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A NOTE ON REAL HYPERSURFACES OF A COMPLEX HYPERBOLIC SPACE

By

He-Jin KIM

Introduction.

A Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form. The complete and simply connected complex space form of complex dimension n consists of a complex projective space P^nC, a complex Euclidean space C^n or a complex hyperbolic space H^nC, according as $c>0$, $c=0$ or $c<0$.

Many subjects for real hypersurfaces of a complex projective space P^nC have been studied [1], [4], [5] and [6]. One of which, done by Kimura [6], asserts the following interesting result.

THEOREM K. *There are no real hypersurfaces of P^nC with parallel Ricci tensor on which $J\xi$ is principal, where ξ denotes the unit normal and J is the complex structure of P^nC.*

A Riemannian curvature of a Riemannian manifold M is said to be *harmonic* if the Ricci tensor S satisfies the Codazzi equation, that is,

$$\nabla_X S(Y, Z)-\nabla_Y S(X, Z)=0 \tag{0.1}$$

for any tangent vector fields X, Y and Z, where ∇ denotes the Riemannian connection of M. This condition is essentially weaker than that of the parallel Ricci tensor [2]. From this point of view, Kwon and Nakagawa [5] extends recently the following:

THEOREM K-N. *There are no real hypersurfaces with harmonic curvature of P^nC on which $J\xi$ is principal.*

Now we are interested in these problems in the case of $c<0$, that is, the ambient space is a complex hyperbolic space H^nC. Montiel [7] stated that there are no Einstein real hypersurfaces in H^nC, and classified the pseudo-Einstein real hypersurfaces of H^nC. In this paper, we will prove

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THEOREM. *There are no real hypersurfaces with harmonic curvature of H^nC on which $J\xi$ is principal.*

We also obtain Kimura's theorem when the ambient space is a complex hyperbolic space as a corollary.

1. Preliminaries.

We begin with recalling fundamental formulas on real hypersurfaces of a complex hyperbolic space. Let M be a real hypersurface of a complex hyperbolic space H^nC $(n\geqq 2)$, endowed with the Bergman metric tensor g of constant holomorphic sectional curvature -4, and let J be the complex structure of H^nC. For any X tangent to M, we put

$$JX=PX+\omega(X)\xi, \tag{1.1}$$

where PX and $\omega(X)\xi$ are, respectively, the tangent and normal components of M. Then P is a tensor field of type (1, 1) and ω a 1-form over M. We denote by E the tangent vector field $-J\xi$. Then it is well known that M admits an almost contact metric structure (P, E, ω, g). Let σ be a second fundamental form of M and A a shape operator derived from ξ. The covariant derivative $\nabla_X P$ of the structure tensor P is denoted by $\nabla_X P(Y)=\nabla_X(PY)-P\nabla_X Y$. Then it follows from the Gauss and Weingartan formulas that the structure (P, E, ω, g) satisfies

$$\begin{aligned} &\nabla_X P(Y)=-g(AX, Y)E+\omega(Y)AX, \\ &\nabla_X E=PAX \end{aligned} \tag{1.2}$$

for any tangent vectors X and Y on M, where ∇ denotes the Riemannian connection of the hypersurface.

Since H^nC is of constant holomorphic sectional curvature -4, the Gauss and Codazzi equations are respectively given:

$$\begin{aligned} R(X, Y)Z=&-\{g(Y, Z)X-g(X, Z)Y+g(PY, Z)PX-g(PX, Z)PY \\ &+2g(X, PY)PZ\}+g(AY, Z)AX-g(AX, Z)AY, \end{aligned} \tag{1.3}$$

$$\nabla_X A(Y)-\nabla_Y A(X)=-\{\omega(X)PY-\omega(Y)PX+2g(X, PY)E\}. \tag{1.4}$$

By the Gauss equation, The Ricci tensor S of M is given by

$$S(X, Y)=-\{(2n+1)g(X, Y)-3\omega(X)\omega(Y)\}+hg(AX, Y)-g(AX, AY), \tag{1.5}$$

where h denotes the trace of the shape operator A.

From now on, we assume that the structure vector field E is principal, thot is, E is eigenvector of A associated with eigenvalue α. Then equation (1.2) implies that

$$\nabla_X A(E)=d\alpha(X)E+\alpha PAX-APAX, \tag{1.6}$$

which together with (1.4) yields

$$\begin{aligned} &2APA=\alpha(AP+PA)-2P, \\ &\beta(AP+PA)=0, \qquad d\alpha=\beta\omega, \end{aligned} \tag{1.7}$$

where $\beta=d\alpha(E)$. Taking account of (1.4) (1.6) and (1.7), it is easy to see that

$$\begin{aligned} &\nabla_X A(E)=\alpha(PA-AP)X/2+PX+\beta\omega(X)E, \\ &\nabla_E A(X)=\alpha(PA-AP)X/2+\beta\omega(X)E. \end{aligned} \tag{1.8}$$

2. Proof of the Theorem.

At first we determine the hypersurface M satisfying (0.1). Using (1.5), we see that (0.1) is equivalent to

$$\begin{aligned} &h\{g(\nabla_X A(Y)-\nabla_Y A(X), Z)+g(\nabla_X A(Y)-\nabla_Y A(X), AZ)-g(\nabla_X A(Z), AY) \\ &+g(\nabla_Y A(Z), AX)\}+(\nabla_X h)g(AY, Z)-(\nabla_Y h)g(AX, Z)+3\{g(PAX, Y)\omega(Z) \\ &+g(PAX, Z)\omega(Y)-g(PAY, X)\omega(Z)-g(PAY, Z)\omega(X)\}=0 \end{aligned} \tag{2.1}$$

for any vector fields X, Y and Z tangent to M. Putting $Z=E$ in (2.1) and taking account of (1.8), we have

$$\alpha(PA^2+A^2P)/2+2(PA+AP)-\alpha APA-2(\alpha-h)P=0. \tag{2.2}$$

Similarly, putting $X=E$ in (2.1), we also obtain

$$-(3PA-AP)+\alpha(PA-AP)(\alpha-A)/2-(h-\alpha)P+\gamma A-\alpha dh\otimes E=0, \tag{2.3}$$

where $\gamma=dh(E)$.

Now first of all we prove that the principal curvature α is constant. Suppose that there exist points x at which $\beta(x)\neq 0$. Then we have $AP+PA=0$ and $APA=-P$ by means of (1.7). Taking a principal vector X orthogonal to E with principal curvature λ, we find $\lambda=\pm 1$ and $-\lambda$ is also a principal curvature. This implies that $h=\alpha$ and hence $\alpha P=0$ at x by means of (1.7) and (2.2), which together with (2.3) yields $\lambda=0$. A contradiction. So we have $\beta=d\alpha(E)=0$ on M. Moreover using (1.7), we have $d\alpha(X)=0$ for any X orthogonal to E. Consequently, we can say that α is constant. Moreover it is non-zero. In fact, suppose that $\alpha=0$. Then we can verify, making use of (2.2) and (2.3), that it follows that

$$-4PA-2hP+\gamma A=0.$$

Let X be a principal vector with principal curvature λ which is orthogonal to E. Then by means of above equation, we have $(4\lambda+2h)PX-\gamma\lambda X=0$, which implies that $2\lambda+h=0$ and $\gamma\lambda=0$, because X and PX are mutually orthogonal. This implies that the trace of A satisfies $h=\alpha+(2n-2)\lambda=-(n-1)h$, which means that $\lambda=h=0$, and hence M is totally geodesic. Thus it is a contradiction.

Next, the constancy of the mean curvature h will be proved. Replacing X and Z by E and making use of (1.8), equation (2.1) becomes

$$\alpha(\gamma\omega-dh)=0 \tag{2.4}$$

Since α is non-zero constant, equation (2.4) yields

$$\operatorname{grad} h=\gamma E\,,$$

from which we have

$$d\gamma(X)\omega(Y)-d\gamma(Y)\omega(X)=-\gamma g((PA+AP)X, Y)$$

for any X and Y, because of the fact that $g(\nabla_X \operatorname{grad} h, Y)=g(\nabla_Y \operatorname{grad} h, X)$. Suppose that there exist points x at which $\gamma(x)\neq 0$. Putting $Y=E$ in the above equation, we have $d\gamma=d\gamma(E)\omega$ and hence it implies that $PA+AP=0$. Making use of the same discussion as above, we get $P=0$, which is a contradiction. Thus γ vanishes identically and by (2.4) h must be constant.

LEMMA. *Let M be a real hypersurfaces with harmonic curvature of H^nC. If the structure vector E is principal, then all principal curvatures are constant and the number of distinct principal curvatures is at most* 5.

PROOF. Let X be a principal vector orthogonal to E with principal curvature λ. Then it follows from (1.7) that

$$(2\lambda-\alpha)APX=(\alpha\lambda-2)PX. \tag{2.5}$$

Fix any point q of M such that

$$\lambda_1(q)=\cdots=\lambda_s(q)=\alpha/2\,,\qquad \lambda_{s+1}(q)\neq\alpha/2,\ \cdots,\ \lambda_{2n-2}(q)\neq\alpha/2\,,$$

where $0\leqq s\leqq 2n-2$. Then there exists a neighborhood W_λ of q such that $\lambda_r\neq\alpha/2$ on W_λ, where $r\geqq s+1$. For $\lambda=\lambda_r$, $Y=PX$ is also a principal vector on the open set W_λ and its corresponding principal curvature is given by $\mu=(\alpha\lambda-2)/(2\lambda-\alpha)$. Hence (2.3) is reduced to

$$(3\lambda-\mu)-\alpha^2(\lambda-\mu)/2+\alpha(\lambda-\mu)\lambda/2+(h-\alpha)=0. \tag{2.6}$$

Accordingly the principal curvature $\lambda=\lambda_r$ is the roots of the following cubic equation with constant coefficients:

(2.7) $$\alpha x^3-2(\alpha^2-3)x^2+(\alpha^3-5\alpha+2h)x-(\alpha h-2)=0.$$

It means that the number of distinct principal curvatures for any fixed point q is at most 5 and λ_r are constant on W_λ.

Next we will show that all principal curvatures are constant. Suppose that there exist a point y in W_λ and an index a at which $\lambda_a(y)\neq\alpha/2$, $a\leqq s$. Then y is a distinct point from q. Let W_a be the set consisting of points of W_λ at which $\lambda_a\neq\alpha/2$. By the same discussion as above λ_a are constant on W_a and hence the continuity of λ_a shows that W_a is closed. Without loss of generality, we may assume that W_λ is connected. In fact, we may take a connected components of W_λ if necessary. Since W_a is open and closed in the connected set W_λ, we conclude W_a is empty, that is, $\lambda_a=\alpha/2$ for any $a\leqq s$ on W_λ. Accordingly all principal curvatures are constant in W_λ and hence W_λ is equal to M, that is, all principal curvatures are constant on M.

Finally, we are going to prove the main theorem mentioned in the Introduction. Let X be a principal vector orthogonal to E with principal curvature $\lambda(\neq\alpha/2)$. Then PX is also a principal vector with principal curvature $\mu=(\alpha\lambda-2)/(2\lambda-\alpha)$. It follows from (2.7) that λ satisfies

$$\alpha\lambda^3-2(\alpha^2-3)\lambda^2+(\alpha^3-5\alpha+2h)\lambda-(\alpha h-2)=0.$$

Suppose that $\lambda\neq\mu$. It follows from (2.6) that

(2.3) $$\alpha\lambda^2-2(\alpha^2-4)\lambda+\alpha(\alpha^2-5)=0.$$

From two equations obtained above it follows that

(2.9) $$2\lambda^2-2h\lambda+\alpha h-2=0.$$

We assert that the operator P commutes with the shape operator A. If $s=2n-2$, then the property $PA=AP$ is trivial. So suppose that $0<s<2n-2$. Since there exists at least one principal vector associated with principal curvature $\alpha/2$ by means of the supposition, the equation (2.5) emplies $\alpha=\pm2$ and hence we get $\lambda\neq\mu$ for the principal curvature λ different from $\alpha/2$. In fact, if $\lambda=\mu$, we see $\lambda^2-\alpha\lambda+1=0$, which means that $\lambda=\pm1=\alpha/2$. Then, from (2.8) and (2.9) we have $h=2(\alpha^2-4)/\alpha=0$ and $\lambda=-\mu=\pm1$. On the other hand, h is given by $h=(s+2)\alpha/2$, a contradiction. Accordingly we may only consider the case of $s=0$. Now, for a real hypersurface M of a complex hyperbolic space H^nC, one can construct a Lorentzian hypersurface N of an anti-de Sitter space S_1^{2n+1} which is a principal S^1-bundle over M with totally geodesic fibers and the projection $\pi: N\rightarrow M$ in such a way that the diagram

$$\begin{array}{ccc} N & \xrightarrow{i'} & S_1^{2n+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{i} & H^nC \end{array}$$

is commutative (i and i' being respective isometric immersions). Let $\mu_1, \cdots, \mu_{2n-1}$ be principal curvatures of M at any point x such that $\mu_1 = \alpha$. Since the structure vector E is assumed to be principal, let $E_1, \cdots, E_{2n-1}$ be an orthonormal basis of T_xM with $AE_1 = \alpha E_1$ and $AE_a = \mu_a E_a (a=2, \cdots, 2n-1)$. Then horizontal lift $E_a{}^*$ and a unit vector E' form an orthonormal basis of T_zN, $\pi(z) = x$, with respect to the shape operator A' of N is represented by

$$\left(\begin{array}{cc|ccc} 0 & -1 & & & \\ 1 & \alpha & & 0 & \\ \hline & & \mu_2 & & \\ & 0 & & \ddots & \\ & & & & \mu_{2n1} \end{array}\right)$$

where the first submatrix corresponds to the restriction of A' to the Lorentzian plane spanned by $\{E', E_1{}^*\}$. See Montiel [7]. This means that N is an isoparametric hypersurface of S_1^{2n+1} and hence a theorem due to Hahn [3] implies $\lambda\mu = 1$. Thus the principal curvatures λ and μ satisfy $\lambda\mu = \alpha^2 - 5$ and $\lambda + \mu = 4/\alpha$ from (2.8), which implies that $4n - 2 = 0$ by the definition of the mean curvature, a contradiction. Hence we have $\lambda = \mu$, which implies $PA = AP$.

Therefore, we obtain $\lambda = (\alpha - h)/2$ by means of (2.6) and hence, in spite of $s=0$ or $s>0$, we have $\alpha = h$, which enables us to obtain $\lambda = 0$. Making use of (2.5) again, we have $PA = AP = 0$ and hence $P = 0$ by means of (1.7), which is a contradiction. Thus the theorem is completely proved.

COROLLARY. *There are no real hypersurfaces of H^nC with parallel Ricci tensor on which the structure vector E is principal.*

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Kyungpook National University
Taegu, 635
Korea