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PLURIHARMONIC MAPS INTO PRINCIPAL FIBER BUNDLE AND VERTICAL TORSION

By

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Introduction.

Harmonic maps into Lie groups are deeply studied as classical solutions of the principal chiral model in many recent works of mathematics and theoretical physics. The purpose of this paper is to investigate a geometrical linkage among harmonic maps and pluriharmonic maps into structure group, total space of a principal fiber bundle by the medium of their *vertical* data.

Let Σ, P and M be Riemannian manifolds and $\pi: P\rightarrow M$ be a Riemannian submersion. A smooth map $\phi:\Sigma\rightarrow P$ is called *vertically harmonic* if ϕ is a critical point of the vertical energy;

$$E^{V}(\phi)=\int_{\Sigma}|(d\phi)^{V}|^{2}d\Sigma$$

for arbitrary compactly supported vertical variation through ϕ, where $(d\phi)^{V}$: $T\Sigma\rightarrow\phi^{-1}\,Ker\,\pi_{*}$ denotes the *vertical differential* which is the vertical component of $d\phi: T\Sigma\rightarrow\phi^{-1}TP$.

The notion of vertically harmonic map includes that of usual harmonic map as as a vertically harmonic graph map ([10]) and a Yang-Mills connection as a harmonic section ([9]). In [10], C.M. Wood characterizes vertical harmonicity of ϕ in terms of *vertical tension field* $\tau^{V}(\phi)$ via *vertical torsion* $T^{V,\phi}$ introduced by him (cf. definition in § 1).

At first by connecting a vertical tension field with a usual one, we obtain the following theorem.

THEOREM A. *Let π be a Riemannian submersion with totally geodesic fibers. If $\phi:\Sigma\rightarrow P$ is harmonic, then ϕ is always vertically harmonic.*

Let G be a Lie group which admits a bi-invariant Riemannian metric $\langle\,,\rangle$ and (P, π, M) be a smooth principal G-bundle with a connection form ω over a Riemannian manifold (M, g_{M}). The horizontally lifted metric ${}^{\omega}g_{P}$ on P by ω

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is defined as follows;

$$({}^{\omega}g_P)_u(X, Y)=(g_M)_{\pi(u)}(\pi_{*u}X, \pi_{*u}Y)+\langle\omega_u(X), \omega_u(Y)\rangle \quad \text{for } X, Y\in T_uP,\ u\in P.$$

$\pi:(P, {}^{\omega}g_P)\to(M, g_M)$ becomes to be a Riemannian submersion with totally geodesic fibers ([8]). Vertical torsion in this case can be considered as a just obstruction for generating a harmonic map into G.

THEOREM B. (i) *For a smooth map* $\phi:\Sigma\to P$, $\phi:\Sigma\to(P, {}^{\omega}g_P)$ *is vertically harmonic if and only if* $\phi^*\omega$ *satisfies the Hodge gauge equation;* $\delta(\phi^*\omega)=0$.

(ii) *Let* Σ *be a connected Riemannian manifold. Choose a based point* $x_0\in\Sigma$ *and assume that* $Hom(\pi_1(\Sigma), G)=\{1\}$. *For a vertically harmonic map* $\phi:\Sigma\to(P, {}^{\omega}g_P)$, $T^{V,\phi}=0$ *if and only if there exists a unique harmonic map* $\varphi:\Sigma\to G$ *such that* $\varphi^*\theta=\phi^*\omega$ *and* $\varphi(x_0)=e$, *where* δ, $\pi_1(\Sigma)(=\pi_1(\Sigma, x_0))$, $Hom(\cdot,\cdot)$, θ *and* e *denote the codifferential with respect to the Riemannian metric of* Σ, *the fundamental group of* Σ, *the set of group homomorphisms, the Maurer-Cartan form on* G *and the identity element of* G, *respectively.*

Let Σ be a complex manifold instead of a Riemannian manifold. $\phi^{-1}Ker\,\pi_*$-valued 2-form $T^{V,\phi}$ extends by complex linearily to the $\phi^{-1}Ker\,\pi_*^{\mathbf{C}}$-valued 2-form $T^{V,\phi}$. Relative to the complex structure of Σ, we have the decomposition of the vector space of $\phi^{-1}Ker\,\pi_*^{\mathbf{C}}$-valued 2-forms on Σ. By restricting $T^{V,\phi}$ to to (1, 1)-factor, we define $\phi^{-1}Ker\,\pi_*^{\mathbf{C}}$-valued (1, 1)-form $(T^{V,\phi})^{(1,1)}$.

Here we shall introduce the notion of *vertical pluriharmonicity*, which is vertical version of pluriharmonicity;

$\phi:\Sigma\to(P, {}^{\omega}g_P)$ is *vertically pluriharmonic* if ${}^{\phi}D^{V''}(\partial\phi)^V=0$, where ${}_{\phi}D^{V''}(\partial\phi)^V$ is the *vertically* (0, 1)-*exterior covariant derivative* of $d\phi:T\Sigma^C\to\phi^{-1}TP^C$ (cf. definition in § 1).

Vertical pluriharmonicity is characterized in terms of $(T^{V,\phi})^{(1,1)}$ and we have the following theorem which is an analogue of theorem A and B.

THEOREM C. *Let* $\phi:\Sigma\to(P, {}^{\omega}g_P)$ *be a pluriharmonic map. Then*

(i) ϕ *is vertically pluriharmonic if and only is* $(T^{V,\phi})^{(1,1)}=0$,

(ii) *assume that* Σ *is connected and* $Hom(\pi_1(\Sigma), G)=\{0\}$, *then* $T^{V,\phi}=0$ *if and only if for each* $x_0\in\Sigma$ *there exists a unique pluriharmonic map* $\varphi:\Sigma\to G$ *such that* $\varphi^*\theta=\phi^*\omega$ *and* $\varphi(x_0)=e$,

(iii) *let* Σ *be a compact Kähler surface, then* ϕ *is vertically pluriharmonic if and only if* $T^{V,\phi}=0$.

Real four-dimensional self-duality is crucial to (iii) ([1]).

It is a well-known fact that pluriharmonicity is equivalent to harmonicity when Σ is a Riemann surface. To the contrary, even over a Riemann surface, vertically pluriharmonicity *does not* coincide with vertically harmonicity because of vertical torsion, which appears in the difference between Theorem A and Theorem $C(i)$.

Finally, by the definition of $T^{V,\phi}$, it is obvious that $T^{V,\phi}=0$ if ϕ is *horizontal*; $(d\phi)^V=0$ for a smooth map ϕ. To prove the converse for weakly stable pluriharmonic map with respect to the total energy, we make use of an energy descending deformation along loop parameter of real extended solution into a based loop group ΩG ([4], [7]). In this case, the vertical torsion can be considered as an obstruction for projecting down ϕ to a pluriharmonic map into M.

THEOREM D. *Let Σ be a compact connected Kähler manifold with $Hom(\pi_1(\Sigma), G)=\{1\}$ and $\phi: \Sigma \to (P, {}^{\omega}g_P)$ be a weakly stable pluriharmonic map. If $T^{V,\phi}=0$, then ϕ is horizontal, and therefore, $\pi\circ\phi: \Sigma\to M$ is pluriharmonic.*

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1. Second Fundamental Form and Vertical Torsion.

Let Σ, P and M be Riemannian manifolds and $\pi: P\to M$ be a Riemannian submersion with totally geodesic fibers. The Riemannian connections of Σ and P are denoted by ${}^{\Sigma}\nabla$ and ${}^{P}\nabla$. Relative to the Riemannian structure of P, we have the orthogonal decomposition

$$TP=Ker\,\pi_*\oplus(Ker\,\pi_*)^{\perp}\,;\; X=X^V+X^H\,.$$

The induced connection in $Ker\,\pi_*$ from ${}^{P}\nabla$ is denoted by ${}^{P}\nabla^V$. Each fiber of P is totally geodesic so that

LEMMA 1.1. *For any $X, Y\in C^{\infty}(TP)$,*

$$({}^{P}\nabla_X Y)^V=({}^{P}\nabla_X Y^V)^V+A(X^H, Y^H)\,,$$

where A and $C^{\infty}(\cdot)$ denote the O'Neill's tensor A ([5]) *and the vector space of smooth sections, respectively.*

Let $\phi: \Sigma\to P$ be a smooth map. The induced connections through ϕ from ${}^{P}\nabla$ and ${}^{P}\nabla^V$ are denoted by ${}^{\phi}\nabla$ and ${}^{\phi}\nabla^V$, respectively. By pulling back Lemma 1.1, we get

LEMMA 1.2. *For any* $X, Y \in C^\infty(T\Sigma)$,

(i) $({}^\phi\nabla_X((d\phi)Y))^V = {}^\phi\nabla_X^V((d\phi)^V Y) + A((d\phi)^H X, (d\phi)^H Y)$,

(ii) $(({}^\phi\nabla_X d\phi)Y)^V = ({}^\phi\nabla_X^V(d\phi)^V)Y + A((d\phi)^H X, (d\phi)^H Y)$,

where $(d\phi)X = (d\phi)^V X + (d\phi)^H X$ *etc.*.

The vertical tension field $\tau^V(\phi)$ and the vertical torsion $T^{V,\phi}$ of ϕ are defined as follows ([10]);

$$\tau^V(\phi) = Trace_\Sigma^\phi \nabla^V(d\phi)^V\,,$$

$$T^{V,\phi}(X, Y) = {}^\phi\nabla_X^V((d\phi)^V Y) - {}^\phi\nabla_Y^V((d\phi)^V X) - (d\phi)^V([X, Y])$$

for $X, Y \in C^\infty(T\Sigma)$.

LEMMA 1.3. *For any* $X, Y \in C^\infty(T\Sigma)$,

(i) $T^{V,\phi}(X, Y) = -2A((d\phi)^H X, (d\phi)^H Y)$,

(ii) $({}^\phi\nabla_X((d\phi)Y))^V = {}^\phi\nabla_X^V((d\phi)^V Y) - 1/2T^{V,\phi}(X, Y)$,

(iii) $(({}^\phi\nabla_X d\phi)Y)^V = ({}^\phi\nabla_X^V(d\phi)^V)Y - 1/2T^{V,\phi}(X, Y)$,

(iv) $(\tau(\phi))^V = \tau^V(\phi)$,

where $\tau(\phi)$ *denotes the usual tension field of* ϕ.

PROOF. (i); By using Lemma 1.2, we compute

$$\begin{aligned}
T^{V,\phi}(X, Y) &= {}^\phi\nabla_X^V((d\phi)^V Y) - {}^\phi\nabla_Y^V((d\phi)^V X) - (d\phi)^V([X, Y]) \\
&= ({}^\phi\nabla_X((d\phi)Y))^V - A((d\phi)^H X, (d\phi)^H Y) \\
&\quad - ({}^\phi\nabla_Y((d\phi)X))^V + A((d\phi)^H Y, (d\phi)^H X) - (d\phi)^V([X, Y]) \\
&= \{({}^\phi\nabla_X d\phi)Y - ({}^\phi\nabla_Y d\phi)X\}^V \\
&\quad - \{A((d\phi)^H X, (d\phi)^H Y) - A((d\phi)^H Y, (d\phi)^H X)\}.
\end{aligned}$$

The second fundamental form ${}^\phi\nabla d\phi$ of ϕ is symmetric and A is skew-symmetric with respect to horizontal vectors ([5]) so that

$$T^{V,\phi}(X, Y) = -2A((d\phi)^H X, (d\phi)^H Y)\,.$$

(ii); Substitute (i) to Lemma 1.2, we get (ii).
(iii); Add $-(d\phi)^V({}^\Sigma\nabla_X Y)$ to the both sides of (ii).
(iv); Take a trace of (iii).

Since vertical harmonicity for ϕ is equivalent to the vanishing of $\tau^V(\phi)$ ([10, Theorem 2]), it follows from Lemma 1.3 (iv) that

THEOREM 1.4 (THEOREM A). *If ϕ is harmonic, then ϕ is vertically harmonic.*

Extend $d\phi$, ${}^{\phi}\nabla$, ${}^{\phi}\nabla^{V}$, $T^{V,\phi}$ and A by complex linearity, then Lemma 1.2 holds for any $X, Y\in C^{\infty}(T\Sigma^{C})$.

Let Σ be a complex manifold instead of a Riemannian manifold. Relative to the complex structure of Σ, we have the decomposition

$$T\Sigma^{C}=T^{(1,0)}\Sigma\oplus T^{(0,1)}\Sigma\,.$$

The *usual* and *vertical* (0, 1)-*exterior covariant derivatives* ${}^{\phi}D''\partial\phi$ and ${}^{\phi}D^{V''}(\partial\phi)^{V}$ of $d\phi$ are defined as follows;

$$({}^{\phi}D''_{\bar{Z}}\partial\phi)W={}^{\phi}\nabla_{\bar{Z}}((d\phi)W)-(d\phi)(\bar{\partial}_{\bar{Z}}W)\,,$$

$$({}^{\phi}D^{V''}_{\bar{Z}}(\partial\phi)^{V})W={}^{\phi}\nabla^{V}_{\bar{Z}}((d\phi)^{V}W)-(d\phi)^{V}(\bar{\partial}_{\bar{Z}}W)\,,$$

for $Z, W\in C^{\infty}(T^{(1,0)}\Sigma)$, where $\bar{\partial}$ denotes the $\bar{\partial}$-operator of $T^{(1,0)}\Sigma$. From Lemma 1.3 (ii),

LEMMA 1.5. *For any $Z, W\in C^{\infty}(T^{(1,0)}\Sigma)$*

(i) $({}^{\phi}\nabla_{\bar{Z}}((d\phi)W))^{V}={}^{\phi}\nabla_{\bar{Z}}((d\phi)^{V}W)-1/2T^{V,\phi}(\bar{Z}, W)$,

(ii) $(({}^{\phi}D''_{\bar{Z}}\partial\phi)W)^{V}=({}^{\phi}D^{V''}_{\bar{Z}}(\partial\phi)^{V})(W)-1/2(T^{V,\phi})^{(1,1)}(\bar{Z}, W)$.

ϕ is called *pluriharmonic* (resp. *vertically plurinarmonic*) if the usual (resp. vertical) (0, 1)-exterior covariant derivative vanishes.

Lemma 1.5 (ii) implies that

THEOREM 1.6. *Let $\phi:\Sigma\to P$ be a pluriharmonic map. Then ϕ is vertically pluriharmonic if and only if $(T^{V,\phi})^{(1,1)}=0$.*

Let G be a Lie group with a bi-invariant Riemannian metric and (P, π, M) be a smooth principal G-bundle with a connection form ω. $\pi:(P, {}^{\omega}g_{P})\to M$ is a Riemannian submersion with totally geodesic fibers ([8, Theorem 3.5]) so that Theorem C (i) is deduced from Theorem 1.6.

Evaluating $T^{V,\phi}$ by ω, we have

PROPOSITION 1.7.

(i) $\omega\circ T^{V,\phi}=2\phi^{*\omega}\Omega$,
where ${}^{\omega}\Omega$ is the curvature form of ω,

(ii) *$T^{V,\phi}=0$ if and only if $\phi^{*}\omega$ is integrable.*

PROOF. (i); From [3, Chapter 2, Corollary 5.3] and [5, Lemma 2], $\omega\circ A(X^H, Y^H)=-{}^{\omega}\Omega(X, Y)$ for all $X, Y\in T_uP$, $u\in P$. By Lemma 1.3 (i), for all $X, Y\in T_x\Sigma$, $x\in\Sigma$

$$\begin{aligned}\omega(T^{V,\phi}(X, Y)) &= -2\omega(A((d\phi)^H X, (d\phi)^H Y)) \\ &= 2{}^{\omega}\Omega((d\phi)^H X, (d\phi)^H Y) \\ &= 2{}^{\omega}\Omega((d\phi)X, (d\phi)Y) \\ &= 2(\phi^{*\omega}\Omega)(X, Y),\end{aligned}$$

since ${}^{\omega}\Omega$ is a tensorial 2-form.

(ii); By the structure equation ([3, Chapter 2, Theorem 5.2]),

$$\phi^{*\omega}\Omega=\phi^*(d\omega+1/2[\omega\wedge\omega])=d(\phi^*\omega)+1/2[\phi^*\omega\wedge\phi^*\omega],$$

Combining (i), we get (ii).

Evaluation by ω for $\phi^{-1}\,Ker\,\pi_*$-valued differential forms *does not* depend upon the choice of ω, therefore, we denote it by ${}^{\phi}I$ in §2.

2. Transfer of Vertical Data.

Let G be a Lie group with a bi-invariant Riemannian metric $\langle\,,\rangle$ and $(\mathfrak{g}, [\,,])$ be the Lie algebra of G. ${}^G\nabla$ denotes the Riemannian connection of $\langle,\rangle$. The tangent bundle TG is identified with $G\times\mathfrak{g}$ through the Maurer-Cartan form θ;

$$T_gG \longrightarrow \{g\}\times\mathfrak{g}\,;\ X\longmapsto(g, \theta_g(X)), \qquad g\in G.$$

This trivialization is denoted by ${}^GI$ and induces a natural identification between smooth sections ${}^GI: C^\infty(TG)\to C^\infty(G\times\mathfrak{g})$. Here we give a connection ${}_{\theta}\nabla^{(0)}$ by

$${}_{\theta}\nabla^{(0)}s=ds+1/2[\theta, s], \qquad \text{for } s\in C^\infty(G\times\mathfrak{g}).$$

The following lemma is well-known (cf. [2, Chapter 2]):

LEMMA 2.1.

$${}^G\nabla={}^GI^{-1}\circ{}_{\theta}\nabla^{(0)}\circ{}^GI.$$

Let (P, π, M) be a smooth principal G-bundle with a connection form ω over a Riemannian manifold M.

${}^PI: Ker\,\pi_*\to P\times\mathfrak{g}$ denotes the trivialization defined by;

$$Ker\,\pi_{*u} \longrightarrow \{u\}\times\mathfrak{g}\,;\ X\longmapsto(u, A), \qquad u\in P,$$

where $X=(d/dt)(u\,exp\,tA)|_{t=0}$, $A\in\mathfrak{g}$.

This trivalization induces a natural identification between smooth sections

${}^{G}I : C^{\infty}(Ker\ \pi_{*}) \to C^{\infty}(P\times \mathfrak{g})$.

We introduce a connection ${}_{\omega}\nabla^{(0)}$ in $P\times\mathfrak{g}$ as follows;

$${}_{\omega}\nabla^{(0)} s = ds + 1/2[\omega, s], \qquad \text{for } s\in C^{\infty}(P\times\mathfrak{g}).$$

The following proposition is an analogue of Lemma 2.1.

PROPOSITION 2.2.

$${}^{P}\nabla^{V} = {}^{P}I^{-1} \circ {}_{\omega}\nabla^{(0)} \circ {}^{P}I .$$

PROOF. The above equation follows from Lemma 2.1 since each fiber is totally geodesic.

For a C^{∞}-manifold Σ and a smooth map $\phi : \Sigma\to P$, $\phi^{-1}\,Ker\ \pi_{*}$, $\Sigma\times\mathfrak{g}^{\phi}$, I and ${}^{\phi}_{\omega}\nabla^{(0)}(={}_{\phi*\omega}\nabla^{(0)})$ denote the corresponding induced objects through ϕ to $Ker\ \pi_{*}$, $P\times\mathfrak{g}^{P}$, I and ${}_{\omega}\nabla^{(0)}$, respectively. From Proposition 2.2, we have

PROPOSITION 2.3.

$${}^{\phi}\nabla^{V} = {}^{\phi}I^{-1} \circ {}_{\omega*\phi}\nabla^{(0)} \circ {}^{\phi}I .$$

THEOREM 2.4. *Let Σ be a Riemannian manifold, then*

(i) $\omega(({}^{\phi}\nabla_{X}(d\phi)^{V})Y) = ({}^{\Sigma}\nabla_{X}\phi^{*}\omega)(Y) + 1/2[(\phi^{*}\omega)(X), (\phi^{*}\omega)(Y)]$

for all $X\in T_{x}\Sigma,\ x\in\Sigma,\ Y\in C^{\infty}(T\Sigma)$,

(ii) $\omega(\tau^{V}(\phi)) = -\delta(\phi^{*}\omega)$.

PROOF. (i); By using Proposition 2.3,

$$\omega({}^{\phi}\nabla^{V}_{X}((d\phi)^{V}Y)) = {}^{\phi}I({}^{\phi}\nabla^{V}_{X}((d\phi)^{V}Y)) = {}^{\phi}_{\omega}\nabla^{(0)}_{X}({}^{\phi}I((d\phi)^{V}Y))$$

$$= {}_{\phi*\omega}\nabla^{(0)}_{X}(\omega((d\phi)Y)) = {}_{\phi*\omega}\nabla^{(0)}_{X}((\phi^{*}\omega)(Y)),$$

therefore,

$$\omega(({}^{\phi}\nabla^{V}_{X}(d\phi)^{V})Y) = \omega)^{\phi}\nabla^{V}_{X}((d\phi)^{V}Y)) - \omega((d\phi)({}^{\Sigma}\nabla_{X}Y))$$

$$= X((\phi^{*}\omega)(Y)) + 1/2[(\phi^{*}\omega)(X), (\phi^{*}\omega)(Y)] - (\phi^{*}\omega)({}^{\Sigma}\nabla_{X}Y)$$

$$= ({}^{\Sigma}\nabla_{X}\phi^{*}\omega)(Y) + 1/2[(\phi^{*}\omega)(X), (\phi^{*}\omega)(Y)]$$

(ii); Take the trace of the equation of (i), we get $\omega(\tau^{V}(\phi)) = -\delta(\phi^{*}\omega)$.

Theorem B(i) is deduced from Theorem 2.4 (ii).

Return to Proposition 2.3, the equality means that vertical notions are independent of the choice of g_{M} and depend only upon the data of Σ, ϕ and ω. Namely, we may define ${}^{\phi}\nabla^{V}$ by ${}^{\phi}I^{-1} \circ {}_{\phi*\omega}\nabla^{(0)} \circ {}^{\phi}I$ without the Riemannian data of M. Combining Proposition 2.3, we obtain

LEMMA 2.5. *Let* (Q_1, π_1, F_1), (Q_2, π_2, N_2) *be smooth principal G-bundles and* η *be a connection form on* Q_2 *and* $\Psi: Q_1 \rightarrow Q_2$ *be a bundle homomorphism. For a smooth map* $f_1: \Sigma \rightarrow Q_1$, *set* $f_2 = \Psi \circ f_1: \Sigma \rightarrow Q_2$. *Then*

(i) ${}^{f_1}_{\Psi^*\eta}\nabla^{(0)} = {}^{f_2}_{\eta}\nabla^{(0)}$ *in* $\Sigma \times \mathfrak{g}$,

(ii) $(\Psi^*\eta)((df_1)^V Y) = \eta((df_2)^V Y)$ *for all* $Y \in T_x\Sigma$, $x \in \Sigma$,

(iii) $(\Psi^*\eta)({}^{f_1}\nabla^V_X((df_1)^V Y)) = \eta({}^{f_2}\nabla^V_X((df_2)^V Y))$

for all $X \in T_x\Sigma$, $x \in \Sigma$, $Y \in C^\infty(T\Sigma)$,

(iv) *over a Riemannian manifold* Σ,

$$(\Psi^*\eta)(({}^{f_1}\nabla^V_X(df_1)^V)Y) = \eta(({}^{f_2}\nabla^V_X(df_2)^V)Y)$$

for all $X, Y \in T_x\Sigma$, $x \in \Sigma$,

(v) *over a complex manifold* Σ,

$$(\Psi^*\eta)(({}^{f_1}D^{V''}_{\bar{Z}}(\partial f_1)^V)W) = \eta(({}^{f_2}D^{V''}_{\bar{Z}}(\partial f_2)^V)W)$$

for all $Z, W \in T^{(1,0)}_x\Sigma$, $x \in \Sigma$.

PROOF. (i), (ii); trivial, (iii); Using (i) and (ii),

$$\begin{aligned}(\Psi^*\eta)({}^{f_1}\nabla^V_X((df_1)^V Y)) &= {}^{f_1}I({}^{f_1}\nabla^V_X((df_1)^V Y)) = {}^{f_1}_{\Psi^*\eta}\nabla^{(0)}_X({}^{f_1}I((df_1^V Y)) \\ &= {}^{f_1}_{\Psi^*\eta}\nabla^{(0)}_X((\Psi^*\eta)(df_1)^V Y) = {}^{f_2}_{\eta}\nabla^{(0)}_X(\eta((df_2)^V(Y)) \\ &= {}^{f_2}_{\eta}\nabla^{(0)}_X({}^{f_2}I((df_2)^V Y)) = {}^{f_2}I({}^{f_2}\nabla^V_X(df_2)^V Y)) \\ &= \eta({}^{f_2}\nabla^V_X((df_2)^V Y))\end{aligned}$$

(iv), (v); It follows from (iii).

Set $\Phi = \pi \circ \phi: \Sigma \rightarrow M$. For a principal G-bundle (P, π, M) with a connection form ω $(\Phi^{-1}P, {}^{\Phi}\pi, \Sigma)$ denotes the induced principal G-bundle with a connection form $\bar{\Phi}^*\omega$, where $\bar{\Phi}: \Phi^{-1}P \rightarrow P$ is the induced bundle homomorphism by $(\Phi^{-1}P)_x = P_{\Phi(x)}$, $x \in \Sigma$. Since $\phi(x)$ lies in the fiber $P_{\Phi(x)}$, the smooth map $\bar{\phi}: \Sigma \rightarrow \Phi^{-1}P$; $x \mapsto \phi(x) \in (\Phi^{-1}P)_x$ is a global section of $\Phi^{-1}P$. $\bar{\Phi} \circ \bar{\phi} = \phi$ so that Lemma 2.5 deduces the following proposition.

PROPOSITION 2.6.

(i) *Let* Σ *be a Riemannian manifold. Then* ϕ *is vertically harmonic if and only if* $\bar{\phi}$ *is vertically harmonic.*

(ii) *Let* Σ *be a complex manifold. Then* ϕ *is vertically pluriharmonic if and only if* ϕ *is vertically pluriharmonic.*

For a global section $\bar{\phi}$ there exists a unique bundle isomorphism $I_{\bar{\phi}}: \Phi^{-1}P \rightarrow$

$\Sigma \times G$ such that $(I_{\bar{\phi}} \circ \bar{\phi})(x) = (x, e)$ for all $x \in \Sigma$. Notice that $\bar{\Phi}^* \omega = I_{\bar{\phi}}^*(ad(\iota \circ p_2) \cdot (p_1^* \phi^* \omega) + p_2^* \theta)$, where $\iota : G \to G$; $g \mapsto g^{-1}$, $p_1 : \Sigma \times G \to \Sigma$; $(x, g) \mapsto x$ and $p_2 : \Sigma \times G \to G$; $(x, g) \mapsto g$.

This means that $\phi^* \omega$ and ${}^{\phi^* \omega}F = d(\phi^* \omega) + 1/2[\phi^* \omega \wedge \phi^* \omega] (= \phi^{*\omega} \Omega)$ are cocycle representations for $\bar{\Phi}^* \omega$ and ${}^{\bar{\Phi}^* \omega}\Omega$, respectively. Using Proposition 1.7, we can restate Theorem 1.6 in terms of ${}^{\phi^* \omega}F$.

PROPOSITION 2.7. *Let Σ be a complex manifold and $\phi : \Sigma \to (P, {}^{\omega}g_P)$ be a pluriharmonic map. Then ϕ is vertically pluriharmonic if and only if ${}^{\phi^* \omega}F^{(1,1)} = 0$.*

3. Self-duality in Kähler Surface.Σ

In this section, we assume that Σ is a *Kähler surface*, that is, a complex two-dimensional Kähler manifold.

Let G be a Lie group with a bi-invariant Riemannian metric $\langle , \rangle$. $* : \mathfrak{g} \otimes \wedge^2 \Sigma \to \mathfrak{g} \otimes \wedge^2 \Sigma$ denotes the *Hodge star operator* with respect to the Kähler metric and the natural orientation of Σ, and induces the operator acting $\mathfrak{g}$-valued 2-forms $* : C^\infty(\mathfrak{g} \otimes \wedge^2 \Sigma) \to C^\infty(\mathfrak{g} \otimes \wedge^2 \Sigma)$ where $\wedge^2 \Sigma = \wedge^2 T^* \Sigma$.

Note that $*^2 = 1$. For a $\mathfrak{g}$-valued 2-form ζ, ζ is called *self-dual* if ζ is *a* $+1$-eigen section of $*$.

Extend ζ by complex linearity. ζ is self-dual if and only if $\zeta^{(1,1)}$ is proportional to the Kähler form of Σ (cf. [1]).

Let M be a Riemannian manifold and (P, π, M) be a smooth principal G-bundle with a connection form ω. Combining the above fact and Proposition 2.7, we have

PROPOSITION 3.1. *Let $\phi : \Sigma \to (P, {}^{\omega}g_P)$ be a pluriharmonic map. If ϕ is vertically pluriharmonic, then ${}^{\phi^* \omega}F$ is self-dual.*

We define a real 4-form $\langle {}^{\phi^* \omega}F \wedge {}^{\phi^* \omega}F \rangle$ on Σ as follows;

$$\langle {}^{\phi^* \omega}F \wedge {}^{\phi^* \omega}F \rangle = \sum_{a,b} \langle F_a, F_b \rangle \xi_a \wedge \xi_b ,$$

where ${}^{\phi^* \omega}F = \Sigma_a F_a \otimes \xi_a$ (F_a is a $\mathfrak{g}$-valued function, ξ_a is a real 2-form).

Theorem C (iii) is deduced from the following theorem.

THEOREM 3.2. *Let Σ be a campact Kähler surface and $\phi : \Sigma \to (P, {}^{\omega}g_P)$ be a pluriharmonic map. If ϕ is vertically pluriharmonic, then ${}^{\phi^* \omega}F = 0$, and therefore, $T^{V,\phi} = 0$.*

PROOF. By Proposition 3.1, ${}^{\phi^*\omega}F$ is self-dual, it follows that

$$|{}^{\phi^*\omega}F|^2=\langle {}^{\phi^*\omega}F\wedge *{}^{\phi^*\omega}F\rangle=\langle {}^{\phi^*\omega}F\wedge {}^{\phi^*\omega}F\rangle$$

$\bar{\Phi}^*\omega$ is a connection form in a trivial bundle $\Phi^{-1}P$. According to the Chern-Weil theory,

$$\int_\Sigma |{}^{\phi^*\omega}F|^2 d\Sigma=\int_\Sigma \langle {}^{\phi^*\omega}F\wedge {}^{\phi^*\omega}F\rangle d\Sigma=0$$

so that ${}^{\phi^*\omega}F=0$.

4. Integrability Condition for Vertical Differential.

Let Σ be a connected C^∞-manifold and $\phi:\Sigma\to P$ be a smooth map, where P is the total space of a smooth principal G-bundle with a connection form ω and G is a Lie group with a bi-invariant Riemannian metric $\langle,\rangle$. Choose a based point $x_0\in\Sigma$. $\pi_1(\Sigma)=\pi_1(\Sigma, x_0)$ denotes the fundamental group of Σ. In this section, we assume that $Hom(\pi_1(\Sigma), G)=\{1\}$.

If $\phi^*\omega$ is integrable, then the corresponding connection $\bar{\Phi}^*\omega$ in $\Phi^{-1}P$ is flat, and therefore, there exists a unique bundle isomorphism $I_{\phi^*\omega}:\Phi^{-1}P\to\Sigma\times G$ such that $\bar{\Phi}^*\omega=(I_{\phi^*\omega})^*(p_2^*\theta)$ and $I_{\phi^*\omega}(\bar{\phi}(x_0))=(x_0, e)$ (cf. [3, Chapter 2, §9], [6]).

Set $\varphi=p_2\circ I_{\phi^*\omega}\circ\bar{\phi}:\Sigma\to G$, then $I_{\phi^*\omega}\circ\bar{\phi}$ is the graph map of φ, that is, $I_{\phi^*\omega}\circ\bar{\phi}=(id._\Sigma\times\varphi)\circ\Delta:\Sigma\to\Sigma\times G$, where $id._\Sigma:\Sigma\to\Sigma$; $x\mapsto x$ and $\Delta:\Sigma\to\Sigma\times\Sigma$; $x\mapsto(x, x)$.

LEMMA 4.1. *$\varphi^*\theta=\phi^*\omega$.*

PROOF. By Lemma 2.5 (ii), for all $Y\in T_x\Sigma$, $x\in\Sigma$

$$\begin{aligned}(\phi^*\omega)(Y)&=\omega((d\phi)Y)=\omega((d\phi)^V Y)\\&=(\bar{\Phi}^*\omega)((d\bar{\phi})^V Y)\\&=(I_{\phi^*\omega}^{-1})^*(\bar{\Phi}^*\omega)((d((id._\Sigma\times\varphi)\circ\Delta))^V Y)\\&=(p_2^*\theta)((d((id._\Sigma\times\varphi)\circ\Delta))^V Y)\\&=(p_2^*\theta)(d((id._\Sigma\times\varphi)\circ\Delta)Y)\\&=\theta(d(p_2\circ(id._\Sigma\times\varphi)\circ\Delta)Y)\\&=\theta((d\varphi)Y)=(\varphi^*\theta)(Y).\end{aligned}$$

Combining Proposition 1.7, Lemma 2.5 (iii) and Proposition 2.6, we obtain the following theorem.

THEOREM 4.2. *Let $\phi:\Sigma\to P$ be a smooth map with $T^{V,\phi}=0$. Then*

(i) *over a Riemannian manifold* Σ,
ϕ *is vertically harmonic if and only if* φ *is harmonic,*
(ii) *over a complex manifold* Σ,
ϕ *is vertically pluriharmonic if and only if* φ *is pluriharmonic.*

By the *Maurer-Cartan equation* $d\theta+1/2[\theta\wedge\theta]=0$,

$$d(\phi^*\omega)+1/2[\phi^*\omega\wedge\phi^*\omega]=\varphi^*(d\theta+1/2[\theta\wedge\theta])=0\,,$$

From Proposition 1.7 (ii), have $T^{V,\phi}=0$. Therefore, Theorem B (ii) (resp. C (ii)) follows from Lemma 4.1 and Theorem 4.2 (i) (resp. 4.2 (ii)).

5. Stability and Vertical Torsion.

In this section, let G, M, (P, π, M) and Σ be a Lie group which admits a bi-variant Riemannian metric $\langle\,,\rangle$, a Riemannian manifold, a smooth principal G-bundle with a connection form ω and a compact connected Kähler manifold with $Hom(\pi_1(\Sigma), G)=\{1\}$, respectively. It is easy to check the following lemma.

LEMMA 5.1. *Let* $\phi: \Sigma\to P$ *be a smooth map with* $T^{V,\phi}=0$.
(i) φ *is constant if and only if* ϕ *is horizontal.*
(ii) *For any variation* φ_t *through* $\varphi_0=\varphi$,
$\Phi\circ I^{-1}_{\phi^*\omega}\circ(id._\Sigma\times\varphi_t)\circ\Delta$ *is a vertical variation of* ϕ.
(iii) *Conversely, any vertical variation* ϕ *through* $\phi_0=\phi$ *is of the form* $\bar{\Phi}\circ I^{-1}_{\bar{\phi}^*\omega}\circ(id._\Sigma\times\varphi_t)\circ\Delta$,
(iv) $E(\phi_t)=E(\varphi_t)+E(\Phi)$,
where $E(\cdot)$ *denotes the total energy.*

Combining theorem 1.6, proposition 1.7 (ii) and theorem 4.2 (ii), we have

PROPOSITION 5.2. *Let* $\phi: \Sigma\to(P, {}^\omega g_P)$ *be a pluriharmonic map with* $T^{V,\phi}=0$. *Then there exists a unique pluriharmonic map* $\varphi: \Sigma\to G$ *such that* $\varphi^*\theta=\phi^*\omega$ *and* $\varphi(x_0)=e$.

It is known ([4], [7]) that if $\varphi: \Sigma\to G$ is a non-constant pluriharmonic map, then φ has an energy descending variation which is a real loop into G, and therefore φ is unstable as a harmonic map.

Existence of energy descending variation implies the following theorem by using Lemma 5.1 and Proposition 5.2.

THEOREM 5.3. *Let* $\phi: \Sigma\to(P, {}^\omega g_P)$ *be a pluriharmonic map. If* $T^{V,\phi}=0$ *and*

ϕ is not horizontal, then ϕ is unstable. Equivalently, if ϕ is weakly-stable, then $T^{V,\phi}\neq 0$ or ϕ is horizontal.

Theorem D follows from Theorem 5.3.

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