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SECONDARY COMPOSITION OPERATIONS IN HOMOTOPY PAIR THEORY

By

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Abstract. We study certain secondary composition operations that generalise the Toda brackets in order to develop a technique for systematic computation of stable and unstable homotopy pair groups of sphere classes.

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0. Introduction.

A long term goal of topologists has been the problem of enumerating and computing the set of homotopy classes $\pi(X, Y)$ of (pointed) continuous maps from a space X to a space Y. If X (or Y) has a suitable structure then $\pi(X, Y)$ acquires a group structure but the problem of computing such groups, even when X and Y are compact polyhedra, is still far out of reach. Indeed even when X and Y are spheres a complete solution to the problem has not been achieved but much information is available [17], [19]. A more general problem (that also promises to be relatively tractable in the sphere case) is to determine various *homotopy coherence class* sets and groups.

The notion of an n-cube of spaces is due to Brown and Loday [1] while homotopy coherence has been studied by a number of authors including Vogt [20], Cordier [2], [3], Cordier–Porter [4], [5]. A 1-cube of spaces with vertices X and Y is simply a continuons map $X \to Y$. The corresponding homotopy coherence class set is the classical homotopy set $\pi(X, Y)$. A 2-cube of spaces with vertices (X, Y) and (E, B) is a diagram of maps

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$$
(0.1)\qquad \begin{array}{ccc} X & \xrightarrow{\ \phi\ } & E \\ {\scriptstyle f}\big\downarrow & \overset{h_t}{\nearrow} & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow[\ \phi\ ]{} & B \end{array}
$$

where h_t is a homotopy with $h_0=\phi f$ and $h_1=g\phi$. The homotopy coherence class of the 2-cube 0.1 (a definition is given in [13]) is an element of the homotopy coherence class set $\pi\begin{pmatrix} X & E \\ Y & B\end{pmatrix}$. Instead of specifying only fixed vertices, one may wish to fix certain of the maps. For example if we fix the maps f and g then the corresponding homotopy coherence class set is $\pi(f, g)$, the *homotopy pair set* in the sense of [6], [8]. Alternatively, if we fix the vertex space X and the maps ϕ and g, then the corresponding homotopy coherence class set is the set $\pi(X\,;\,\phi, g)$. It is easy to see that $\pi(X\,;\,\phi, g)$ is in bijective correspondence with the set $\pi(X, Z)$, where Z denotes the standard homotopy pullback of the triad determined by ϕ and g.

The motivating force of this paper and its sequelae is the belief that the problem of computing homotopy coherence class sets will be found to be manageable when the vertex spaces are spheres and that the solution of such problems will be useful in (and should be undertaken as preliminary steps toward) the solution of the problem of computing $\pi(X, Y)$ when X and Y are finite CW-complexes. As evidence contributory to the belief we present here a discussion of the homotopy pair case. Specifically we indicate, in principle, how the homotopy pair set $\pi(f, g)$ can be determined when f and g are maps between spheres. For the purposes of the discussion we assume to be known (i) the homotopy groups of spheres (ii) the results of the composition operation and secondary homotopy composition operation applied to elements of the homotopy groups of spheres (iii) the results of the suspension operation and Hopf-James invariants applied to such elements. In brief we assume known the sort of information yielded by Toda's "composition method" [19] and find that the family of groups $\pi_n(f, g)=\pi(\Sigma^n f, g)$ $n>0$ can be determined by a kind of extension of the composition method.

As principal computational tool we use the Mayer-Vietoris sequence

$$
(0.2)\qquad \cdots \to \pi(\Sigma Y, B)\oplus\pi(\Sigma X, E) \xrightarrow{\begin{pmatrix}\Sigma f\\ -g\end{pmatrix}} \pi(\Sigma X, B) \xrightarrow{\ \nabla\ } \pi(f, g) \xrightarrow{(c,\, d)} \pi(Y, B)\oplus\pi(X, E)
$$

of [11 ; Proposition 3.11], but to settle group extension problems (when they arise) at $\pi(f, g)$ we use the secondary composition operations referred to in our title. These are discussed in section 1 and 3. They take the form of bracket operations, generalizing the classical Toda brackets [19], but in which one or more component may be permitted to be a homotopy pair class. They turn out to enjoy composition, additivity and suspension properties analogous to the classical Toda brackets and, in many cases, they can be computed either by reduction to Toda brackets or by analogous methods. As has been indicated elsewhere [7] the category of homotopy pairs offers a convenient setting for the definition and derivation of properties of Toda brackets. In section 1 we use it to define and study the new brackets.

In section 4 we present some examples in which the new brackets (with particular elements) are evaluated in the process of computing homotopy pair groups associated with the Hopf map $S^3 \to S^2$.

1. Homotopy pair brackets.

Recall that the objects of the *category of homotopy pairs* [8], [6] are (pointed) continuous maps and the morphisms from f to g are equivalence classes of diagrams 0.1. Specifically the square 0.1 is $\sim$—related to the composite (i.e. outer) square

$$
(1.1)\qquad
\begin{array}{ccc}
X & \xrightarrow{\ \psi_1\ } & E \\
\| & \psi_t \nearrow & \| \\
X & \xrightarrow{\ \psi=\psi_0\ } & E \\
f \downarrow & h_t \nearrow & \downarrow g \\
Y & \xrightarrow{\ \phi=\phi_0\ } & B \\
\| & \phi_{1-t} \nearrow & \| \\
Y & \xrightarrow[\ \phi_1\ ]{} & B
\end{array}
$$

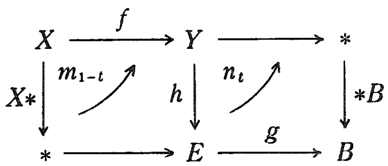
where ϕ_t and ψ_t are homotopies, and also to the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ \psi\ } & E \\
f \downarrow & h'_t \nearrow & \downarrow g \\
Y & \xrightarrow{\ \phi\ } & B
\end{array}
$$

if h_t and h'_t belong to the same *track* (i. e. relative homotopy class of homotopies). It is convenient to denote the *homotopy pair class* (i. e. $\sim$—equivalence class) of the square 0.1 by $\{\phi, \psi, h_t\}$. The equalities $d\{\phi, \psi, h_t\}=\{\psi\}$ and $c\{\phi, \psi, h_t\}=\{\phi\}$ define the *domain* and *codomain restriction operators* $d: \pi(f, g)\rightarrow\pi(X, E)$ and $c: \pi(f, g)\rightarrow\pi(Y, B)$. Also basic in homotopy pair theory is the *Puppe operator* $P: \pi(f, g)\rightarrow\pi(Pf, Pg)$. Recall that in Puppe's notation [16], $Pf: Y\rightarrow C_f$ refers to the inclusion of Y into the homotopy cofibre of f. As discussed in [8], P becomes (via the formula [16; (9)]) an endofunctor of the category of homotopy pairs of which the Puppe operator is the associated morphism function.

Let $f: X\rightarrow Y$, $h: Y\rightarrow E$, $g: E\rightarrow B$ be maps such that $hf\simeq *$ and $gh\simeq *$, and let $m_t: X\rightarrow E$ and $n_t: Y\rightarrow B$ be nullhomotopies of hf, gh respectively. Then the composite square

(1.2)



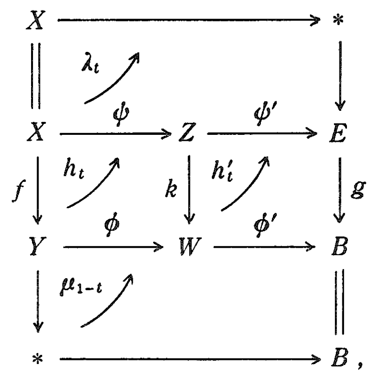
defines an element ξ of the homotopy pair set $\pi(X*, *B)$. Then the operator $cP: \pi(X*, *B)\rightarrow\pi(\Sigma X, B)$ applied to ξ selects the homotopy class of the map induced by 1.2 from the cofibre of $X*$ to the cofibre of $*B$. The Toda bracket set

(1.3) $$\{\{g\}, \{h\}, \{f\}\}\subseteqq\pi(\Sigma X, B)$$

is defined to be the set $\{cP(\xi) \mid \text{nullhomotopies } m_t, n_t\}$, which turns out to be a double coset of the subgroups $\pi(\Sigma Y, B)\circ\{\Sigma f\}$ and $\{g\}\circ\pi(\Sigma X, E)$ in $\pi(\Sigma X, B)$.

Now let $\alpha=\{\phi, \psi, h_t\}\in\pi(f, k)$ and $\beta=\{\phi', \psi', h'_t\}\in\pi(k, g)$ be homotopy pair classes such that $d(\beta\circ\alpha)=0$ and $c(\beta\circ\alpha)=0$. Then there exist diagrams

(1.4)



where λ_t and μ_t are nullhomotopies of $\psi'\psi$ and $\phi'\phi$ respectively. If $\xi\in\pi(X*, *B)$ is the element defined by the composite square 1.4 then we define the *homotopy pair bracket set*

(1.5) $$\{\beta,\ \alpha\}=\{cP(\xi)\mid \text{nullhomotopies } \lambda_t,\ \mu_t\}\subseteqq\pi(X,\ B)\,.$$

The (straightforward) proof of the following proposition is left to the reader.

1.6. PROPOSITION. *The bracket set $\{\beta,\ \alpha\}$ as defined above is independent of the choice of representative squares $(\phi,\ \psi,\ h_t)$ and $(\phi',\ \psi',\ h'_t)$, and is a double coset of the subgroups $\pi(\Sigma Y,\ B)\circ\{\Sigma f\}$ and $\{g\}\circ\pi(\Sigma X,\ E)$.*

1.7. REMARK. If we choose α, β to be the classes associated with the squares

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{\ h\ } & E \end{array}\ ,\qquad \begin{array}{ccc} Y & \xrightarrow{\ h\ } & E \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ E & \xrightarrow{\ g\ } & B \end{array}$$

regarded as commuting via constant homotopies then, in the case $hf\simeq*$ and $gh\simeq*$, we recover $\{\{g\},\ \{h\},\ \{f\}\}=-\{\beta,\ \alpha\}$.

A disadvantage of the homotopy pair bracket operation 1.5 is that its elements β and α are homotopy pair classes. For such classes canonical notation is not always available—at least ab initio. To remedy this difficulty we consider also somewhat coarser operations of the following type. Let $\alpha\in\pi(f,\ k)$, $\beta_1\in\pi(W,\ B)$, $\beta_2\in\pi(Z,\ E)$ be classes satisfying

$$\beta_2\circ d\alpha=0,\qquad \beta_1\circ c\alpha=0,\qquad \{g\}\circ\beta_2=\beta_1\circ\{k\}$$

and let

(1.8) $$\{\{g\}^{\beta_2}_{\beta_1},\ \alpha\}\subseteqq\pi(\Sigma X,\ B)$$

be the set of all elements represented by diagrams of type 1.4, where $\{\psi'\}=\beta_2$, $\{\phi'\}=\beta_1$ and $\alpha=\{\phi,\ \psi,\ h_t\}$. Since we have

(1.9) $$\{\{g\}^{\beta_2}_{\beta_1},\ \alpha\}=\cup\{\{\beta,\ \alpha\}\mid d\beta=\beta_2,\ c\beta=\beta_1\}$$

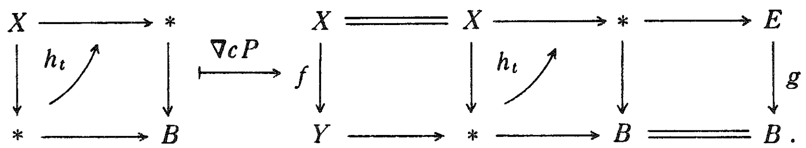
it is clear that the indeterminacy of the bracket 1.8 is larger than the indeterminacy of 1.5. For a detailed discussion see §5.

2. The Mayer-Vietoris sequence.

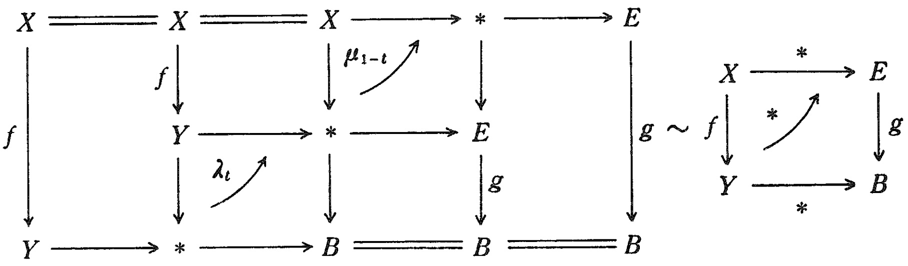
In the sequel we shall most frequently be working with the sequence 0.2 in the region somewhat to the left of the segment displayed. Then it takes the form

$$\cdots \to \pi_n(Y, B)\oplus\pi_n(X, E) \xrightarrow{\binom{.\Sigma^n f}{-g.}} \pi_n(X, B) \xrightarrow{\nabla} \pi_{n-1}(f, g) \tag{2.1}$$
$$\xrightarrow{(c, d)} \pi_{n-1}(Y, B)\oplus\pi_{n-1}(X, E)$$

and its exactness can be inferred from Wall's principles [21] applied to the braid diagram [11; 3.10]. To make the argument essentially self-contained (and also because we wish to add certain refinements to observations already in print) we give a direct proof of exactness for the segment $n=1$ shown in 0.2. As indicated in [9], $cP: \pi(X*, *B)\to\pi(\Sigma X, B)$ is a bijection. (Actually an antiisomorphism if we endow $\pi(X*, *B)$ with the track addition.) The operator ∇cP acts by appending commutative squares on the right and left:



Exactness at $\pi(f, g)$ (as pointed set) is then obvious. To check that $\nabla\binom{.\Sigma f}{-g.}=0$ it is sufficient to observe that



which is clear. Moreover if $\nabla cP\{*, *, h_t\}=0$ in $\pi(f, g)$ then, by the $\sim$ relation, $h_t\simeq v_t f+g w_t$ for homotopies $v_t: Y\to B$ and $w_t: X\to E$. (Here $+$ refers to concatenation and $\simeq$ to the relation of relative homotopy). It follows that $Pc\{*, *, h_t\}$ belongs to the image of $\binom{.\Sigma f}{-g.}$. The exactness at $\pi(\Sigma Y, B)\oplus\pi(\Sigma X, E)$ is easy to check provided one understands that (by definition)

$$\left(\begin{array}{ccc} Y & \longrightarrow & * \\ \downarrow & \overset{\lambda_t}{\nearrow} & \downarrow \\ * & \longrightarrow & B \end{array}, \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \overset{\mu_t}{\nearrow} & \downarrow \\ * & \longrightarrow & E \end{array}\right) \overset{\binom{\cdot\Sigma f}{-g\cdot}}{\longmapsto} \begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \overset{\lambda_t f+g\mu_{1-t}}{\nearrow} & \downarrow \\ * & \longrightarrow & B . \end{array}$$

The exactness at $\pi_1(X, B)$ admits of some enrichment. First, by an argument similar to one given above, one can prove the following (c. f. [12; Theorem 3.5]).

2.2. PROPOSITION. *The images of two elements under* ∇ *coincide if and only if they belong to the same double coset of the subgroups* $\pi(\Sigma Y, B)\circ\{\Sigma f\}$ *and* $\{g\}\circ\pi(\Sigma X, E)$.

If the spaces X and Y are suspensions then the exactness can be further enriched by an action of $\pi_1(X, B)$ on $\pi(f, g)$ enabling $\pi(f, g)$ to be enumerated.

2.3. PROPOSITION. *If* $X=\Sigma X'$ *and* $Y=\Sigma Y'$ *then there is a bijection*

$$\pi(f, g) \longleftrightarrow \bigcup_{\alpha\in\pi(X,B)} \ \bigcup_{(\beta,\gamma)\in g^{-1}\cdot(\alpha)\times(\cdot f)^{-1}\alpha} C(\alpha, \beta, \gamma),$$

where $C(\alpha, \beta, \gamma)$ *is the cokernel of the homomorphism*

$$\begin{pmatrix}\Gamma(\gamma, f)\\ \nabla(g, \beta)\end{pmatrix}: \pi_1(Y, B)\oplus\pi_1(X, E) \longrightarrow \pi_1(X, B),$$

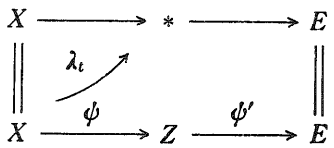
where $\Gamma(\gamma, f)$ *and* $\nabla(g, \beta)$ *are the homomorphisms described by Rutter* [18; 3.2, 1.2].

The bijection of Proposition 2.3 is consequence of the classification theorem given [12; 3.6] in the basepoint free case (see also [12; Remark 3.9]).

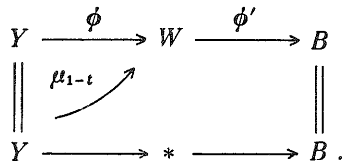
When attempting to make computations using exact sequences of groups, one soon encounters problems of group extension. Typically these have been resolved only after resort to some secondary operation (cf. [19; Proposition 1.9]). In the case of the Mayer-Vietoris sequence 2.1, the appropriate operation to consider appears to be the coarser homotopy pair bracket 1.8.

2.4. PROPOSITION. *Let* $\alpha\in\pi(f, k)$, $\beta_1\in\pi(W, B)$, $\beta_2\in\pi(Z, E)$ *be classes (as in* 1.8) *satisfying* $\beta_2\circ d\alpha=0$, $\beta_1\circ c\alpha=0$ *and* $\{g\}\circ\beta_2=\beta_1\circ\{k\}$. *Then in the sequence* 0.2 *we have* $((c, d)^{-1}(\beta_1, \beta_2))\circ\alpha=\nabla\{\{g\}^{\beta_2}_{\beta_1}, \alpha\}$.

PROOF. First note that applying ∇cP to the square 1.4 has the effect of replacing the upper rectangle by the rectangle



and of replacing the lower rectangle by



Let $\beta \in \pi(k, g)$ be a typical element of $(c, d)^{-1}(\beta_1, \beta_2)$. (Such elements exist since $\{g\} \circ \beta_2 = \beta_1 \circ \{k\}$.) Then $\{\beta, \alpha\}$ is well-defined, and, in view of the $\sim$ relation, $\beta \circ \alpha = \nabla \{\beta, \alpha\} \subseteqq \nabla \{\{g\}^{\beta_2}_{\beta_1}, \alpha\}$. Thus $((c, d)^{-1}(\beta_1, \beta_2) \circ \alpha \subseteqq \nabla \{\{g\}^{\beta_2}_{\beta_1}, \alpha\}$. Applying 1.9, we have $\nabla \{\{g\}^{\beta_2}_{\beta_1}, \alpha\} \subseteqq \cup \{\nabla \{\beta, \alpha\} \mid d\beta = \beta_2, c\beta = \beta_1\} \subseteqq ((c, d)^{-1}(\beta_1, \beta_2)) \circ \alpha$, which completes the proof.

2.5. REMARK. We retain the usual convention with regard to secondary operations that if a bracket would be undefined because of the non-vanishing of some composition then the bracket set is empty.

2.6. REMARK. We wish to apply Proposition 2.4 most frequently in the case in which α is a multiple of an identity class so that $d\alpha$ is a suspension class. Then corollary 5.3 provides full information concerning the indeterminacy of $\{\{g\}^{\beta_2}_{\beta_1}, \alpha\}$.

3. Bracket properties.

The homotopy pair brackets enjoy properties analogous to those of the Toda brackets [19]. If $\{\beta, \alpha\}$ coincides with its indeterminacy we write $\{\beta, \alpha\} = 0$. Corresponding to [19; Proposition 1.2] we have:

3.1. PROPOSITION. *Given* $\alpha \in \pi(f, k)$, $\beta \in \pi(k, g)$, $\gamma \in \pi(g, h)$,

(0) $\{0, \alpha\} = 0$ *and* $\{\beta, 0\} = 0$;

(i) *If* $d(\gamma \circ \beta \circ \alpha) = 0$ *and* $c(\gamma \circ \beta \circ \alpha) = 0$ *then*

$$\{\gamma \circ \beta, \alpha\} = \{\gamma, \beta \circ \alpha\};$$

(ii) *If* $d(\beta \circ \alpha) = 0$ *and* $c(\beta \circ \alpha) = 0$ *then*

$$\{\gamma\circ\beta,\ \alpha\} \supseteqq c\gamma\circ\{\beta\circ\alpha\}\ ;$$

(iii) *If* $d(\gamma\circ\beta)=0$ *and* $c(\gamma\circ\beta)=0$ *then*

$$\{\gamma,\ \beta\circ\alpha\} \supseteqq \{\gamma,\ \beta\}\circ\Sigma d\alpha\ ;$$

(iv) *If* $d(\beta\circ\alpha)=0$ *and* $c(\gamma\circ\beta)=0$ *then*

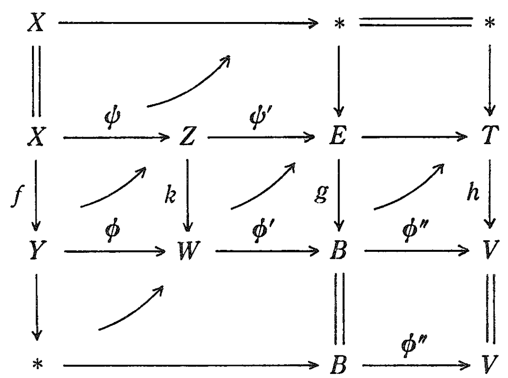
$$\{c\gamma,\ c\beta,\ c\alpha\circ\{f\}\} \subseteqq \{c\gamma,\ c\beta\circ\{k\},\ d\alpha\} \supseteqq \{\{h\}\circ d\gamma,\ d\beta,\ d\alpha\}\ ;$$

and

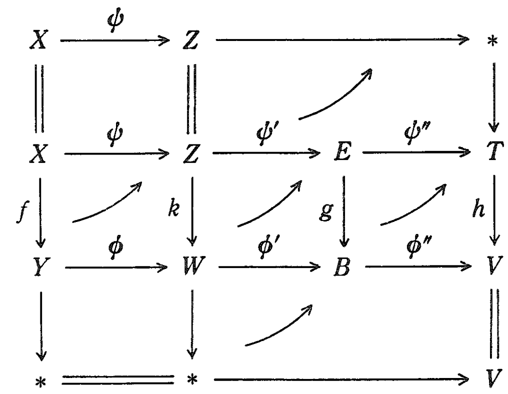
$$\{c\gamma,\ c\beta,\ c\alpha\circ\{f\}\} \cap \{\{h\}\circ d\gamma,\ d\beta,\ d\alpha\} \cap \{\gamma\circ\beta,\ \alpha\} \neq \emptyset\ ,$$

where the triple brackets are Toda brackets.

PROOF. (0) and (i) are obvious. (ii) The following diagram describes a typical element of $c\gamma\circ\{\beta,\ \alpha\}$, which also belongs to $\{\gamma,\ \beta\circ\alpha\}=\{\gamma\circ\beta,\ \alpha\}$.



(iii) Similar to (ii). The following diagram is relevant.



(iv) It is easy to check from $d(\beta\circ\alpha)=0$ and $c(\gamma\circ\beta)=0$ that the Toda brackets are defined. Then

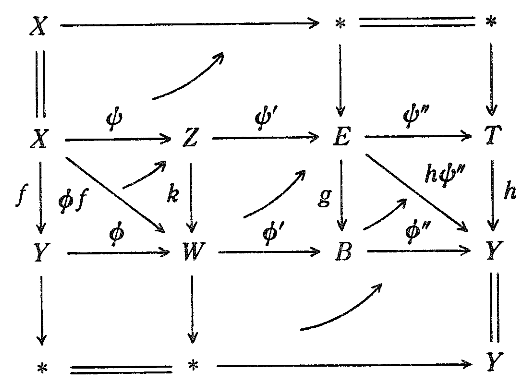
$$\{c\gamma,\ c\beta,\ c\alpha\circ\{f\}\}=\{c\gamma,\ c\beta,\ \{k\}\circ d\alpha\}\subseteq\{c\gamma,\ c\beta\circ\{k\},\ d\alpha\},$$

by [19; Proposition 1.2 (ii)], and

$$\{c\gamma,\ c\beta\circ\{k\},\ d\alpha\}=\{c\gamma,\ \{g\}\circ d\beta,\ d\alpha\}\supseteq\{c\gamma\circ\{g\},\ d\beta,\ d\alpha\},$$

[19; Proposition 1.2 (iii)].

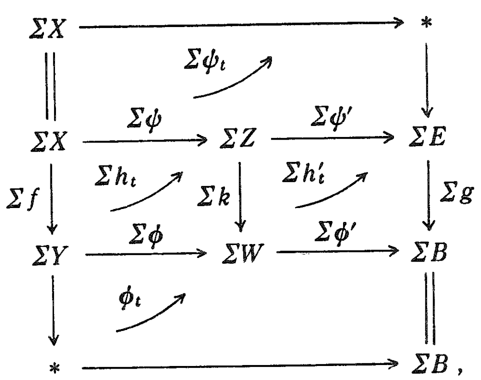
Since $d(\beta\circ\alpha)=0$ and $c(\gamma\circ\beta)=0$ there is a diagram



that represents an element common to the three cosets.

Next we consider analogues of Toda's brackets of type $\{-,\ -,\ -\}_n$ and their interaction with suspension. Let $\alpha\in\pi(f,\ k)$, $\beta\in\pi(k,\ g)$ be such that $d(\beta\circ\alpha)=0$ in $\pi(X,\ E)$ and $\Sigma c(\beta\circ\alpha)=0$ in $\pi(\Sigma Y,\ \Sigma B)$. By considering the set of all elements of the form

(3.2)



where ϕ_t is a nullhomotopy of $\phi'\phi$, $h_t:\phi f\simeq k\phi$ and $h'_t:\phi' k\simeq g\phi'$, we obtain a bracket

(3.3) $$\{\Sigma\beta,\ \Sigma\alpha\}^1$$

with (reduced) indeterminacy $(\Sigma^2 f)^*\pi(\Sigma^2 Y, \Sigma B)\oplus(\Sigma g)_*\Sigma(\pi(\Sigma X, E))$. Similarly, if α and β are such that $\Sigma d(\beta\circ\alpha)=0$ and $c(\beta\circ\alpha)=0$, with the obvious modification of diagram 3.2 we obtain a bracket $\{\Sigma\beta, \Sigma\alpha\}_1$ with indeterminacy $(\Sigma^2 f)^*\Sigma(\pi(\Sigma Y, B))\oplus(\Sigma g)_*\pi(\Sigma^2 X, \Sigma E)$. More generally it is clear that brackets of type $\{\Sigma^r\beta, \Sigma^r\alpha\}_n^m$ $(n, m\leqq r)$ and of type $\{\{\Sigma^r g\}_{\Sigma^r\beta_1}^{\Sigma^r\beta_2}, \Sigma^r\alpha\}_n^m$ can be defined.

Let $\Omega_0: \pi(\Sigma^2 X, \Sigma B)\to\pi(\Sigma X, \Omega\Sigma B)$ denote the adjoint isomorphism and let $i_g\in\pi(g, \Omega\Sigma g)$ denote the homotopy pair class of the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ i_E\ } & \Omega\Sigma E \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Omega\Sigma g} \\ B & \xrightarrow{\ i_B\ } & \Omega\Sigma B \end{array}$$

(which commutes by the constant homotopy). An analogue of [19; Proposition 1.3] is the following.

3.4. Proposition. *If* $\alpha\in\pi(f, k)$ *and* $\beta\in\pi(k, g)$ *then*

(i) $\Omega_0\{\Sigma\beta, \Sigma\alpha\}=-\{i_g\circ\beta, \alpha\}$.

(ii) $\{\Sigma^r\beta, \Sigma^r\alpha\}_n^m=-\Sigma\{\Sigma^{r-1}\beta, \Sigma^{r-1}\alpha\}_{n-1}^{m-1}$.

A proof of 3.4(i) can be constructed by modifying the argument in [7; Proof of 1.8]. The arguments required for 3.4(ii) are standard. The details are left to the reader.

We next consider additivity properties, obtaining the following analogue of [19; Proposition 1.6].

3.5. Proposition. *Let* $\alpha, \alpha_1, \alpha_2\in\pi(f, k)$; $\beta, \beta_1, \beta_2\in\pi(k, g)$.

(i) *If* $n\geqq 1$, *or* $f=\Sigma f'$ *and* $n=0$ *then*

$$\{\Sigma^n\beta, \Sigma^n\alpha_1\}^n+\{\Sigma^n\beta, \Sigma^n\alpha_2\}^n\supseteqq\{\Sigma^n\beta, \Sigma^n\alpha_1+\Sigma^n\alpha_2\}^n.$$

(ii) *If* $n\geqq 1$, *or* $\alpha=\Sigma\alpha'$ *and* $n=0$ *then*

$$\{\Sigma^n\beta_1, \Sigma^n\alpha\}^n+\{\Sigma^n\beta_2, \Sigma^n\alpha\}^n\supseteqq\{\Sigma^n\beta_1+\Sigma^n\beta_2, \Sigma^n\alpha\}^n.$$

(iii) (i) *and* (ii) *with* $\{-, -\}_n$ *replacing* $\{-, -\}^n$.

We shall prove (i) in the case $f=\Sigma f'$ and $n=0$, leaving the remainder to the reader. Suppose that the diagrams

$$\begin{array}{ccc} \Sigma X' & \xrightarrow{\psi_i} & Z \\ {\scriptstyle f=\Sigma f'}\downarrow & \nearrow{\scriptstyle h_t^i} & \downarrow{\scriptstyle k} \\ \Sigma Y' & \xrightarrow[\phi_i]{} & W \end{array} \quad (i=1, 2), \qquad \begin{array}{ccc} Z & \xrightarrow{\psi'} & E \\ {\scriptstyle k}\downarrow & \nearrow{\scriptstyle h_t'} & \downarrow{\scriptstyle g} \\ W & \xrightarrow[\phi']{} & B \end{array}$$

are representatives of the classes α_i $(i=1, 2)$ and β, respectively. Recall [8; Lemma 1.2] that

$$(3.6) \qquad \alpha_1+\alpha_2=\{\phi_1\oplus\phi_2,\ \psi_1\oplus\psi_2,\ h_t^1\oplus h_t^2\},$$

where, for example, $\psi_1\oplus\psi_2 : \Sigma X'\to Z$ is given by

$$(\psi_1\oplus\psi_2)(x, s)=\begin{cases} \psi_1(x, 2s) & \left(0\leqq s\leqq \frac{1}{2}\right) \\ \psi_2(x, 2s-1) & \left(\frac{1}{2}\leqq s\leqq 1\right) \end{cases} \quad (x\in X',\ s\in I).$$

Now let λ_t^i, respectively μ_{1-t}^i, $i=1, 2$, be nullhomotopies of $\psi'\psi_i$, respectively $\phi'\phi_i$. Then the diagram

$$(3.7) \qquad \begin{array}{ccccc} \Sigma X' & \longrightarrow & \longrightarrow & \longrightarrow & * \\ \| & \nearrow{\scriptstyle \lambda_t^1\oplus\lambda_t^2} & & & \downarrow \\ \Sigma X' & \xrightarrow{\psi_1\oplus\psi_2} & Z & \xrightarrow{\psi'} & E \\ {\scriptstyle f=\Sigma f'}\downarrow & \nearrow{\scriptstyle h_t^1\oplus h_t^2} & \downarrow{\scriptstyle k} & \nearrow{\scriptstyle h_t'} & \downarrow{\scriptstyle g} \\ \Sigma Y' & \xrightarrow[\phi_1\oplus\phi_2]{} & W & \xrightarrow[\phi']{} & B \\ \downarrow & & \nearrow{\scriptstyle \mu_t} & & \| \\ * & \longrightarrow & \longrightarrow & \longrightarrow & B \end{array}$$

represents an element of the coset $\{\beta, \alpha_1+\alpha_2\}$. However, it can be checked that

$$\phi'(h_t^1\oplus h_t^2)+h_t'(\psi_1\oplus\psi_2)=(\phi' h_t^1+h_t'\psi_1)\oplus(\phi' h_t^2+h_t'\psi_2)$$

and hence that the element represented by 3.7 also belongs to $\{\beta, \alpha_1\}+\{\beta, \alpha_2\}$. Since the indeterminacy of $\{\beta, \alpha_1+\alpha_2\}$ is contained in the sum of the respective indeterminacies of $\{\beta, \alpha_1\}$ and $\{\beta, \alpha_2\}$, this establishes the desired inclusion.

In the unstable range, interaction of Toda brackets with the Hopf-James invariant is often the key to their detection. Recall that, for a connected cell complex B, there is a homotopy equivalence $B_\infty\to\Omega\Sigma B$ [15], where B_∞ denotes James' reduced product space. Then, given a map $g: E\to B$ there exists a diagram

$$\begin{array}{ccc} E_\infty & \longrightarrow & \Omega\Sigma E \\ {\scriptstyle g_\infty}\downarrow & & \downarrow{\scriptstyle \Omega\Sigma g} \\ B_\infty & \longrightarrow & \Omega\Sigma B \end{array}$$

that commutes via a canonical homotopy. In view of [8; Theorem 1.3] the diagram yields a homotopy pair equivalence $g_\infty \to \Omega\Sigma g$, and hence a bijection

$$\Omega_1 : \pi(\Sigma k, \Sigma g) \longrightarrow \pi(k, g_\infty),$$

the homotopy pair analogue of [19; (2.4)]. Since there is a commutative diagram

$$\begin{array}{ccc} E_\infty & \xrightarrow{h_E} & (E\#E)_\infty \\ {\scriptstyle g_\infty}\downarrow & & \downarrow{\scriptstyle (g\#g)_\infty} \\ B_\infty & \xrightarrow[h_B]{} & (B\#B)_\infty, \end{array}$$

where h_E and h_B are James maps, the diagram together with Ω_1 induces an operator

$$H: \pi(\Sigma k, \Sigma g) \longrightarrow \pi(\Sigma k, \Sigma(g\#g)), \tag{3.8}$$

the homotopy pair version of the Hopf-James invariant homomorphism $H: \pi(\Sigma^2 X, \Sigma B) \to \pi(\Sigma^2 X, \Sigma(B\#B))$. Then we have the following c. f. [19; Proposition 2.3].

3.9. Proposition. (i) *If* $\beta \in \pi(\Sigma k, \Sigma g)$ *and* $\alpha \in \pi(\Sigma f, \Sigma k)$ *then* $H\{\beta, \alpha\} \subseteqq \{\boldsymbol{H}\beta \circ \alpha\}$.

(ii) *If* $\beta_1 \in \pi(\Sigma W, \Sigma B)$, $\beta_2 \in \pi(\Sigma Z, \Sigma E)$ *and* $\alpha \in \pi(\Sigma f, \Sigma k)$ *then* $H\{\{\Sigma g\}^{\beta_2}_{\beta_1}, \alpha\} \subseteqq \{\{\Sigma(g\#g)\}^{H\beta_2}_{H\beta_1}, \alpha\}$.

Proof. (i) This is a consequence of Proposition 3.1(ii) and the definition of $\boldsymbol{H}$. For (ii) we have $H\{\{\Sigma g\}^{\beta_2}_{\beta_1}, \alpha\} = H(\cup\{\{\beta, \alpha\} \mid d\beta = \beta_2, c\beta = \beta_1\}) \subseteqq \{\{\boldsymbol{H}\beta, \alpha\} \mid d\boldsymbol{H}\beta = H\beta_2, c\boldsymbol{H}\beta = H\beta_1\} \subseteqq \{\{\Sigma(g\#g)\}^{H\beta_2}_{H\beta_1}, \alpha\}$, as required.

To obtain an analogue of [19; Proposition 2.6] we may recall [7] that there is a partially exact sequence

$$\to \pi(\Sigma X, K) \xrightarrow{\Sigma} \pi(\Sigma^2 X, \Sigma K) \xrightarrow{H} \pi(\Sigma^2 X, \Sigma(K\#K)) \xleftarrow{\Delta^{\leftarrow}} \pi(X, K), \quad K = E \text{ or } B,$$

where $\Delta^{\leftarrow}$ is a partial function defined (with a degree of indeterminacy) using Toda brackets.

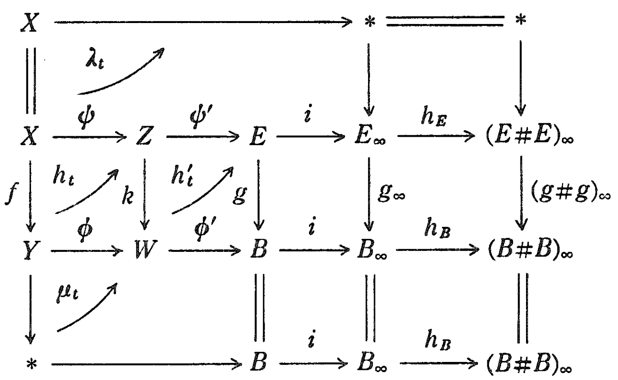
3.10. PROPOSITION. (i) *If* $\Sigma\alpha\in\pi(\Sigma f, \Sigma k)$, $\Sigma\beta\in\pi(\Sigma k, \Sigma g)$ *are such that* $d\Sigma(\beta\circ\alpha)=0$ *and* $c(\beta\circ\alpha)=0$ *then* $H\{\Sigma\beta, \Sigma\alpha\}_1=-\Sigma\{g\# g\}\circ\Delta^{\leftarrow}(d\beta\circ d\alpha)$.

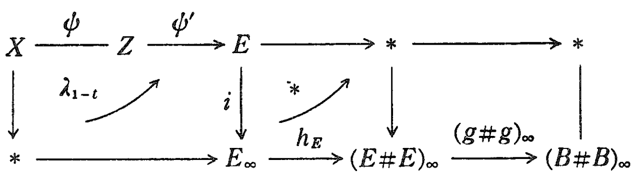
(ii) *If* $\Sigma\alpha$ *and* $\Sigma\beta$ *are such that* $d(\beta\circ\alpha)=0$ *and* $c\Sigma(\beta\circ\alpha)=0$ *then* $H\{\Sigma\beta, \Sigma\alpha\}^1 =\Delta^{\leftarrow}(c\beta\circ c\alpha)\circ\Sigma^2\{f\}$.

PROOF. Recall [7; §4] that if $\Sigma\mu=0$ then

(3.11) $$\Delta^{\leftarrow}\mu=-\Omega_1^{-1}\{^\circ\{h_B\}, \{i\}, \mu\},$$

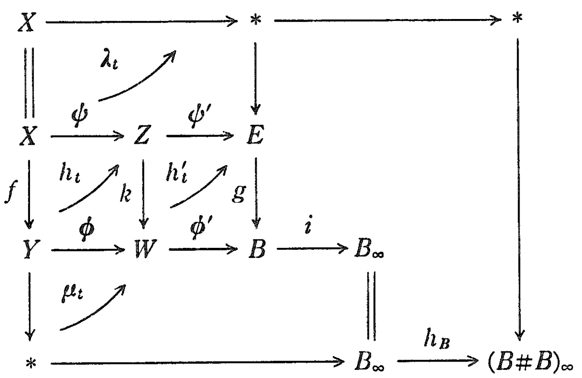
where $i: B\to B_\infty$ is the natural inclusion. (The little circle indicating that the Toda bracket in 3.11 has the reduced indeterminacy consequent on the utilisation of a preferred nullhomotopy of $h_B i$, in this case the constant homotopy). To verify the equality in 3.10(i), first note that the respective indeterminacies coincide. (The indeterminacy of $\Delta^{\leftarrow}$ is the image of H). Then we may observe that the following diagrams represent elements of opposite sign.

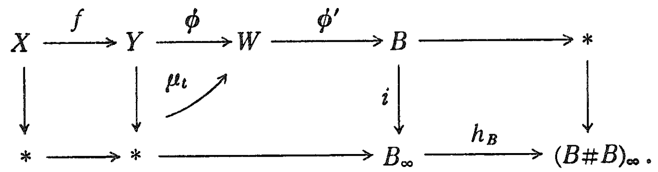




Modifying 3.4(i) to its Ω_1 version, we see that applying Ω_1^{-1} to the element represented by the first diagram yields an element in the coset $-H\{\Sigma\beta, \Sigma\alpha\}_1$. On the other hand, 1.2 and 3.11 indicate that applying Ω_1^{-1} to the element represented by the second diagram yields an element of $-\Sigma\{g\# g\}\circ\Delta^{\leftarrow}(d\beta\circ d\alpha)$. Thus the cosets in 3.10(i) have a common element. In the case of 3.10(ii) we

may again check that the indeterminacies are equal. Now the elements represented by the following diagrams coincide.





Applying Ω_1^{-1} to the element represented we see that it belongs to $-H\{\Sigma\beta, \Sigma\alpha\}^1 \cap -\Delta^{\leftarrow}(c\beta\circ c\alpha)\circ\Sigma^2\{f\}$, which completes the proof.

3.12. COROLLARY. (i) *If* $\alpha\in\pi(f, k)$, $\beta_1\in\pi(W, B)$ *and* $\beta_2\in\pi(Z, E)$ *are such that* $\Sigma\beta_2\circ\Sigma d\alpha=0$ *and* $\beta_1\circ c\alpha=0$ *then*

$$H\{\{\Sigma g\}^{\Sigma\beta_2}_{\Sigma\beta_1}, \Sigma\alpha\}_1=-\Sigma(g\#g)\circ\Delta^{\leftarrow}(\beta_2\circ d\alpha)\,.$$

(ii) *If* $\alpha\in\pi(f, k)$, $\beta_1\in\pi(W, B)$ *and* $\beta_2\in\pi(Z, E)$ *are such that* $\beta_2\circ d\alpha=0$ *and* $\Sigma\beta_1\circ\Sigma c\alpha=0$ *then*

$$H\{\{\Sigma g\}^{\Sigma\beta_2}_{\Sigma\beta_1}, \Sigma\alpha\}^1=\Delta^{\leftarrow}(\beta_1\circ c\alpha)\circ\Sigma^2\{f\}\,.$$

PROOF. (i) $H\{\{\Sigma g\}^{\Sigma\beta_2}_{\Sigma\beta_1}, \Sigma\alpha\}_1=\cup\{H\{\Sigma\beta, \Sigma\alpha\}_1 \mid d\beta=\beta_2,\ c\beta=\beta_1\}$

$$=-\Sigma(g\#g)\circ\Delta^{\leftarrow}(\beta_2\circ d\alpha)\,.$$

The proof of (ii) is similar.

Finally, we have the following "reduction" property.

3.13. PROPOSITION. *If* $\alpha\in\pi(f, k)$, $\beta_2\in\pi(Z, E)$, $\{g\}\circ\beta_2=0$ *and* $\beta_2\circ d\alpha=0$ *then* $\{\{g\}^{\beta_2}_0, \alpha\}\cap\{\{g\}, \beta_2, d\alpha\}\neq\emptyset$.

PROOF. Let $\gamma\in\pi(g, g)$ be the identity class and let $\beta\in\pi(k, g)$ be such that

$d\beta=\beta_2$ and $c\beta=0$. (Such β exist since $\{g\}\circ\beta_2=0$.) Then $d(\beta\circ\alpha)=0$ and $c(\gamma\circ\beta)=0$. By Proposition 3.1(iv), $\{\{g\}, \beta_2, d\alpha\}\cap\{\beta, \alpha\}\neq\emptyset$, which implies the desired result.

3.14. REMARK. It seems to be possible to prove a (less useful) reduction of brackets of type $\{\{g\}^0_{\beta_1}, \alpha\}$ to matrix Toda brackets.

4. Some computations.

Let $h: S^3\to S^2$ be a representative of the Hopf class. In this section, to illustrate the type of argument required to apply the propositions of sections 2 and 3, we compute $\pi^m_n=\pi(\Sigma^{m+n}h, \Sigma^m h)$ in the cases $m=0$, $n=0$; $m=1$, $n=1$; $m=1$, $n=6$; $m=1$, $n=8$. These results independently confirm a calculation in [9] and partially complement corresponding results for the stable range in [10].

We use the notation $\begin{bmatrix}\gamma\\ \delta\end{bmatrix}$ for an element of π^m_n (if such exists) with $d\begin{bmatrix}\gamma\\ \delta\end{bmatrix}=\gamma$ and $c\begin{bmatrix}\gamma\\ \delta\end{bmatrix}=\delta$. For elements of the homotopy groups of spheres we use the notation due to Toda [19].

4.1. PROPOSITION. (i) *There is a bijection*

$$\boldsymbol{Z}\longleftrightarrow\pi^0_0=\left\{\begin{bmatrix}r^2\\ r\end{bmatrix}\middle| r\in\boldsymbol{Z}\right\};$$

(ii) $\pi^1_1\approx\boldsymbol{Z}_4\oplus\boldsymbol{Z}_3$, *with the 2-component generated by* $\begin{bmatrix}\eta_4\\ \eta_3\end{bmatrix}$.

(iii) $\pi^1_6\approx\boldsymbol{Z}_2\oplus\boldsymbol{Z}_8\oplus\boldsymbol{Z}_3\oplus\boldsymbol{Z}_3\oplus\boldsymbol{Z}_3$;

(iv) $\pi^1_8\approx\boldsymbol{Z}_4\oplus\boldsymbol{Z}_3$, *with the 2-component generated by* $\begin{bmatrix}\varepsilon_4\\ \varepsilon_3\end{bmatrix}$.

PROOF. (i) It is well-known that $-h.: \pi_3(S^3)\to\pi_3(S^2)$ is an isomorphism. By [14; Theorem 3.6], the function $.h: \pi_2(S^2)\to\pi_3(S^2)$ is such that $.h(r\iota_2)=r^2\eta_2$. It follows from Proposition 2.3 that $\pi^0_0=\pi(h, h)\leftrightarrow\cup(r\in\boldsymbol{Z})C_r$, where C_r is the cokernel of the homomorphism

$$\begin{bmatrix}\Gamma(r\iota_2, h)\\ \nabla(h, r^2\iota_3)\end{bmatrix}: \pi_3(S^2)\oplus\pi_4(S^3)\longrightarrow\pi_4(S^2).$$

However we claim that C_r is trivial; to check this it is sufficient to verify that $\Gamma(r\iota_2, h)$ is surjective. Since S^4 and S^3 are homotopy-abelian H'-spaces, we may apply [18; Theorem 3.4.3]. Accordingly $\Gamma(r\iota_2, h)(\zeta)=(\Sigma h)^*\zeta+\sum_n(-1)^n(\Sigma w)^{*-1}[r\iota_2, \zeta]_{x_n}\circ\Sigma H_{x_n}(h)$ where H_{x_n} is a Hopf-Hilton invariant associated with the "basic pair" x_n and $[r\iota_2, \zeta]_{x_n}$ the associated iterated Whitehead product. In particular

$$(\Sigma w)^{*-1}[r\iota_2, \zeta]_{x_1} \circ \Sigma H_{x_1}(h) = \pm [r\iota_2, \zeta] = 0 \quad \text{in } \pi_4(S^2),$$

since ζ has to be a multiple of η_2 and $[\iota_2, \eta_2]=0$. Thus $\Gamma(r\iota_2, h)(t\eta_2)=t\eta_2^2$ and $\Gamma(r\iota_2, h)$ is surjective.

(ii) It will be convenient to use the abbreviated notation λ_n for the homomorphism $\begin{pmatrix} \cdot\Sigma^n f \\ \text{-}g\cdot \end{pmatrix}$ in the sequence 2.1 with $g=\Sigma h$. The exactness of 2.1 gives rise to short exact sequences.

$$\text{(4.2)} \qquad \operatorname{coker} \lambda_{n+1} \rightarrowtail \pi_n^1 \twoheadrightarrow \ker \lambda_n .$$

Then $\lambda_1 : \pi_4(S^3) \oplus \pi_5(S^4) \to \pi_5(S^3)$ is such that $\lambda_1(\eta_3, \eta_4) = \eta_3 \circ \eta_4 - \eta_3 \circ \eta_4 = 0$, so that $\ker \lambda_1 \approx \boldsymbol{Z}_2$ is generated by (η_3, η_4). Moreover $\lambda_2 : \pi_5(S^3) \oplus \pi_6(S^4) \to \pi_6(S^3) \approx \boldsymbol{Z}_4 \oplus \boldsymbol{Z}_3 = \{\nu'\} \oplus \{\alpha_1\}$. Since $2\nu' = \eta_3^3$ [19; Proposition 5.6], coker $\lambda_2 \approx \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$ generated by ν', α_1. To determine the extension we need to study $\{\eta_3{}^{\eta_4}_{\eta_3}, 2\iota_{\Sigma 2h}\}$. By 3.12 (ii) we have $H\{\eta_3{}^{\eta_4}_{\eta_3}, 2\iota_{\Sigma 2h}\}^1 = \Delta^{\leftarrow}(2\eta_2) \circ \eta_5 = \pm \eta_5$. (As discussed in [7; §4], we can here recognise $\Delta^{\leftarrow}$ as Toda's Δ^{-1}.) Since $H\nu' = \eta_5$ [19; 5.3], it follows that $\nu' \in \{\eta_3{}^{\eta_4}_{\eta_3}, 2\iota_{\Sigma 2h}\}^1 \subseteqq \{\eta_3{}^{\eta_4}_{\eta_3}, 2\iota_{\Sigma 2h}\}$. By Proposition 2.4, $\nabla\nu' = 2\begin{bmatrix} \eta_4 \\ \eta_3 \end{bmatrix} \neq 0$, so that the extension is non-trivial.

(iii) Using information from [19] it is easy to check that $\ker \lambda_6 \approx \boldsymbol{Z}_8 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3 \oplus \boldsymbol{Z}_3$, with the 2-component generated by $(0, \nu_4^2)$ and that coker $\lambda_7 \approx \boldsymbol{Z}_2$ generated by $\varepsilon_3 \in \pi_{11}(S^3)$. To check that the extension is split we have to prove that

$$\text{(4.3)} \qquad \varepsilon_3 \notin \{\eta_3{}^{\nu_4^2}_{0}, 8\iota_{\Sigma 7h}\} \cup \{\eta_3{}^{2\nu_4^2}_{0}, 4\iota_{\Sigma 7h}\} \cup \{\eta_3{}^{4\nu_4^2}_{0}, 2\iota_{\Sigma 7h}\}.$$

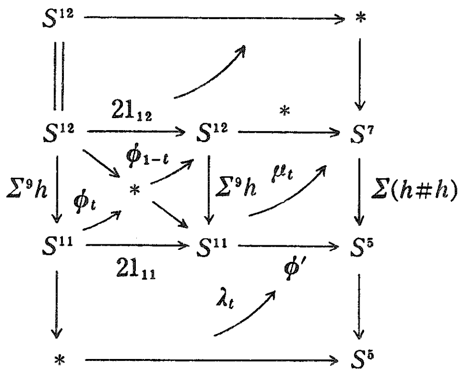
By Proposition 3.13 $\{\eta_3{}^{4\nu_4^2}, 2\iota_{\Sigma 7h}\}$ and $\{\eta_3, 4\nu_4^2, 2\iota_{10}\}$ have a common element and, from information in [19], it can be checked that their respective indeterminacies are trivial. Since $4\nu_7 = \eta_7^3$ [19; (5.5)], we have

$$\{\eta_3, 4\nu_4^2, 2\iota_{10}\} = \{\eta_3, \nu_4 \circ \eta_7^3, 2\iota_{10}\} \supseteqq \{\eta_3, \nu_4 \circ \eta_7^2, \eta_9 \circ 2\iota_{10}\} = 0$$

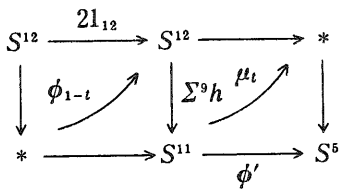
[19; Proposition 1.2]. Similarly it can be checked that $\{\eta_3{}^{2\nu_4^2}_{0}, 4\iota_{\Sigma 7h}\} = \{\eta_3, 2\nu_4^2, 4\iota_{10}\}$. But $\{\eta_3, 2\nu_4^2, 4\iota_{10}\} \subseteqq \{\eta_3, 4\nu_4^2, 2\iota_{10}\} = 0$ (as above). Also similarly it can be checked $\{\eta_3{}^{\nu_4^2}_{0}, 8\iota_{\Sigma 7h}\} = \{\eta_3, \nu_4^2, 8\iota_{10}\} \subseteqq \{\eta_3, 2\nu_4^2, 4\iota_{10}\} = 0$ (as above). Thus 4.3 is proved.

(iv) From [19] we may check that $\ker \lambda_8 = \boldsymbol{Z}_2$, generated by $(\varepsilon_3, \varepsilon_4)$ and that coker $\lambda_9 = \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_3$, with the 2-component generated by $\varepsilon' \in \pi_{13}(S^3)$. By Proposition 2.4, the extension is non-trivial if and only if $\varepsilon' \in \{\eta_3{}^{\varepsilon_4}_{\varepsilon_3}, 2\iota_{\Sigma 9h}\}$. Applying 3.9(ii), we have $H\{\eta_3{}^{\varepsilon_4}_{\varepsilon_3}, 2\iota_{\Sigma 9h}\} \subseteqq \{\eta_5^2{}^{0}_{\nu_5^2}, 2\iota_{\Sigma 9h}\}$. Since the image under H of the indeterminacy of $\{\eta_3{}^{\varepsilon_4}_{\varepsilon_3}, 2\iota_{\Sigma 9h}\}$ is zero, it will be sufficient to prove that $\varepsilon_5 = H\varepsilon' \in \{\eta_5^2{}^{0}_{\nu_5^2}, 2\iota_{\Sigma 9h}\}$. First note that the following diagram, in which ϕ' denotes an element of ν_5^2 and ϕ_t, λ_t and μ_t are nullhomotopies, represents an element of $\{\eta_5^2\circ\nu_5^2, 2\iota_{\Sigma^9 h}\}$.

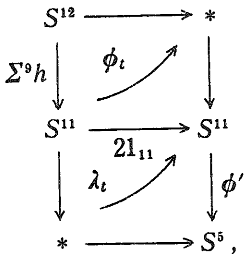
(4.4)



However the diagram



represents an element of $\{\nu_5^2, \eta_{11}, 2\iota_{12}\}$, which has zero indeterminacy, and is in fact zero since $\{\nu_5^2, \eta_{11}, 2\iota_{12}\} \subseteq \{\nu_5, \nu_8\circ\eta_{11}, 2\iota_{12}\} = 0$, [19; Proposition 5.8]. It follows that the element represented by 4.4 can also be obtained from



i.e. belongs to $-\{\nu_5^2, 2\iota_{11}, \eta_{11}\}$. Since $\varepsilon_5 \in \{\nu_5^2, 2\iota_{11}, \eta_{11}\}$ [17; p. 189], this completes the proof.

5. Indeterminacy of the coarse bracket.

Let T denote the left central homotopy commutative square of diagram 1.4 and let $T(\mu_t, h_t', \lambda_t)$ denote the element of $\pi(\Sigma X, B)$ represented by 1.4, that is to say the element obtained by applying the operator cP to the corresponding element of $\pi(X*, *B)$. Let h''_t be an alternative homotopy from $\phi' k$ to $g\phi'$. Then by a technique similar to that used in the proof of [13; Proposition 2.5] we find that

$$(T(\mu_t, h''_t, \lambda_t))^{-1}T(\mu_t, h'_t, \lambda_t)= \begin{array}{ccccc} X & = & X & \longrightarrow & * \\ \| & & {\scriptstyle\phi}\downarrow & \nearrow{\scriptstyle \phi' h_{1-t}+\mu_t f} & \downarrow \\ X & \xrightarrow{\ \phi\ } & Z & \xrightarrow{\ \phi' k\ } & B \\ \downarrow & \nearrow{\scriptstyle \mu_{1-t}f+\phi' h_t} & \downarrow{\scriptstyle\phi' k} & \nearrow{\scriptstyle h'_t+h''_{1-t}} & \| \\ * & \longrightarrow & B & = & B, \end{array}$$

where the diagram is used to denote the element obtained from it by applying the operator cP. Indeed in terms of the bracket operation defined in [13; 5.2] we have

$$T(\mu_t, h'_t, \lambda_t) \in T(\mu_t, h''_t, \lambda_t)\{\xi, \{\phi\}\}, \tag{5.1}$$

where $\xi = \{h'_t + h''_{1-t}\}$ is an element of the $\phi' k$-based track group $\pi_1^Z(B; \phi' k)$ [18; 1.1]. Argueing as in the proof of [13; Proposition 5.4 and Corollary 5.6], we obtain:

5.2. PROPOSITION. *If X is a suspension then the bracket $\{\{g\}_{\beta_1}^{\beta_2}, \alpha\}$ in 1.9 is a coset of the subgroup*

$$\{g\} \circ \pi(\Sigma X, E) + \pi(\Sigma Y, B) \circ \{\Sigma f\} + \{\pi_1^Z(B; \phi' k), d\alpha\} \ \ of\ \ \pi(\Sigma X, B).$$

Further simplification is possible if both X and Z are suspensions and $d\alpha$ is a suspension class. Argueing as in the proof of [13; Proposition 5.7 and Corollary 5.8] we obtain:

5.3. PROPOSITION. *If X and Z are suspensions and if $d\alpha$ is a suspension class then the bracket $\{\{g\}_{\beta_1}^{\beta_2}, \alpha\}$ in 1.9 is a coset of the subgroup*

$$\{g\} \circ \pi(\Sigma X, E) + \pi(\Sigma Y, B) \circ \{\Sigma f\} + \pi(\Sigma Z, B) \circ \Sigma d\alpha\,.$$

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