\phi(L_0)-iX_\phi(t)}u-F_+^*F_0u\|\to 0 \text{ as } t\to\infty.$$

This yields the existence of $\Omega_+(\phi(L_1), \phi(L_0))$ and

$$\Omega_+(\phi(L_1), \phi(L_0))=F_+^*F_0,$$

which implies the unitarity and the intertwining property. In particular if we take $\phi(\lambda)=\lambda$, we have

$$\Omega_+(L_1, L_0)=F_+^*F_0.$$

Similarly we obtain

$$\Omega_-(\phi(L_1), \phi(L_0))=F_-^*F_0$$

if we set

$$(F_-u)(\xi)=|\xi|^{-(n-1)/2}(\mathcal{F}_-u)(-|\xi|, -\xi/|\xi|)$$

with $\mathcal{F}_-$ introduced in Theorem 2.9. Q.E.D.

REMARK. For an interval $I=(\sigma_1^2, \sigma_2^2)$ $(0<\sigma_1<\sigma_2<\infty)$, let

$$J_+(I)u=(2\pi)^{-1/2}\int_{\sigma_1}^{\sigma_2} e^{-p(x,\sigma)-(n-1)\pi i/4}\,\eta(r)(\mathcal{F}_0^+u)(\sigma, \tilde{x})\,d\sigma\,;$$

$$(\mathcal{F}_0^+u)(\sigma, \tilde{x})=(2\pi)^{-n/2}\,\sigma^{(n-1)/2}\int_{\boldsymbol{R}^n} e^{-i\sigma\tilde{x}\cdot y}\,u(y)dy\,.$$

Then we have another type of the invariance principle from Theorems 4.6 and 4.10 similarly as in Mochizuki-Uchiyama [15] (see also Kako [7]):

$$\Omega_+(L_1, L_0)\mathcal{E}_0(I)=\text{s-}\lim_{t\to\infty} e^{it\phi(L_1)}J_+(I)e^{-it\phi(L_0)}\mathcal{E}_0(I),$$

where $\mathcal{E}_0(\lambda)$ is the spectral measure of L_0.

§ 5. Proof of Lemma 4.5.

This section is devoted to the proof of Theorem 4.5, which will be carried out along the ideas in Calderón-Vaillancourt [1] and Mochizuki-Uchiyama [15].

LEMMA 5.1 (*Calderón-Vaillancourt* [1]). *Let I be a bounded interval of $\boldsymbol{R}$ and let $B(r)$ with $r\in I$ be a weakly measurable and uniformly bounded family of operators in a separable Hilbert space $\mathcal{H}$. If*

$$\|B(r)B^*(r')\|\le h^2(r, r'),\ \|B^*(r)B(r')\|\le h^2(r, r')$$

for $r, r'\in I$ with a non-negative function $h(r, r')$ which is the kernel of a bounded operator in $L^2(I)$ with norm bounded by M, then the operator $\int_I B(r)dr$ defined by

$$\left(\int_I B(r)dr\right)f=\int_I B(r)f\,dr,\ f\in\mathcal{H}$$

is a bounded operator in $\mathcal{H}$ with norm bounded by M.

Let $\zeta(x)$ be a $C_0^\infty(\boldsymbol{R}^n)$-function such that $0\le\zeta\le1$, $\zeta(x)=1$ if $|x|\le1$ and $\zeta(x)=0$ if $|x|\ge2$. We set

$$b_\varepsilon(\sigma, \lambda, r, \tilde{x}, s, t)=\zeta(\varepsilon x)^2\chi_2(|x|/(s+t))^2a(\sigma, x)\overline{a(\lambda, x)},\tag{5.1}$$

$$\Theta(\sigma, \lambda, r, \tilde{x}, s, t)=\lambda^2s+\phi(\lambda^2)t-K(x, \lambda)-[\sigma^2s+\phi(\sigma^2)t-K(x, \sigma)],$$

where $r=|x|$, $\tilde{x}=x/|x|$ and $\varepsilon>0$.

LEMMA 5.2. *Let $B_\varepsilon(s, t)$ be the operator defined by*

$$[B_\varepsilon(s, t)\phi](\sigma, \tilde{x})=\int_0^\infty\int_\Sigma e^{i\Theta(\sigma,\lambda,r,\tilde{x},s,t)}b_\varepsilon(\sigma, \lambda, r, \tilde{x}, s, t)\phi(\lambda, \tilde{x})d\lambda dr$$

for $\phi\in L^2(\Sigma\times S^{n-1})$. Then we can find a constant $T''>T'$ having the following property: For any $s>0$ and $t>T''$ there exists a constant ε_0 such that if $s>0$, $t>$

T'' and $0<\varepsilon<\varepsilon_0$, then $B_\varepsilon(s, t)$ is bounded in $L^2(\Sigma\times S^{n-1})$ and satisfies

$$\|B_\varepsilon(s, t)\|\leq C(s+t)^{-2(1+\delta)},$$

where the constant $C>0$ is independent of s, t and ε.

PROOF. For $s>0$ and $t>T'$, put $\varepsilon_0=3/d(s+t)$, where d is the constant given by (4.5). Let $I_{s,t,\varepsilon}$ be the interval defined by $I_{s,t,\varepsilon}=[3d(s+t)/4, 2/\varepsilon]$ for $s>0$, $t>T'$ and $\varepsilon\in(0, \varepsilon_0)$. Note that the support of b_ε in r is contained in $I_{s,t,\varepsilon}$. We define the family of operators in $L^2(\Sigma\times S^{n-1})$ with parameters s, t and ε by

$$[B_\varepsilon(r, s, t)\phi](\sigma, \tilde{x})=\int_\Sigma e^{i\Theta(\sigma,\lambda,r,\tilde{x},s,t)}b_\varepsilon(\sigma, \lambda, r, \tilde{x}, s, t)\phi(\lambda, \tilde{x})d\lambda.$$

Then each $B_\varepsilon(r, s, t)$ is bounded and selfadjoint in $L^2(\Sigma\times S^{n-1})$. Since

$$\|B_\varepsilon(r, s, t)\|\leq \sup_{\tilde{x}\in S^{n-1}}\left\{\int_\Sigma\int_\Sigma|b_\varepsilon(\sigma, \lambda, r, \tilde{x}, s, t)|^2d\sigma d\lambda\right\}^{1/2},$$

it follows from (4.4) and (5.1) that

$$\|B_\varepsilon(r, s, t)\|\leq C$$

for $t>T'$, $\varepsilon\in(0, \varepsilon_0)$ and $r\in I_{s,t,\varepsilon}$. Furthermore Lebesgue's theorem implies that $B_\varepsilon(r, s, t)$ is strongly continuous in $r\in I_{s,t,\varepsilon}$ and thus

$$B_\varepsilon(s, t)=\int_{I_{s,t,\varepsilon}}B_\varepsilon(r, s, t)\,dr.$$

Now we claim that there exist a constant $T''>T'$ and a kernel $h_\varepsilon(r, r', s, t)$ such that if $s>0$, $t>T''$, $\varepsilon\in(0, \varepsilon_0)$ and r, $r'\in I_{s,t,\varepsilon}$, then

$$\|B_\varepsilon(r, s, t)B_\varepsilon(r', s, t)\|\leq h_\varepsilon^2(r, r', s, t), \tag{5.2}$$

$$\int_{I_{s,t,\varepsilon}}\left|\int_{I_{s,t,\varepsilon}}h_\varepsilon(r, r', s, t)f(r')dr'\right|^2dr \leq C(s+t)^{-4(1+\delta)}\int_{I_{s,t,\varepsilon}}|f(r')|^2dr' \tag{5.3}$$

for $f(r)\in L^2(I_{s,t,\varepsilon})$, where the constant $C>0$ is independent of s, t and ε. To this end, consider the kernel function $G_\varepsilon(\sigma, \lambda, r, r', \tilde{x}, s, t)$ of $B_\varepsilon(r, s, t)B_\varepsilon(r', s, t)$:

$$G_\varepsilon(\sigma, \lambda, r, r', \tilde{x}, s, t)=\int_\Sigma e^{i\Psi(\mu,\sigma,\lambda,r,r',\tilde{x},s,t)}g_\varepsilon(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)\,d\mu\,;$$

$$\begin{aligned}\Psi(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)&=\Theta(\sigma, \mu, r, \tilde{x}, s, t)+\Theta(\mu, \lambda, r, \tilde{x}, s, t)\\&=-(K(r\tilde{x}, \mu)-K(r'\tilde{x}, \mu))+(\lambda^2-\sigma^2)s\\&\quad+(\phi(\lambda^2)-\phi(\sigma^2))t+K(r\tilde{x}, \sigma)-K(r'\tilde{x}, \lambda),\end{aligned}$$

$$g_\varepsilon(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)=b_\varepsilon(\sigma, \mu, r, \tilde{x}, s, t)b_\varepsilon(\mu, \lambda, r', \tilde{x}, s, t).$$

Then it follows from an immediate calculation that

$$\partial_\mu \Psi(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)=-(r-r')(1+Z(\mu, r, r', \tilde{x}));$$

$$Z(\mu, r, r', \tilde{x})=\int_0^1 (\partial_r \partial_\mu Y)(\{\tau r+(1-\tau)r'\}\tilde{x}, \mu)\,d\tau,$$

with $Y(x, \sigma)$ defined by (2.2). In virtue of (2.3), there exists a constant $T''>T'$ such that if $t>T''$, r, $r'\in I_{s,t,\varepsilon}$ and $\mu\in\Sigma$, then

$$|Z(\mu, r, r', \tilde{x})|\leq 1/2,$$

$$|\partial_\mu^k Z(\mu, r, r', \tilde{x})|\leq C_k \text{ for any integer } k\geq 1.$$

Hence the following inequalities hold if $t>T''$, r, $r'\in I_{s,t,\varepsilon}$ and μ, σ, $\lambda\in\Sigma$:

$$|\partial_\mu \Psi(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)|\geq\frac{1}{2}|r-r'|,$$

$$|\partial_\mu^k \Psi(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)|\leq C_k|r-r'| \text{ for any integer } k\geq 2.$$

From (4.4) and (5.1) we have for any integer $k\geq 0$

$$|\partial_\mu^k g_\varepsilon(\mu, \sigma, \lambda, r, r', \tilde{x}, s, t)|\leq C_k(s+t)^{-4(1+\delta)}$$

if $t>T''$, $\varepsilon\in(0, \varepsilon_0)$, r, $r'\in I_{s,t,\varepsilon}$ and μ, σ, $\lambda\in\Sigma$. We can now apply Proposition 3.1 to get for any large integer N

$$|G_\varepsilon(\sigma, \lambda, r, r', \tilde{x}, s, t)|\leq C_N(s+t)^{-4(1+\delta)}(1+|r-r'|)^{-N}$$

if $s>0$, $t>T''$, $\varepsilon\in(0, \varepsilon_0)$, r, $r'\in I_{s,t,\varepsilon}$ and μ, σ, $\lambda\in\Sigma$, which implies that for $s>0$, $t>T''$, $\varepsilon\in(0, \varepsilon_0)$ and r, $r'\in I_{s,t,\varepsilon}$,

$$\|B_\varepsilon(r, s, t)B_\varepsilon(r', s, t)\|\leq C_N(s+t)^{-4(1+\delta)}(1+|r-r'|)^{-N},$$

where the constant $C_N>0$ is independent of s, t and ε. Hence the function $h_\varepsilon(r, r', s, t)=\sqrt{C_N}(s+t)^{-2(1+\delta)}(1+|r-r'|)^{-N/2}$ fulfills (5.2) and (5.3) for N large. Applying Lemma 5.1 to $B_\varepsilon(s, t)$, we conclude the proof of the lemma.

Proof of Lemma 4.5. By (2.3) there exists a constant $T>T''$ such that if $t>T$, $|x|\geq 3d(s+t)/4$ and $\sigma\in\Sigma$, then

$$|\partial_\sigma Y(x, \sigma)|\leq\frac{1}{2}|x|, \text{ hence } |\partial_\sigma K(x, \sigma)|\geq\frac{1}{2}|x|.$$

We apply Proposition 3.1 to $[A(s, t)\psi](x)$ to obtain for large N

$$|[A(s, t)\psi](x)|\leq C_{s,t}(1+|x|)^{-N} \text{ if } t>T,$$

that is, $A(s, t)\psi\in L^2(\boldsymbol{R}^n)$ for each $s>0$ and $t>T$. Therefore we can use Lebesgue's theorem and Fubini's theorem to get for $s>0$ and $t>T$

$$\|A(s,\ t)\phi\|^2=\lim_{\varepsilon\downarrow 0}\int_{\boldsymbol{R}^n}|\zeta(\varepsilon x)[A(s,\ t)\phi](x)|^2dx$$

$$=\lim_{\varepsilon\downarrow 0}(B_\varepsilon(s,\ t)\phi,\ \phi)_{L^2(\Sigma\times S^{n-1})}.$$

This and Lemma 5.2 complete the proof of Lemma 4.5.

§6. Proof of Theorem 1.7.

LEMMA 6.1. *Suppose that Assumption* 1.6 *is fulfilled. Then*

$$\|(L_1-L_0)e^{-iW_\pm(t)}u\|\rightarrow 0 \quad as \quad t\rightarrow\infty \tag{6.1$_\pm$}$$

for $u\in\mathcal{D}=\{u\in\mathcal{S}(\boldsymbol{R}^n)$, *the Schwartz space*; $\hat{u}\in C_0^\infty(\boldsymbol{R}^n\backslash\{0\})\}$, *where*

$$e^{-iW_\pm(t)}=e^{\mp itH_0-iX_\pm(t)}.$$

PROOF. It is sufficient to prove only $(6.1)_+$ since $(6.1)_-$ is similarly verified. For $u\in\mathcal{D}$ we set

$$f(x,\ t)=(L_1-L_0)e^{-iW_+(t)}u(x).$$

Let $\{\chi_j\}_{j=1,2}$ be a partition of unity on $\boldsymbol{R}_+$ such that $\chi_1(s)=0$ for $s>1/2$ and $\chi_2(s)=0$ for $s<1/4$. Put

$$f_j(x,\ t)=\chi_j(4|x|/t)f(x,\ t)\ (j=1,\ 2).$$

We shall first estimate $f_1(x,\ t)$. Putting $x=ty$ we have

$$f_1(ty,\ t)=\int_{\boldsymbol{R}^n}e^{i\Psi(\xi,y,t)}a(\xi,\ y,\ t)d\xi\,;$$

$$\Psi(\xi,\ y,\ t)=\Psi_0(\xi,\ y,\ t)-X_+(\xi,\ t),$$

$$\Psi_0(\xi,\ y,\ t)=t(y\cdot\xi-|\xi|),$$

$$a(\xi,\ y,\ t)=(2\pi)^{-n/2}\chi_1(4|y|)[\sum_{j,k=1}^n\{(a^{jk}(ty)-\delta^{jk})\xi_j\xi_k+$$

$$-i(\partial_{x_j}a^{jk})(ty)\xi_k\}+V(ty)]\hat{u}(\xi).$$

Let $\Gamma=\{(\xi,\ y)\,;\ \hat{u}(\xi)\neq 0,\ \chi_1(4|y|)\neq 0\}$. It is easily seen that for $(\xi,\ y)\in\Gamma$ and $t>0$,

$$|\nabla_\xi\Psi_0(\xi,\ y,\ t)|\geq\frac{1}{2}t.$$

On the other hand, by (1.2) there exists a large constant $T>0$ such that

$$|\nabla_\xi X_+(\xi,\ t)|\leq\frac{1}{4}t$$

for $t>T$ and $\xi\in\operatorname{supp}\hat{u}$. Hence the following inequality holds:

$$|\nabla_\xi\Psi(\xi, y, t)|\geq\frac{1}{4}t \text{ for } (\xi, y)\in\Gamma \text{ and } t>T.$$

We also obtain for any α

$$|\partial_\xi^\alpha\Psi(\xi, y, t)|\leq C_\alpha t,\ |\partial_\xi^\alpha a(\xi, y, t)|\leq C_\alpha$$

if $(\xi, y)\in\Gamma$ and $t>T$. We can now apply Proposition 3.1 to get for any integer $N>0$ and $t>T$

$$|f_1(ty, t)|\leq C_N t^{-N}$$

uniformly for $|y|\leq 1/2$, which implies that

$$\|f_1(\cdot, t)\|\leq C_N t^{-N} \tag{6.1}$$

for $t>T$. As for $f_2(x, t)$, we have with some positive constant C

$$\|f_2(\cdot, t)\|\leq C\sup\{|a^{jk}(x)-\delta^{jk}|,\ |\partial_{x_j}a^{jk}(x)|,\ |V(x)|;$$
$$|x|\geq t,\ j,\ k=1, \cdots, n\},$$

which together with the assumption gives

$$\|f_2(\cdot, t)\|\leq C(1+t)^{-\delta}.$$

Combining this and (6.1) we have

$$\|(L_1-L_0)e^{-iW_+(t)}u\|\leq Ct^{-\delta}$$

for $t>T$. This completes the proof.

PROOF OF THEOREM 1.7. The first half of the assertion in the theorem is an immediate consequence of Assumption 1.6 and the following two well known inequalities (cf., e.g., Lax-Phillips [10], p. 95 and Mizohata [12], p. 451):

$$\int_{|x|<R}|f|^2dx\leq\frac{R^2}{2(n-2)}\int_{\boldsymbol{R}^n}|\nabla_x f|^2dx,\ R>0,$$

$$\int_{\boldsymbol{R}^n}\frac{|f|^2}{|x|^2}dx\leq\frac{4}{(n-2)^2}\int_{\boldsymbol{R}^n}|\nabla_x f|^2dx$$

for $n\geq 3$ and $f\in C_0^\infty(\boldsymbol{R}^n)$. We shall show the last half only for W_+ since W_- can be treated in a similar manner. It suffices to show that

$$\lim_{t\to\infty}\|(J-I)U_0(t)J_0^+(t)f\|_{\mathscr{H}_1}=0 \tag{6.2}$$

for $f={}^t(f_1, f_2)$ with $f_1, f_2\in\mathscr{D}$ since $\mathscr{D}$ is dense in $[\mathscr{D}(H_0)]$ and the operators are uniformly bounded. The proof of (6.2) to be carried out below is essentially the same as for Theorem XI. 76 of Reed-Simon [17]. Recall that $J=T_1^{-1}T_0$, and Jf is calculated as

$$Jf={}^t(H_1^{-1}H_0f_1,\ f_2).$$

If $u(t)$ is the first component of $U_0(t)J_0^+(t)f$, then

$$\begin{aligned}\|(J-I)U_0(t)J_0^+(t)f\|^2_{\mathcal{H}_1}&=\|H_1(H_1^{-1}H_0-I)u(t)\|^2\\&=\|(H_0-H_1)u(t)\|^2.\end{aligned}$$

Since

$$u(t)=\frac{1}{2}\{e^{-iW_+(t)}(f_1+iH_0^{-1}f_2)+e^{-iW_-(t)}(f_1-iH_0^{-1}f_2)\},$$

it suffices to show that for $u\in\mathcal{D}$,

$$(6.3)_\pm \qquad \|(H_0-H_1)e^{-iW_\pm(t)}u\|=\|e^{\pm itH_1}(H_0-H_1)e^{-iW_\pm(t)}u\|\to 0 \text{ as } t\to\infty.$$

We shall prove only $(6.3)_+$. Hereafter we denote $e^{-iW_+(t)}$ by $e^{-iW(t)}$. By Theorem 1.4 we know that

$$e^{itH_1}H_0e^{-iW(t)}u\to\Omega_+H_0u \text{ as } t\to\infty.$$

Hence to obtain $(6.3)_+$, we have only to show that

$$(6.4) \qquad e^{itH_1}H_1e^{-iW(t)}u=H_1e^{itH_1}e^{-iW(t)}u\to H_1\Omega_+u$$

since $H_1\Omega_+=\Omega_+H_0$ by the intertwining property. For $u\in\mathcal{D}$ we have

$$\begin{aligned}&\|H_1e^{itH_1}e^{-iW(t)}u-H_1\Omega_+u\|^2\\&=\|H_1e^{itH_1}e^{-iW(t)}u\|^2+\|H_1\Omega_+u\|^2+R(t),\end{aligned}$$

where we have put

$$\begin{aligned}R(t)&=-(H_1e^{itH_1}e^{-iW(t)}u,\ H_1\Omega_+u)-(H_1\Omega_+u,\ H_1e^{itH_1}e^{-iW(t)}u)\\&=-(e^{itH_1}e^{-iW(t)}u,\ H_1^2\Omega_+u)-(H_1^2\Omega_+u,\ e^{itH_1}e^{-iW(t)}u).\end{aligned}$$

Since $R(t)$ converges to $-2\|H_1\Omega_+u\|^2$, in order to conclude (6.4), it remains to show that

$$\varlimsup_{t\to\infty}\|H_1e^{itH_1}e^{-iW(t)}u\|^2\le\|H_1\Omega_+u\|^2.$$

In fact, the left-hand side of the above inequality is calculated as

$$\begin{aligned}\varlimsup_{t\to\infty}\|H_1e^{itH_1}e^{-iW(t)}u\|^2&=\varlimsup_{t\to\infty}(e^{-iW(t)}u,\ L_1e^{-iW(t)}u)\\&=\varlimsup_{t\to\infty}(e^{-iW(t)}u,\ L_0e^{-iW(t)}u)\end{aligned}$$

$$=||H_0 u||^2=||\Omega_+ H_0 u||^2=||H_1 \Omega_+ u||^2.$$

In the second equality we have used Lemma 6.1. Thus we have completed the proof of the theorem.

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Institute of Mathematics
University of Tsukuba
Ibaraki, 305 Japan