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HYPOELLIPTICITY FOR A CLASS OF DEGENERATE ELLIPTIC OPERATORS OF SECOND ORDER

Dedicated to Professor Mutsuhide MATSUMURA on his sixtieth birthday

By

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§1. Introduction and results.

Fedii [1] studied hypoellipticity for operators of the form $L=D_1^2+\phi(x_1)^2D_2^2$ in $\boldsymbol{R}^2$, and proved that L is hypoelliptic in $\boldsymbol{R}^2$ if $\phi(x_1)\in C^\infty(\boldsymbol{R})$ and $\phi(x_1)>0$ for $x_1\neq 0$. Hörmander's results in [2] can not be applicable to L when $\phi(x_1)$ has a zero of infinite order. Compared with higher dimensional cases, the problem in $\boldsymbol{R}^2$ becomes much simpler. So one can expect that one investigates hypoellipticity for more general operators in $\boldsymbol{R}^2$. In this paper we shall give sufficient conditions of hypoellipticity for operators of the form $P(x, D)=D_1^2+\alpha(x)D_2^2+\beta(x, D)$ in $\boldsymbol{R}^2$, where $x=(x_1, x_2)\in\boldsymbol{R}^2$, $\alpha(x)\in C^\infty(\boldsymbol{R}^2)$ is non-negative and $\beta(x, D)$ is a properly supported classical pseudodifferential operator of order 1. In doing so, we need general criteria for hypoellipticity, which are improvements of ones obtained by Morimoto [5] (see Theorem 1.1 below).

Let us define the usual symbol classes $S_{1,0}^{m,\mathrm{loc}}$ and $S_{1,0}^m$. We say that a symbol $p(x, \xi)$ belongs to $S_{1,0}^{m,\mathrm{loc}}$ (resp. $S_{1,0}^m$) if $p(x, \xi)\in C^\infty(T^*\boldsymbol{R}^n)$ and if for any compact subset K of $\boldsymbol{R}^n$ and for any multi-indices α and β (resp. for any multi-indices α and β there is $C_{\alpha,\beta}\equiv C_{K,\alpha,\beta}>0$ (resp. $C_{\alpha,\beta}>0$) such that $|p_{(\beta)}^{(\alpha)}(x, \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}$ for $x\in K$ and $\xi\in\boldsymbol{R}^n$ (resp. for $(x, \xi)\in T^*\boldsymbol{R}^n$), where $m\in\boldsymbol{R}$, $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$, $D_x=-i\partial_x$, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ and $T^*\boldsymbol{R}^n$ is identified with $\boldsymbol{R}^n\times\boldsymbol{R}^n$. We denote by $L_{1,0}^m$ the set of the pseudodifferential operators whose symbols belong to $S_{1,0}^{m,\mathrm{loc}}$. Let $P(x, D)\in L_{1,0}^m$ be a properly supported pseudodifferential operator, and let $z^0=(x^0, \xi^0)\in T^*\boldsymbol{R}^n\setminus 0$ $(\cong\boldsymbol{R}^n\times(\boldsymbol{R}^n\setminus\{0\}))$. It is said that $P(x, D)$ is microhypoelliptic at z^0 if there is a conic neighbourhood $\mathcal{V}$ of z^0 in $T^*\boldsymbol{R}^n\setminus 0$ such that $WF(u)\cap\mathcal{V}=WF(Pu)\cap\mathcal{V}$ if $u\in\mathcal{D}'$, $(\boldsymbol{R}^n)$. We also say that $P(x, D)$ is microhypoelliptic in a conic in a conic set $\mathcal{V}(\subset T^*\boldsymbol{R}^n\setminus 0)$ (resp. in $\Omega(\subset\boldsymbol{R}^n)$ if $P(x, D)$ is microhypoelliptic at each $(x, \xi)\in\mathcal{W}$ (resp. at

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each $(x, \xi)\in\Omega\times(\boldsymbol{R}^n\setminus\{0\})$). We may assume that the symbol $p(x, \xi)$ of $P(x, D)$ belongs to $S^m_{1,0}$. Assume that

(H) *$\xi^0=(0, \cdots, 0, 1)\in\boldsymbol{R}^n$ and there are a conic neighborhood C of z in $T^*\boldsymbol{R}^n\setminus 0$, a conic smooth manifold Σ in $T^*\boldsymbol{R}^n\setminus 0$, $n'\in\boldsymbol{N}$ and a vector subspace V of $\boldsymbol{R}^{n-1}$ such that $z^0\in\Sigma$, $n'\leqq n$, $P(x, D)$ is microhypoelliptic in $C\setminus\Sigma$ and $T_{z^0}\Sigma\cap W=\{0\}$, where $W=\{(\delta x, \delta\xi', 0)\in T_{z^0}(T^*\boldsymbol{R}^n)\,|\,\delta x_j=0(n'<j\leqq n)$ and $\delta\xi'=(\delta\xi_1, \cdots, \delta\xi_{n-1})\in V\}$ and $\delta x=(\delta x_1, \cdots, \delta x_n)$.*

Denote by $\eta(\delta\xi')$ the orthogonal projection of $\delta\xi'\in\boldsymbol{R}^{n-1}$ to the orthogonal complement $V^\perp$ of V, and choose a real-valued symbol $\varphi(x'', \xi)\in S^0_{1,0}$ such that $\varphi(x'', \xi)$ is positively homogenous of degree 0 for $|\xi|\geqq 1$ and $\varphi(x'', \xi)=|x''-x^{0''}|^2+|\eta(\xi')|^2/\xi_n^2$ near $C\cap\{|\xi|\geqq 1\}$, where $x''=(x_{n'+1}, \cdots, x_n)$, $x^{0''}=(x^0_{n'+1}, \cdots, x^0_n)$, and $x''=0$ if $n'=n$. Let $\lambda(\xi)$ be a real-valued symbol in $S^1_{1,0}$ such that $\lambda(\xi)=\langle\xi_n\rangle$ if $\xi_n\geqq|\xi|/2\geqq 1$ and $\langle\xi\rangle/4\leqq\lambda(\xi)\leqq 2\langle\xi\rangle$. We put

$$\begin{aligned}\Lambda(x, \xi)&\equiv\Lambda_\delta(x'', \xi)\equiv\Lambda_\delta(x'', \xi\,;\, a, N, s)\\&=\{-s+a\varphi(x'', \xi)\}\log\lambda(\xi)+N\log(1+\delta\lambda(\xi))\end{aligned}$$

for $0\leqq\delta\leqq 1$, $a\geqq 0$, $N\geqq 0$ and $s\in\boldsymbol{R}$. Note that $|\Lambda^{(\alpha)}_{(\beta)}(x'', \xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}\times\log(1+\langle\xi\rangle)$ and $e^{\pm\Lambda(x'',\xi)}\in\bigcup_{l\in\boldsymbol{R}}S^l_{1,0}$.

Define $P_\Lambda(x, D)$ by

$$P_\Lambda(x, D)=e^{-\Lambda}(x'', D)P(x, D)e^{\Lambda}(x'', D)\,.$$

where $e^{\pm\Lambda}(x'', D)$ are pseudodifferential operators with symbols $e^{\pm\Lambda(x'',\xi)}$.

THEOREM 1.1. *Assume that the condition* (H) *is satisfied, and assume that there are $\chi_k(x, \xi)\in S^0_{1,0}$ $(k=1, 2)$, $l_k\in\boldsymbol{R}$ $(1\leqq k\leqq 3)$, $a_0\geqq 0$, $N_0\geqq 0$, and $s_0\in\boldsymbol{R}$ such that $\chi_k(x, \xi)$ $(k=1, 2)$ are positively homogeneous of degree* 0 *for $|\xi|\geqq 1$, $\chi_k(z)=1$ near z^0 and for any $a\geqq a_0$, any $N\geqq N_0$ and any $s\geqq s_0$ there are $\Psi(x, \xi)\in S^0_{1,0}$, $\delta_0>0$ $(\delta_0\leqq 1)$ and $C>0$ such that $\Psi(x, \xi)$ is positively homogeneous of degree* 0 *for $|\xi|\geqq 1$,* $\operatorname{supp}\Psi\cap\Sigma=\emptyset$ *and*

$$\begin{aligned}\|\chi_1(x, D)v\|_{l_1}\leqq C\{&\|P_\Lambda(x, D)v\|_{l_2}+\|v\|_{l_1-1}\\&+\|(1-\chi_2(x, D))v\|_{l_3}+\|\Psi(x, D)v\|_{l_3}\}\end{aligned}\tag{1.1}$$

if $v\in C_0^\infty$ and $0<\delta\leqq\delta_0$, where $\|u\|_l=\|\langle D\rangle^l u\|$ and $\|u\|$ denotes the L^2 norm of u. Then $z^0\notin WF(u)$ if $u\in\mathcal{D}'$ and $z^0\notin WF(P(x, D)u)$.

REMARK. When one applies Theorem 1.1, one must choose W in the condition (H) suitably. Whether (1.1) can be shown or not may depend on the choice of W.

Next let us restrict our consideration to operators of second order in $\boldsymbol{R}^2$. We assume that $P(x, D)=D_1^2+\alpha(x)D_2^2+\beta(x, D)$ is a properly supported classical pseudodifferential operator in $\boldsymbol{R}^2$ such that $\alpha(x)\in C^\infty(\boldsymbol{R}^2)$ is non-negative and $\beta(x, D)\in L^1_{1,0}$. Let $x^0\in\boldsymbol{R}^2$, and let Σ_0 be a subset of $\boldsymbol{R}^2$ such that $x^0\in\Sigma_0$ and $P(x, D)$ is microhypoelliptic in $U_0\setminus\Sigma_0$ for some neighborhood U_0 of x^0.

THEOREM 1.2. (i) *Assume that* $\Sigma_0\cap U_0=\{x^0\}$. *If there are a neighborhood* U *of* x^{00} *and* $C>0$ *such that*

$$(1.2)\qquad (\mathrm{Re}\,\beta_1(x, 0, \pm 1))^2\leqq C\alpha(x)\qquad \textit{for } x\in U\,,$$

then $P(x, D)$ *is microhypoelliptic at* x^0, *where* $\beta_1(x, \xi)$ *denotes the principal symbol of* $\beta(x, D)$ *which is positively homogeneous of degree* 1. (ii) *Assume that* $\Sigma_0\cap U_0\subset\{x\in\boldsymbol{R}^2\,|\,f(x)=0\}$, *where* $f(x)\in C^1(\boldsymbol{R}^2)$ *is real-valued,* $f(x^0)=0$ *and* $\partial f/\partial x^1(x^0)\neq 0$. *If there are a neighborhood* U *of* x^0, $l\in\boldsymbol{N}$ *and* $C>0$ *such that*

$$(1.3)\qquad (\mathrm{Re}\,\beta_1(x, 0, \pm 1))^2+(\mathrm{Im}\,\beta_1(x, 0, \pm 1))^{2l}\leqq C\alpha(x)\qquad \textit{for } x\in U,$$

then $P(x, D)$ *is microhypoelliptic at* x^0.

Denote by $\mathcal{P}(\boldsymbol{R}^2)$ the power set of $\boldsymbol{R}^2$. We define the mapping $\tau: \mathcal{P}(\boldsymbol{R}^2)\to\mathcal{P}(\boldsymbol{R}^2)$ as follows: For $A\in\mathcal{P}(\boldsymbol{R}^2)$, $\tau(A)$ is a subset of A and $x^0\in A\setminus\tau(A)$ if and only if $\alpha(x^0)>0$ or there are a neighborhood U of x^0 and $f(x)\in C^1(\boldsymbol{R}^2)$ such that (i) $f(x^0)=0$, $\partial f/\partial x^1(x^0)\neq 0$ and $A\cap U\subset\{x\in\boldsymbol{R}^2\,|\,f(x)=0\}$ and (ii) (1.2) holds if $A\cap U=\{x^0\}$ and (1.3) holds if $A\cap U\neq\{x^0\}$. The following Corollary is an immediate consequence of Theorem 1.2.

COROLLARY 1. *$P(x, D)$ is microhypoelliptic in* $\boldsymbol{R}^2\setminus\bigcap_{j=1}^\infty\tau^j(S)$, *where* $S=\{x\in\boldsymbol{R}^2\,|\,\alpha(x)=0\}$.

REMARK. We note that $\tau(\boldsymbol{R}^2)\subset S$. So we have $\bigcap_{j=1}^\infty\tau^j(\boldsymbol{R}^2)=\bigcap_{j=1}^\infty\tau^j(S)$.

Define $\tilde{S}=\bigcup_{A\subset S,\,\tau(A)=A}A$, where $S=\{x\in\boldsymbol{R}^2\,|\,\alpha(x)=0\}$. Then it is easy to see that $\tau(\tilde{S})=\tilde{S}$ and that $A\subset\tilde{S}$ if $A\subset S$ and $\tau(A)=A$. Using transfinite induction, we can prove the following

COROLLARY 2. *$P(x, D)$ is microhypoelliptic in* $\boldsymbol{R}^2\setminus\tilde{S}$. *In particular, if there is not a non-empty subset* A *of* S *satisfying* $\tau(A)=A$, *then* $P(x, D)$ *is microhypolliptic in* $\boldsymbol{R}^2$.

Next assume that $\alpha(0)=0$ and that $S\cap U\subset\{x\in\boldsymbol{R}^2 | x_1=0\}$ for some neighborhood U of the origin in $\boldsymbol{R}^2$, where $S=\{x\in\boldsymbol{R}^2 | \alpha(x)=0\}$. Put

$$A(t)=\inf\{\alpha(s, x_2) \mid (s, x_2)\in[-c_0, c_0]\times[-c_0, c_0] \text{ and } \pm(s-t)\geqq 0\} \quad \text{for } c_0\geqq\pm t\geqq 0,$$

$$B(t)=\sup\{|\mathrm{Re}\,\beta_1(s, x_2, 0, 1)| \mid (s, x_2)\in[-c_0, c_0]\times[-c_0, c_0] \text{ and } \pm t\geqq(t-s)\geqq 0\} \quad \text{for } c_0\geqq\pm t\geqq 0,$$

$$\Gamma(t)=\sup\{|\mathrm{Im}\,\beta_1(s, x_2, 0, 1)| \mid (s, x_2)\in[-c_0, c_0]\times[-c_0, c_0] \text{ and } \pm t\geqq(t-s)\geqq 0\} \quad \text{for } c_0\geqq\pm t\geqq 0,$$

where c_0 is a positive constant satisfying $[-c_0, c_0]\times[-c_0, c_0]\Subset U$. Here $A\Subset B$ means that the closure $\bar{A}$ of A is included in the interior $\mathring{B}$ of B. It is easy to see that $A(t)$, $B(t)$ and $\Gamma(t)$ are Lipschitz continuous functions defined on $[-c_0, c_0]$. Under the above assumptions Theorem 1.2 can be improved as follows:

THEOREM 1.3. (i) *Assume that* $S\cap U=\{0\}$. *If* (1.2) *holds or if there is* $l\in\boldsymbol{N}$ *such that*

$$A_0\equiv\limsup_{t\to 0}|t|^{2l}/A(t)<\infty, \tag{1.4}$$

$$B_0\equiv\limsup_{t\to 0}|t|^{1-l}B(t)<\infty, \tag{1.5}$$

$$2^{l+5}\{1+2^{l+2}/l(l+1)\}A_0B_0^2/l(l+1)<1, \tag{1.6}$$

then $P(x, D)$ *is microhypoelliptic at the origin.* (ii) *Assume that* (1.2) *is valid or* (1.4)-(1.6) *are valid. If* (1.3) *holds or if* $\lim_{t\to 0}t^2\Gamma(t)\log A(t)=0$, *then* $P(x, D)$ *is microhypoelliptic at the orgin.*

The remainder of this paper is organized as follows. In §2 we shall give the proof of Theorem 1.1. Theorem 1.2 and Corollary 2 will be proved in §3. In §4 we shall prove Theorem 1.3. Further remark will be given in §5.

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§2. Proof of Theorem 1.1.

Theorem 1.1 is a variant of Theorem 1.2 in [4]. For completeness we give the proof of Theorem 1.1 in this section. Let $u\in\mathcal{D}'(\boldsymbol{R}^n)$ and put $f=P(x, D)u$. We may assume that $u\in\mathcal{E}'\cap H^{s'}$ for some $s'\in\boldsymbol{R}$, where $H^s(\equiv H^s(\boldsymbol{R}^n))$

denotes the Sobolev space of order s. Let C_1 be a conic neighborhood of z^0 such that $C_1\cap\{|\xi|=1\}\Subset\{(x,\xi)\in C\,|\,\chi_1(x,\xi)=\chi_2(\chi,\xi)=1\}$. Assume that there is a conic neighborhood C_2 of z^0 such that $C_2\Subset C_1$ and $WF(f)\cap C_2=\emptyset$, where $C_2\Subset C_1$ means $C_2\cap\{|\xi|=1\}\Subset C_1$. Then it follows from the assumytion (H) that $WF(u)\cap\partial C_2\cap\widetilde{W}=\emptyset$, where $\widetilde{W}=\{(x_0,\lambda\xi_0)\,|\,\lambda>0\}+W=\{(x^0+x,\lambda\xi^0+\xi)\,|\,(x,\xi)\in W,\lambda>0\}$ and ∂C_2 denotes the boundary of C_2, modifying C if necessary. Choose $\chi(x,\xi)\in S^0_{2,0}$ so that $\chi(x,\xi)$ is positively homogenenous of degree 0 for $|\xi|\geqq 1$. $\chi(z)=1$ near z^0, $\mathrm{supp}\,\chi\cap\{|\xi|=1\}\Subset C_1$ and $WF(f)\cap\mathrm{supp}\,\chi\cap\{|\xi|=1\}=\emptyset$. Then we have $WF(u)\cap\widetilde{W}\cap\mathrm{supp}\,d\chi\cap\{|\xi|=1\}=\emptyset$. Therefore there is $\varepsilon>0$ such that $(x,\xi)\notin WF(u)$ if $(x,\xi)\in\mathrm{supp}\,d\chi$, $|\xi|\geqq 1$ and $\varphi(x'',\xi)\leqq 2\varepsilon$. For a fixed $\sigma>s'$ we can choose $a\geqq a_0$ and $s\geqq s_0$ so that $a\varepsilon-s>l_2+m-1-s'$ and $a\varepsilon/2-s<l_1-\sigma$. Moreover choose $N\geqq N_0$ so that $N>s-s'+\max\{l_2+m,l_1-1,l_3\}$. It follows from calculus of pseudodifferential operators that there is $Q_\delta(x'',\xi)$ $(\equiv Q_\delta(x'',\xi;a,N,s))$ such that

$$\|e^{\Lambda}(x'',D)Q_\delta(x'',D)e^{-\Lambda}(x'',D)g-g\|_\rho\leqq C_{a,N,s,\rho}(g)$$

for $\rho\in\boldsymbol{R}$ and $g\in H^{-\infty}(\equiv\bigcup_{l\in\boldsymbol{R}}H^l)$. Here and after the constants do not depend on δ $(\delta\leqq 1)$ if not stated. Put

$$v_\delta(x)=Q_\delta(x'',D)e^{-\Lambda}(x'',D)\chi(x,D)u\,.$$

Then we have

$$\|e^{\Lambda}(x'',D)v_\delta-\chi(x,D)u\|_\rho\leqq C_{a,N,s,\rho}(u)\,,$$

$$\|P_\Lambda(x,D)v_\delta-e^{-\Lambda}(x'',D)\chi(x,D)f-e^{-\Lambda}(x'',D)[P,\chi]u\|_\rho\leqq C_{a,N,s,\rho}(u)$$

for any $\rho\in\boldsymbol{R}$, where $[P,\chi]u=(P((x,D)\chi(x,D)-\chi(x,D)P(x,D))u$. Since u is in C^∞ near $\{(x,\xi)\,|\,\varphi(x'',\xi)\leqq 3\varepsilon,\ (x,\xi)\in\mathrm{supp}\,d\chi$ and $|\xi|\geqq 1\}$ and $-s+a\varphi(x'',\xi)>l_2+m-1-s'$ if $\varphi(x'',\xi)\geqq\varepsilon$, we have

$$\|P_\Lambda(x,D)v_\delta\|_{l_2}\leqq C_{a,N,s}(u)\,.$$

Noting that $v_\delta\in H^{\max(l_2+m,l_2-1,l_3)}$ for $\delta>0$ and that (1.1) is also valld for $v_\delta\in H^{\max(l_2+m,l_2-1,l_3)}$, we have

$$\|\chi_1(x,D)v_\delta\|_{l_1}\leqq C_{a,N,s}(u)\qquad\text{for } 0<\delta\leqq\delta_0\,,$$

where $\delta_0>0$ is as in Theorem 1.1. In fact, we have $\|(1-\chi_2(x,D))v_\delta\|_{l_3}\leqq C'_{a,N,s}(u)$, since $\mathrm{supp}\,(1-\chi_2(x,\xi))\cap\mathrm{supp}\,\chi\cap\{|\xi|\geqq 1\}=\emptyset$. We have also

$$\|\Psi(x,D)v_\delta\|_{l_3}\leqq C''_{a,N,s}(u)\,,$$

since u is in C^∞ near $\mathrm{supp}\,\Psi\cap\mathrm{supp}\,\chi\cap\{|\xi|\geqq 1\}$ by the assumption (H). Therefore, we have $\|v_\delta\|_{l_1}\leqq\tilde{C}_{a,N,s}(u)$ for $0<\delta\leqq\delta_0$. This implies that $v_\delta\to v_0$ weakly in H^{l_2} as $\delta\to 0$ and that $v_0\in H^{l_2}$. Let $\tilde{\chi}(x,\xi)$ is positively homogeneous of degree 0 for $|\xi|\geqq 1$ and $\operatorname{supp}\tilde{\chi}(x,\xi)\cap\{|\xi|\geqq 1\}\Subset\{(x,\xi)\mid \chi(x,\xi)=1$ and $\varphi(x'',\xi)\leqq\varepsilon/2\}$. Then, noting that $-s+a\varphi)x'',\xi)<l_1-\sigma$ if $\varphi(x'',\xi)\leqq\varepsilon/2$, we have $\tilde{\chi}(x,D)u\in H^{\sigma}$. This proves Theorem 1.1.

§ 3. Proofs of Theorem 1.2. and Corollary 2.

In this section we shall prove Theorem 1.2, applying Theorem 1.1, and Corollary 2 by transfinite induction. Recall that $P(n,D)=D_1^2+\alpha(x)D_2^2+\beta(x,D)$ is an operator in $\boldsymbol{R}^2$, $\alpha(x)\geqq 0$ and $\beta(x,D)\in L_{1,0}^1$. We may assume that $\alpha(x)\in\mathscr{B}^{\infty}(\boldsymbol{R}^2)$ and $\beta(x,\xi)\in S_{1,0}^1$. Let $x^0\in\boldsymbol{R}^2$, and let Σ_0 and U_0 be as in § 1. Assume that $\Sigma_0\cap U_0\subset\{x\in\boldsymbol{R}^2\mid f(x)=0\}$, where $f(x)\in C^1(\boldsymbol{R}^2)$ is real-valued, $f(x^0)=0$ and $\partial f/\partial x_1(x^0)\neq 0$. It is sufficient to prove that $P(x,D)$ is microhypoelliptic at $z^0=(x^0;0,\pm 1)$. We shall show that $P(x,D)$ is microhypoelliptic at $(x^0;0,1)$. Note that microhypoelliptic at $(x^0;0,-1)$ can be similarly proved. Choose a real-valued $\varphi(t)\in\mathscr{B}^{\infty}(\boldsymbol{R})$ so that $\varphi(t)=0$ when $\Sigma_0\cap U_0=\{x^0\}$, and $\varphi(t)=(t-x_2^0)^2$ near $t=x_2^0$ when $\Sigma_0\cap U_0\neq\{x^0\}$. We put

$$\Lambda(x,\xi)\equiv\Lambda_\delta(x,\xi)\equiv\Lambda_\delta(x,\xi;a,N,s)=\{-s+a\varphi(x_2)\}\log\lambda(\xi)+N\log(1+\delta\lambda(\xi))$$

for $0\leqq\delta\leqq 1$, $a\geqq 0$, $N\geqq 0$ and $s\in\boldsymbol{R}$, where $\lambda(\xi)$ is defined in § 1. Then there is a conic neighborhood C of $(x^0,0,1)$ in $T^*\boldsymbol{R}^2\setminus 0$ such that

$$\Lambda(x,\xi)=\begin{cases}-s\log\langle\xi_2\rangle+N\log(1+\delta\langle\xi_2\rangle) & \text{if } \Sigma_0\cap U_0=\{x^0\},\\ \{-s+a(x_2-x_2^0)^2\}\log\langle\xi_2\rangle+N\log(1+\delta\langle\xi_2\rangle) & \text{if } \Sigma_0\cap U_0\neq\{x^0\}\end{cases}$$

near $C\cap\{|\xi|\geqq 2\}$. Write $p_{\xi_2}=p_{\xi_2}(x,\xi)=\partial p/\partial\xi_2(x,\xi),\cdots$. A simple calculation yields

$$\begin{aligned}P_\Lambda(x,\xi)=&(1+q(x,\xi))p(x,\xi)+i(\Lambda_{\xi_2}p_{x_2}-\Lambda_{x_2}p_{\xi_2})+\Lambda_{x_2}\Lambda_{\xi_2\xi_2}p_{x_2}\\&+\Lambda_{\xi_2}\Lambda_{x_2x_2}p_{\xi_2}+\Lambda_{\xi_2}\Lambda_{x_2}p_{\xi_2x_2}-(\Lambda_{x_2}^2+\Lambda_{x_2x_2})p_{\xi_2\xi_2}]\\&+R_1(x,\xi)+R_2(x,\xi),\end{aligned}$$

where $q(x,\xi)=i\Lambda_{x_2}\Lambda_{\xi_2}-(\Lambda_{\xi_2}^2-\Lambda_{\xi_2\xi_2})(\Lambda_{x_2}^2+\Lambda_{x_2x_2})/2\in S_{1,0}^{-1+\rho}$ $(\rho>0)$, $R_1(x,\xi)\in S_{1,0}^2$, and $R_2(x,\xi)\in S_{1,0}^0$, $\operatorname{supp}R_1\cap C\cap\{|\xi|\geqq 2\}=\emptyset$, $|R_{1(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{2-|\alpha|}$ and $|R_{2(\beta)}^{(\alpha)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{-|\alpha|}$. Hereafter the constants do not depend on δ if not stated. Since $|1+q(x,\xi)|\geqq 1/2$ for $|\xi|\geqq C_{a,N,s}\gg 1$, there is $Q(x,\xi)\in S_{1,0}^0$ such that $Q(x,\xi)(1+q(x,\xi))=1$ for $|\xi|\geqq C_{a,N,s}$. Define $\tilde{P}_\Lambda(x,D)=Q(x,D)P_\Lambda(x,D)$. Then we have

$$\begin{aligned}\tilde{P}_\Lambda(x,\xi)=\xi_1^2=\alpha(x)\xi_2^2+\mathrm{Re}\,\beta_1(x,\xi)+\Lambda_{x_2}\,\mathrm{Im}\,\beta_1(x,\xi)+2\Lambda_{\xi_2}\Lambda_{x_2x_2}\alpha(x(\xi_2\\ +2\Lambda_{\xi_2}\Lambda_{x_2}\alpha_{x_2}(x)\xi_2-(\Lambda_{x_2}^2+\Lambda_{x_2x_2})\alpha(x)-2i\Lambda_{x_2}\alpha(x)\xi_2\\ +ir(x,\xi)+R_1'(x,\xi)+R_2'(x,\xi)\qquad \text{for } |\xi|\geqq 1\,,\end{aligned}$$

where $\beta_1(x,\xi)$ denotes the principal symbol of $\beta(x,\xi)$, $r(x,\xi)\in S^1_{1,0}$ is real-valued, $|r^{(\alpha)}_{(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\langle\xi\rangle^{1-|\alpha|}$, and $R_1'(x,\xi)$ and $R_2'(x,\xi)$ have the same properties as $R_1(x,\xi)$ and $R_2(x,\xi)$, respectively. Write $\beta_1(x,\xi)=\beta_1(x,0,1)\xi_2+\tilde{\beta}_0(x,\xi)\xi_1$, where $\tilde{\beta}_0(x,\xi)$ is positively homogeneous of degree 0. Then we have

$$\begin{aligned}(3.1)\qquad \tilde{P}_\Lambda(x,\xi)=\xi_1^2+\alpha(x)\xi_2^2+\mathrm{Re}\,\beta_1(x,0,1)\xi_2+e_0(x,\xi)\xi_1+e_1(x,\xi)\alpha(x)\log\lambda(\xi)\\ +e_2(x,\xi)\alpha(x)(\log\lambda(\xi))^2+e_3(x,\xi)\,\mathrm{Im}\,\beta_1(x,0,1)\log\lambda(\xi)\\ +e_4(x,\xi)\alpha_{x_2}(x)\log\lambda(\xi)+ie_5(x,\xi)\alpha(x)\xi_2\log\lambda(\xi)\\ +ir(x,\xi)+R_1'(x,\xi)+R_2'(x,\xi)\qquad \text{for } |\xi|\geqq 1\,,\end{aligned}$$

where $e_j(x,\xi)\in S^0_{1,0}$ $(0\leqq j\leqq 5)$ are real-valued, $e_k(x,\xi)=0$ if $1\leqq k\leqq 5$ and $\Sigma_0\cap U_0=\{x^0\}$, and $e_3(x,\xi)\equiv e_3(x)$ does not depend on ξ.

LEMMA 3.1. *Assume that there are* $\chi(x,\xi)\in S^0_{1,0}$, $\Psi(x)\in\mathscr{B}^\infty(\boldsymbol{R}^2)$, $a_0\geqq 0$, $N_0\geqq 0$ *and* $s_0\in\boldsymbol{R}$ *such that* $\chi(x,\xi)$ *is positively homogeneous of degree* 0 *for* $|\xi|\geqq 1$, $\chi(x,\xi)=1$ *near* $(x_0,0,1)$, $\operatorname{supp}\Psi\cap\Sigma_0=\emptyset$ *and the bollowing property holds; for any* $a\geqq a_0$, *any* $N\geqq N_0$ *and any* $s\geqq s_0$ *there are* $\delta_0>0$ $(\delta_0\leqq 1)$, $C_0>$ *and* $C>0$ *such that* (3.2) $\mathrm{Re}\,(\tilde{P}_\Lambda(x,D)v,v)\geqq C_0\|D_1v\|^2-C\{\|v\|^2+\|(1-\chi(x,D))v\|_2^2+\|\Psi(x)v\|_2^2\}$ *if* $v\in C_0^\infty$ *and* $0<\delta\leqq\delta_0$. *Then* $(x^0,0,1)\notin WF(u)$ *if* $u\in\mathscr{D}'$ *and* $(x^0,0,1)\notin WF(P(x,D)u)$.

PROOF. Note that the condition (H) in §1 is satisfied with $\Sigma=\{(x,\xi)\in T^*\boldsymbol{R}^2\setminus 0\mid x\in\Sigma_0$ and $\xi_1=0\}$ and $W=\{(\delta x,\delta\xi_1,0)\in T_{z^0}(T^*\boldsymbol{R}^2)\mid\delta x_2=0\}$, where $z^0=(x^0,0,1)$. Applying the implicit function theorem, we can write $\{x\in U\mid f(x)=0\}=\{(g(x_2),x_2)-|x_2-x_2^0|\leqq c\}$, where U is a neighborhood of x^0, $g(t)\in C^1(x_2^0-c,x_2^0+c)$ and $c>0$. Choose $\Psi(t)\in C_0^\infty(\boldsymbol{R})$ so that $0\leqq\Psi(t)\leqq 1$, $\Psi(t)=1$ if $|t|\leqq 1/2$ and $\operatorname{supp}\Psi(t)\subset\{|t|\leqq 1\}$. For $d>0$, write $v=v_1+v_2+v_3$, where $v\in C_0^\infty$, $v_1=\Psi((x_1-g(x_2))/d)\Psi((x_2-x_2^0)/c)v$, $v_2=(1-\Psi((x_1-g(x_2))/d))\Psi((x_2-x_2^0)/c)v$, $v_3=(1-\Psi((x_2-x_2^0)/c))v$. Applying Poincare's inequality to v_1, we have $\|v_1\|\leqq\sqrt{2}\,d\|D_1v\|$. Since $\operatorname{supp}(1-\Psi((x_1-g(x_2))/d))\Psi((x_2-x_2^0)/c)\cap\Sigma_0=\emptyset$, there are $\Psi_d(x)\in\mathscr{B}^\infty(\boldsymbol{R}^2)$ and $C_d>0$ such that $\operatorname{supp}\Psi_d\cap\Sigma_0=\emptyset$ and $\|v_2\|\leqq C_d\|\Psi_d(x)v\|_1$. Therefore, for any $\varepsilon>0$ there are $d>0$ and $C>0$ such that

$$(3.3)\qquad \|v\|^2\leqq\varepsilon\|D_1v\|^2+C\{\|(1-\Psi((x_2-x_2^0)/c))v\|^2+\|\Psi_d(x)v\|_1^2\}\qquad\text{for } v\in C_0^\infty\,.$$

Since $\mathrm{Re}\,(\tilde{P}_A(x, D)v, v) \leqq C\|P_A(x, D)v\|^2 + \|v\|^2$, it follows from (3.2) and (3.3) that

$$\|v\| \leqq C_d\{\|P_A(x, D)v\| + \|(1-\chi(x, D))v\|_2 + \|(1-\Psi((x_2 - x_2^0)/c))v\|_1$$
$$+ \|\Psi(x)v\|_2 + \|\Psi_d(x)v\|_1\}$$

if $v \in C_0^\infty$ and $0 < \delta \leqq \delta_0$, where $0 < d \ll 1$. So we can apply Theorem 1.1 and prove the lemma.

Next we shall prove that (3.2) holds in the cases (i) and (ii) in Theorem 1.2, respectively. We need the following

LEMMA 3.2. *For any* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$|\mathrm{Re}\,(e_1(x, D)\alpha(x)(\log \lambda(D))v, v)|$$
$$\leqq \varepsilon(\alpha(x)D_2 v, D_2 v) + C_\varepsilon\{\|v\|^2 + \|(1-\chi(x, D))v\|_1^2\} \qquad \text{for } v \in C_0^\infty .$$

PROOF. Choose $\chi(x, \xi) \in S_{1,0}^0$ so that $\mathrm{supp}\,\chi \Subset \mathcal{C} \cap \{|\xi| \geqq 2\}$. Then we can write

$$e_1(x, D)\alpha(x) \log \lambda(D) \equiv T_1^{(-1+\rho)}\alpha(x)D_2 + T_2^{(\rho)}(1-\chi(x, D)) \bmod L_{1,0}^{-1+\rho}$$

if $0 < \rho < 1$, where $T_j^{(*)}$ $(j=1, 2)$ means the pseudodifferential operators with the symbols in $S_{1,0}^*$. Hence, for any $\varepsilon > 0$ we have

$$|\mathrm{Re}\,(e_1(x, D)\alpha(x)(\log \lambda(D))v, v)|$$
$$\leqq |\mathrm{Re}\,(T_1^{(-1+\rho)}\alpha(x)D_2 v, v)| + |\mathrm{Re}\,(T_2^{(\rho)}(1-\chi(x, D))v, v)| + C\|v\|_{-1+\rho}\|v\|$$
$$\leqq \varepsilon\|\alpha(x)D_2 v\|^2 + C_\varepsilon\|v\|^2 + C\|(1-\chi(x, D))v\|_\rho^2$$
$$\varepsilon C'(\alpha(x)D_2 v, D_2 v) + C'_\varepsilon\{\|v\|^2 + \|(1-\chi(x, D))v\|_1^2\} \qquad \text{for } v \in C_0^\infty .$$

The proof is complete.

REMARK. By the same method, we can show that

$$\mathrm{Re}\,(e_2(x, D)\alpha(x)(\log \lambda(D))^2 v, v) \quad \text{and} \quad \mathrm{Re}\,(e_4(x, D)\alpha_{x_2}(x)(\log \lambda(D))v, v)$$

have the estimates of the same form as the above. To estimate $\mathrm{Re}\,(e_4(x, D)\alpha_{x_2}(x)(\log \lambda(D))v, v)$, we must use the well-known fact for non-negative functions that $|\alpha_{x_2}(x)| \leqq C\sqrt{\alpha(x)}$ near the origin. Moreover we can prove that $\mathrm{Re}\,(ie_5(x, D)\alpha(x)(\log \lambda(D))v, v)$ has the estimate of the same form as the above, since $ie_5(x, \xi)\alpha(x) \log \lambda(\xi)$ is purely imaginary.

From (3.1) and Lemma 3.2 we obtain

$$\text{(3.4)} \quad \operatorname{Re}(\tilde{P}_\Lambda(x, D)v, v) \geqq (1-\varepsilon)\{\|D_1v\|^2+(\alpha(x)D_2v, D_2v)\}+\operatorname{Re}(\operatorname{Re}\beta_1(x,0,1)D_2v, v)$$
$$+\operatorname{Re}(e_3(x, D)\operatorname{Im}\beta_1(x, 0, 1)(\log\lambda(D))v, v)$$
$$-C_\varepsilon\{\|v\|^2+\|(1-\chi(x, D))v\|_2^2\} \qquad \text{for } v\in C_0^\infty,$$

where $\varepsilon>0$.

From now on, we shall prove (3.2) in the cases (i) and (ii) of Theorem 1.2 respectively, by using (3.4).

Assume that $\Sigma_0\cap U_0=\{x^0\}$. Then we have $e_k(x, \xi)=0$ $(1\leqq k\leqq 5)$. Hence the third term in the right hand side of (3.5) vanishes. It follows from (1.2) that for $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\text{(3.5)} \quad \operatorname{Re}(\operatorname{Re}\beta_1(x, 0, 1)D_2v, v)+C\varepsilon(\alpha(x)D_2v, D_2v)\geqq -C_\varepsilon\|v\|^2 \qquad \text{for } v\in C_0^\infty.$$

Therefore (3·2) holds.

Next assume that $\Sigma_0\cap U_0\neq\{x^0\}$. First we note that

$$\text{(3.6)} \quad |\operatorname{Re}(e_3(x, D)\operatorname{Im}\beta_1(x, 0, 1)(\log\lambda(D))v, v)|$$
$$\leqq\{\|\operatorname{Im}\beta_1(x, 0, 1)(\log\lambda(D))v\|^2+\|e_3(x, D)^*v\|^2\}/2 \qquad \text{for } v\in C_0^\infty.$$

Then we have the following

LEMMA 3.3. *If* (1.3) *is valid, then for any* $\varepsilon>0$ *there exists* $C_\varepsilon>0$ *such that*

$$\|\operatorname{Im}\beta_1(x, 0, 1)(\log\lambda(D))v\|^2\leqq\varepsilon(\alpha(x)D_2v, D_2v)$$
$$+C_\varepsilon\{\|v\|^2+\|(1-\chi(x, D))v\|_2^2\} \qquad \textit{for } v\in C_0^\infty.$$

PROOF. Let us first prove that for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\text{(3.7)} \quad \|h(x)(\log\lambda(D))v\|^2$$
$$\leqq\varepsilon\{(h(x)^{2^{k+1}}\lambda(D)v, \lambda(D)v)+C_{k+1}\|v\|^2\}+C_\varepsilon\|v\|_{-1/2}^2 \qquad \text{for } v\in C_0^\infty,$$

where $h(x)=\operatorname{Im}\beta_1(x, 0, 1)$. Let ρ be a positive number. Nyting that

$$h(x)\log\lambda(D)=(\log\lambda(D))\lambda(D)^{-\rho}h(x)\lambda(D)^\rho+[h(x), (\log\lambda(D))\lambda(D)^{-\rho}]\lambda(D)^\rho,$$

we have

$$\|h(x)(\log\lambda(D))v\|^2\leqq\varepsilon\|h(x)\lambda(D)^\rho v\|^2+C_\varepsilon\|v\|_{-1/2}^2 \qquad \text{for } v\in C_0^\infty.$$

If ρ satisfies $2\rho\leqq 1$, we have

$$\|h(x)\lambda(D)^\rho v\|^2\leqq\|h(x)\lambda(D)^{2\rho}v\|^2+C\|v\|^2.$$

Moreover if ρ satisfies $2^k\rho\leqq 1$, where k is any positive integer, then there exists $C_k>0$ such that

$$\|h(x)\lambda(D)^\rho v\|^2\leqq\|h(x)^{2^k}\lambda(D)^{2^k\rho}v\|^2+C_k\|v\|^2 \qquad \text{for } v\in C_0^\infty.$$

Taking $2^k\rho=1$, we have (3.7). Next we shall prove that

$$(3.8)\qquad (h(x)^{2^{k+1}}\lambda(D)v,\ \lambda(D)v)\leqq(h(x)^{2^{k+1}}D_2v,\ D_2v)+C\|v\|^2$$
$$+C_{k+1}\|(1-\chi(x,\ D))v\|_2^2 \qquad \text{for } v\in C_0^\infty.$$

Choose $\mu(x,\xi)\in S_{1,0}^0$ so that $\mu(x,\xi)$ is positively homogeneous of degree 0 for $|\xi|\geqq 1$ and $\mu(x,\xi)=1$ near $\operatorname{supp}\chi(x,\xi)$. We may assume that $\operatorname{supp}\chi\subset U\times\{\xi\in\boldsymbol{R}\mid \xi_2\geqq 2|\xi|/3\geqq 2\}$, where U is some neighborhood of x^0. Write $(h(x)^{2^{k+1}}\lambda(D)v,\ \lambda(D)v)=(h(x)^{2^{k+1}}D_2v,\ D_2v)+(\lambda(D)\mu(x,D)^*[\mu(x,D),\ h(x)^{2^{k+1}}]\lambda(D)v,\ v)+(h(x)^{2^{k+1}}(\mu(x,\ D)\lambda(D)-D_2)v,\ \mu(x,\ D)\lambda(D)v)+(h(x)^{2^{k+1}}D_2v,\ (\mu(x,\ D)\lambda(D)-D_2)v)+(\lambda(D)(1-\mu(x,\ D)^*\mu(x,\ D))h(x)^{2^{k+1}}\lambda(D)v,\ v)\equiv(h(x)^{2^{k+1}}D_2v,\ D_2v)+I_1+I_2+I_3+I_4$. Hence it is sufficient to prove that $|I_j|\leqq C_{k+1}\{\|v\|^2+\|(1-\chi(x,\ D))a\|_2^2\}$ $(1\leqq j\leqq 4)$. Since $\operatorname{supp}\sigma([\mu(x,\ D),\ h(x)^{2^{k+1}}])\cap\operatorname{supp}\chi=\emptyset$, where $\sigma(R)$ denotes the symbol of $R(x,\ D)$, we have

$$|I_1|\leqq C_{k+1}\{\|v\|^2+\|(1-\chi(x,\ D))v\|_1^2\}.$$

Noting that $(\mu(x,\ \xi)\lambda(\xi)-\xi_2)\chi(x,\ \xi)\equiv 0$, we have

$$|I_2|\leqq C_{k+1}\{\|v\|^2+\|(1-\chi(x,\ D))v\|_2^2\}.$$

Concerning I_3, the proof is similar to the above one. Since $\operatorname{supp}(1-\mu(x,\ \xi)^2)\cap\operatorname{supp}\chi(x,\ \xi)=\emptyset$, we have

$$|I_4|\leqq\{\|v\|^2+\|(1-\chi(x,\ D)(v\|_2^2\}.$$

This proves (3.8), which completes the proof.

From (3.4), (3.6) and Lemma 3.3 it follows that (3.2) holds. Lemma 3.1 gives $(x^0,\ 0,\ 1)\notin WF(u)$ if $u\in\mathcal{D}'$ and $(x^0,\ 0,\ 1)\notin WF(P(x,\ D)u)$. When $\Sigma_0\cap U_0\neq\{x^0\}$, applying the same argument with x^0 replaced by $x_1\in\Sigma_0\cap U_0$, we can prove Theorem 1.2, where U_0 is a small neighborhood of x^0.

Next let us prove Corollary 2 of Theorem 1.2. Put $\tilde{\omega}=\operatorname{card}(\mathcal{P}(\boldsymbol{R}^2))$, i.e., $\tilde{\omega}$ is the cardinal number of $\mathcal{P}(\boldsymbol{R}^2)$. For any ordinal number $\zeta<\tilde{\omega}$ we define the mapping $\tau_\zeta\colon\mathcal{P}(\boldsymbol{R}^2)\to\mathcal{P}(\boldsymbol{R}^2)$ by $\tau_0(A)=A$ and

$$\tau_\zeta(A)=\begin{cases}\bigcap_{\zeta'<\zeta}\tau_{\zeta'}(A) & \text{if } \zeta \text{ is a limit ordinal number.}\\ \tau(\tau_{\zeta'}(A)) & \text{if } \zeta=\zeta'+1,\end{cases}$$

where $A\subset\boldsymbol{R}^2$.

LEMMA 3.4. *Let A be a subset of $\boldsymbol{R}^2$, and $A=\bigcup_{B\subset A,\tau(B)=B}B$. Then* (i) $\tau(\tilde{A})=\tilde{A}$. (ii) *There exists $\zeta(<\tilde{\omega})$ such that $\tau_\zeta(A)=\tau_{\zeta+1}(A)$.* (iii) *There exists $\zeta_0(<\tilde{\omega})$ such that $\bigcap_{\zeta<\tilde{\omega}}\tau_\zeta(A)=\tau_{\zeta_0}(A)$. Moreover, we have $\tilde{A}=\bigcap_{\zeta<\tilde{\omega}}\tau_\zeta(A)=\tau_{\zeta_0}(A)$.*

PROOF. (i) Let B be a subset of $\tilde{A}$ satisfying $\tau(B)=B$. Then we have

$B\subset\tilde{A}$. Therefore, it follows from $\tau(\tilde{A})\supset\tau(B)=B$ that $\tilde{A}\supset\tau(\tilde{A})\supset\bigcup_{B\subset A,\tau(B)=B}B=\tilde{A}$. (ii) We assume tha† $\tau_\zeta(A)\supsetneqq\tau_{\zeta+1}(A)$ for any $\zeta<\tilde{\omega}$. Then for any $\zeta<\tilde{\omega}$ there exists $x_\zeta\in\tau_\zeta(A)\setminus\tau_{\zeta+1}(A)$. It is obvious that $x_\zeta\neq x_{\zeta}$, if $\zeta\neq\zeta'$. Hence $\operatorname{card}(\{x_\zeta|\zeta<\tilde{\omega}\})=\tilde{\omega}$. On the other hand, $\{x_\zeta|\zeta<\tilde{\omega}\}\subset A$, which leads the contradiction. (iii) From the assertion (ii) there exists $\zeta_0<\tilde{\omega}$ such that $\tau_{\zeta_0}(A)=\tau_{\zeta_0+1}(A)$ Then we can show that

$$\tau_\zeta(A)=\tau_{\zeta_0}(A) \qquad \text{if } \zeta_0\leqq\zeta(<\tilde{\omega}) \tag{3.9}$$

In fact, if $\zeta=\zeta_0$, (3.9) is trivial. Now if we assume that $\tau_\zeta(A)=\tau_{\zeta_0}(A)$ if $\zeta_0\leqq\zeta<\zeta_1$, it follows from the definition of $\tau_\zeta(A)$ that $\tau_{\zeta_1}(A)=\tau_{\zeta_0}(A)$. Transfinite induction gives (3.9). Hence $\tau_{\zeta_0}(A)=\bigcap_{\zeta<\tilde{\omega}}\tau_\zeta(A)$. We can also prove that $\tau_\zeta(A)\supset\tilde{A}$ for any $\zeta<\tilde{\omega}$ by transfinite induction. Then we have $\tilde{A}\subset\bigcap_{\zeta<\tilde{\omega}}\tau_\zeta(A)=\tau_{\zeta_0}(A)$. On the other hand, we have also $\tau_{\zeta_0}(A)\subset\tilde{A}$ in view of the definition of $\tilde{A}$. Hence $\tilde{A}=\bigcap_{\zeta<\tilde{\omega}}\tau_\zeta(A)=\tau_{\zeta_0}(A)$.

Now we can prove Corollary 2. We note that if $P(x, D)$ is microhypollip†ic in $\boldsymbol{R}^2\setminus S$, so is $P(x, D)$ in $\boldsymbol{R}^2\setminus\tau(S)$ in view of Theorem 1.2. Hence if suffices to prove that $P(x, D)$ is microhypoelliptic in $\boldsymbol{R}^2\setminus\tau_\zeta(S)$ for $\zeta<\tilde{\omega}$. We can prove the above assertion by transfinite induction. In fact, the assertion is trivial if $\zeta=0$. Now we assume that $P(x, D)$ is microhypoelliptic in $\boldsymbol{R}^2\setminus\tau_{\zeta'}(S)$ for $\zeta'<\zeta$. When there exists ζ' such that $\zeta=\zeta'+1$, it follows from $\tau_\zeta(S)=\tau(\tau_{\zeta'}(S))$ and the above argument that $P(x, D)$ is microhypoelliptic in $\boldsymbol{R}^2\setminus\tau_\zeta(S)$. Assume that ζ is a limit ordinal number. If $x\in\tau_\zeta(S)$, then it follows from $\tau_\zeta(S)=\bigcap_{\zeta'<\zeta}\tau_{\zeta'}(S)$ that there exists $\zeta'<\zeta$ such that $x\in\tau_{\zeta'}(S)$. Hence $P(x, D)$ is microhypoelliptic at x. Therefore $P(x, D)$ is microhypoelliptic in $\boldsymbol{R}^2\setminus\tau_\zeta(S)$. The proof is complete.

§4. Proof of Theorem 1.3.

First we shall prove the following

LEMMA 4.1. *If* (1.2) *holds or if there exists* $l\in\boldsymbol{N}$ *such that* (1.4)-(1.6) *are valid, then there exist constants* $h>0$, $C_0>0$ $(C_0<1)$ *and* $C>0$ *such that*

$$\begin{aligned} C_0\{\|D_1v\|^2+(\alpha(x(D_2v, D_2v))\}+\operatorname{Re}(\operatorname{Re}\beta_1(x, 0, 1)D_2v, v) \\ \geqq -C\{\|v\|^2+\|(1+\chi(x, D))v\|_2^2\} \end{aligned} \tag{4.1}$$

if $v\in C_0^\infty$ *and* $\operatorname{supp} v\subset\{x \mid |x_1|<h\}$, *where* $\chi(x, \xi)$ *is positively homogeneous of degree* 0 *for* $|\xi|\geqq 1$ *and* $\operatorname{supp}\chi(x, \xi)\subset U\times\{\xi|\xi_2\geqq 2|\xi|/3\geqq 2\}$.

We have already proved in §3 that (4.1) is valid if (1.2) holds. Therefore, we now assume that there exists $l\in \boldsymbol{N}$ such that (1.4)-(1.6) are valid. Let $\Psi_1(\xi)\in S_{1,0}^0$ be a real-valued symbol such that $\Psi_1(\xi)$ is positively homogeneous of degree 0 for $|\xi|\geqq 1$, $0\leqq\Psi_1(\xi)\leqq 1$, $\Psi_1(\xi)=1$ if $\xi_2\geqq|\xi|/3$ and $|\xi|\geqq 1$, and $\operatorname{supp}\Psi_1\subset\{\xi|\xi_2\geqq|\xi|/6$ and $|\xi|\geqq 1/2\}$. Set $\Psi_2(\xi)=1-\Psi_1(\xi)$. We need several lemmas to prove Lemma 4.1.

LEMMA 4.2. *Write* $\beta(x)=\operatorname{Re}\beta_1)x, 0, 1)$. *Then we have*

$$|(\beta(x)D_2v, v)|\leqq|(\beta(x)D_2\Psi_1(D)v, \Psi_1(D)v)|$$
$$+C\{\|v\|^2+\|(1-\chi(x, D)v\|_1^2\} \qquad \textit{for } v\in C_0^\infty.$$

PROOF. Since $\boldsymbol{R}^2\times\operatorname{supp}\Psi_2(\xi)\cap\operatorname{supp}\chi(x, \xi)=\emptyset$, the lemma easily follows.

Let $\Psi(\xi)\in S_{1,0}^0$ be a real-valued symbol such that $\Psi(\xi)$ is positively homogeneous of degree 0 for $|\xi|\geqq 1/2$, $\Psi(\xi)=1$ on $\operatorname{supp}\Psi_1$ and $\operatorname{supp}\Psi\subset\{\xi|\xi_2\geqq|\xi|/7$ and $|\xi|\geqq 1/3\}$. Note that $\Psi_1(D)=\Psi(D)\Psi_1(D)$. Put $\mathcal{D}(\xi)=\xi_2^{1/2}\Psi(\xi)\in S_{1,0}^{1/2}$.

LEMMA 4.3. *We have*

$$|(\beta(x)D_2\Psi_1(D)v, \Psi_1(D)v)|$$
$$\leqq|(\beta(x)\mathcal{D}(D)\Psi_1(D)v, \mathcal{D}(D)\Psi_1(D)v)|+C\|v\|^2 \qquad \textit{for } v\in C_0^\infty.$$

PROOF. Since $D_2\Psi_1(D)=\mathcal{D}(D)^2\Psi_1(D)$, we have

$$(\beta(x)D_2\Psi_1(D)v, \Psi_1(D)v)=(\beta(x)\mathcal{D}(D)\Psi_1(D)v, \mathcal{D}(D)\Psi_1(D)v)$$
$$+([\beta(x), \mathcal{D}(D)]\mathcal{D}(D)\Psi_1(D)v, \Psi_1(D)v).$$

It is obvious that $[\beta(x), \mathcal{D}(D)]\in L_{1,0}^{-1/2}$, which proves the lemma.

We may assume that $B(t)$ is defined on $\boldsymbol{R}$. For example, we define $B(t)=0$ if $|t|>c_0$.

LEMMA 4.4. *Set* $\tilde{v}=\int\exp(-ix_2\xi_2)v(x)dx_2$, *where* $v\in C_0^\infty$. *Then we have*

$$|(\beta(x)D_2v, v)|\leqq 2\int_0^\infty\Big(\int B(x_1)\xi_2|\tilde{v}(x_1, \xi_2)|^2dx_1\Big)d\xi \tag{4.2}$$
$$+C\{\|v\|^2+\|(1-\chi(x, D))v\|_1^2\} \qquad \textit{for } v\in C_0^\infty,$$

where $đ\xi_2=(2\pi)^{-1}d\xi_2$.

PROOF. By Lemma 4.2 and 4.3 we have

$$|(\beta(x)D_2v, v)|\leqq|(\beta(x)\mathcal{D}(D)\Psi_1(D)v, \mathcal{D}(D)\Psi_1(D)v)| \tag{4.3}$$
$$+C\{\|v\|^2+\|(1-\chi(x, D))v\|_1^2\}.$$

Modifying U if necessary, we may assume that $|\beta(x)|\leqq B(x_1)$ for $x\in U$. Since $\operatorname{supp}\chi\subset U\times\boldsymbol{R}^2$, it is easy to see that

$$|(\beta(x)\mathcal{D}(D)\Psi_1(D)v,\ \mathcal{D}(D)\Psi_1(D)v)|\leqq(B(x_1)\mathcal{D}(D)\Psi_1(D)v,\ \mathcal{D}(D)\Psi_1(D)v)$$
$$+C\{\|v\|^2+\|(1-\chi(x,\ D))v\|_1^2\}.$$

From Parseval's formula it follows that

$$(B(x_1)\mathcal{D}(D)\Psi_1(D)v,\ \mathcal{D}(D)\Psi_1(D)v)=\int_0^\infty\Bigl(\int B(x_1)\xi_2|\tilde{w}_1(x_1,\ \xi_2)|^2dx_1\Bigr)d\!\!{}^-\xi_2, \tag{4.4}$$

where $w_1(x)=\Psi_1(D)v(x)$ and $\tilde{w}_1(x_1,\ \xi_2)=\int\exp(-ix_2\xi_2)w_1(x)dx_2$. In fact, we have $\mathcal{D}(\xi)=(\xi_2)_+^{1/2}\Psi(\xi)$, where $(\xi_2)_+=\max\{\xi_2,\ 0\}$. Therefore we have

$$\int\exp(-ix_2\xi_2)\mathcal{D}(D)\Psi_1(D)v(x)dx_2=(\xi_2)_+^{1/2}\tilde{w}_1(x_1,\ \xi_2).$$

Put $w_2(x)=\Psi_2(D)v(x)$ and $\tilde{w}_2(x_1,\ \xi_2)=\int\exp(-ix_2\xi_2)w_2(x)dx_2$. Then we have

$$\int_0^\infty\Bigl(\int B(x_1)\xi_2|\tilde{w}_2(x_1,\ \xi_2)|^2dx_1\Bigr)d\!\!{}^-\xi_2\leqq C\int|\xi_2|\,|\tilde{w}_2(x_1,\ \xi_2)|^2dx_1d\!\!{}^-\xi_2$$
$$\leqq C'\|w_2\|_{1/2}^2$$
$$\leqq C''\{\|v\|^2+\|(1-\chi(x,\ D))v\|_1^2\},$$

since $\boldsymbol{R}^2\times\operatorname{supp}\Psi_2(\xi)\cap\operatorname{supp}\chi=\emptyset$. This, together with (4.3) and)4.4), gives (4.2).

From now on, we shall estimate $E=\int_0^\infty\Bigl(\int B(x_1)\xi_2|\tilde{v}(x_1,\ \xi_2)|^2dx_1\Bigr)d\!\!{}^-\xi_2$. Put $E(\xi_2)=\int B(x_1)\xi_2|\tilde{v}(x_1,\ \xi_2)|^2dx_1$. We fix $\xi_2\gg1$ and take $\chi_1(t)\in C_0^\infty(\boldsymbol{R})$ so that $0\leqq\chi_1(t)\leqq1$, $\chi_1(t)=1$ if $|t|\leqq1$, and $\chi_1(t)=0$ if $|t|\geqq2$. By the assumption, there exists $h>0$, $A_1>0$, $B_1>0$ such that $|t|^{2l}/A(t)\leqq A_1$ for $|t|\leqq h$, $|t|^{1-l}B(t)\leqq B_1$ for $|t|\leqq h$ and $2^{l+5}\{1+2^{l+2}/l(l+1)\}A_1B_1^2/l(l+1)<1$. Put $\varphi(t)=\chi_1(K^{-1}(\xi_2/(A_1B_1))^{1,(l+1)}t)$, where $K=\{(2^2+2^{l+4}/l(l+1))^{1/(l+1)}+(2^3A_1B_1^2/l(l+1))^{-1/(l+1)}/2\}/2$. Hereafter we assume that $\operatorname{supp}v\subset\{x\,|\,|x_1|<h\}$. Write

$$E(\xi_2)\leqq2\Bigl\{\int B(x_1)\xi_2|(1-\varphi(x_1))\tilde{v}(x_1,\ \xi_2)|^2dx_1$$
$$+\int B(x_1)\xi_2|\varphi(x_1)\tilde{v}(x_1,\ \xi_2)|^2dx_1\Bigr\}\equiv2(E^1(\xi_2)+E^2(\xi_2)).$$

Since $\operatorname{supp}(1-\varphi(x_1))\subset\{x_1\mid K^{-1}(\xi_2/(A_1B_1))^{1/(l+1)}|x_1|\geqq1\}$ and $|x_1|^{1-l}B(x_1)\leqq B_1$ for $|x_1|\leqq h$, we have

$$B(x_1)\leqq B_1|x_1|^{l-1}\leqq K^{-(l+1)}A_1^{-1}\xi_2|x_1|^{l+1}|x_1|^{l-1}\leqq K^{-(l+1)}A(x_1)\xi_2$$

in supp $(1-\varphi(x_1))$ if $|x_1|\leqq h$. In the last inequality we have used $|x_1|\times A(x_1)^{-1}\leqq A_1$ for $|x_1|\leqq h$. Hence we have

$$E^1(\xi_2)\leqq K^{-(l+1)}\int A(x_1)|\xi_2|^2|\tilde{v}(x_1, \xi_2)|^2dx_1 .$$

To estimate $E^2(\xi_2)$, we take $a>0$, which depends on ξ_2, such that $aK^{-1}(\xi_2/(A_1B_1))^{1/(l+1)}=2$. Then we obtain

$$\int_0^a B(x_1)\xi_2|\varphi(x_1)\tilde{v}(x_1, \xi_2)|^2dx_1$$

$$\leqq(2K)^{l+1}A_1B_1^2\{l(l+1)\}^{-1}\int_0^\infty|D_1(\varphi(x_1)\tilde{v}(x_1, \xi_2))|^2dx_1 .$$

We have also

$$\int_0^\infty|D_1(\varphi(x_1)\tilde{v}(x_1, \xi_2))|^2dx_1$$

$$\leqq 2\left\{C_1^2K^{-2(l+1)}(A_1B_1^2)^{-1}\int_0^\infty A(x_1)|\xi_2|^2|\tilde{v}|^2dx_1+\int_0^\infty|D_1\tilde{v}|^2dx_1\right\} ,$$

where put $C_1=\text{supp}|\chi_1'(t)|$. In fact, we have

$$(K^{-1}(\xi_2/(A_1B_1))^{1/(l+1)})^{2l}A_1A(t)\geqq 1 \qquad \text{if } \varphi'(t)\neq 0 .$$

Here we note that we can take $C_1(>1)$ so that C_1 is close enough to 1. In the some manner we have the similar estimates for

$\int_{-a}^0 B(x_1)\xi_2|\varphi(x_1)\tilde{v}|^2dx_1$ and $\int_{-\infty}^0|D_1(\varphi(x_1)\tilde{v})|^2dx_1$. We may assume that $h\leqq c_0$. Summing up the above estimates, we have the following

LEMMA 4.5. *Assume that* $v\in C_0^\infty$ *and* supp $v\subset\{\ |\ |x_1|<h\}$. *Then we have*

$$\int_0^\infty\left(\int B(x_1)\xi_2|\tilde{v}(x_1, \xi_2)|^2dx_1\right)d\xi_2 \tag{4.5}$$

$$\leqq K^{-(l+1)}(1+2^{l+2}/l(l+1))\int A(x_1)|\xi_2|^2|\tilde{v}(x_1, \xi_2)|^2dx_1d\xi_2$$

$$+2(2K)^{l+1}A_1B_1^2\{l(l+1)\}^{-1}\int|D_1\tilde{v}(x_1, \xi_2)|^2dx_1d\xi_2 .$$

Now we can prove Lemma 4.1. Note that

$$\int A(x_1)|\xi_2|^2|\tilde{v}(x_1, \xi_2)|^2dx_1d\xi_2\leqq(\alpha(x)D_2v, D_2v),$$

$$\int|D_1\tilde{v}(x_1, \xi_2)|^2dx_1d\xi_2\leqq\|D_1v\|^2 ,$$

where supp $v\subset\{x\ |\ |x_1|<h\}$. Therefore, it follows from (4.2) and (4.5) that

(4.1) holds. In fact, we have

$$\{2^2(1+2^{l+2}/l(l+1))\}^{1/(l+1)}<K<(2^3A_1B_1^2/l(l+1))^{-1/(l+1)}/2.$$

This gives $2^2K^{-(l+1)}(1+2^{l+2}/l(l+1)(<1$ and $2^3(2K)^{l+1}A_1B_1^2/l(l+1)<1$.

Let us prove the assertion (i) of Theorem 1.3. Now take $\varphi(t)\in\mathscr{B}^\infty$ in $A_\delta(x,\xi)$ so that $\varphi(t)=0$. Note that $e_3(x,\xi)=0$ in (3.6). (3.6) and (4.1) show that (3.1) holds. This proves the assertion (1) of Theorem 1.3.

LEMMA 4.6. *If* (1.3) *holds or* $\lim_{t\to 0}t^2\Gamma(t)\log A(t)=0$, *then for any* $\varepsilon>0$ *there exists* $C_\varepsilon>0$ *and* $\Psi(x)\in\mathscr{B}^\infty(\boldsymbol{R}^2)$ *such that* $\operatorname{supp}\Psi\cap\{x\mid x_1=0\}=\emptyset$ *and*

$$|\mathrm{Re}\,(e_3(x,D)\,\mathrm{Im}\,\beta_1(x,0,1)(\log\lambda(D))v,v)| \tag{4.6}$$
$$\leqq\varepsilon\{\|D_1v\|^2+(\alpha(x)D_2v,D_2v)\}+C_\varepsilon\|v\|^2+C\{\|(1-\chi(x,D))v\|_2^2+\|\Psi(x)v\|_2^2\}$$

for $v\in C_0^\infty$, *where* $\chi(x,\xi)$ *is the same as in Lemma* 4.1.

In Lemma 3.3 in §3 we have already proved that (4.6) holds under the assumption (1.3). Therefore, from now on we shall prove that (4.6) holds if $\lim_{t\to 0}t^2\Gamma(t)\log A(t)=0$. We may assume that $U\subset\{x\mid |x_1|<c_0\}$. $L_{\xi_2}=D_1^2+A(x_1)\xi_2^2$. A simple modification of the proof of Proposition 3.1 in [3] gives the following

LEMMA 4.7. *For any* $\varepsilon>0$ *there exists* $n_0>0$ *such that*

$$\int\Gamma(x_1)(\log|\xi_2|)|\tilde{v}(x_1,\xi_2)|^2dx_1\leqq\varepsilon L_{\xi_2}\tilde{v}(x_1,\xi_2)\cdot\overline{\tilde{v}(x_1,\xi_2)}dx_1.$$

for $v\in C_0^\infty(U)$ *and for all* $\xi_2\geqq n_0$, *where* $\tilde{v}(x_1,\xi_2)=\int\exp(-ix_2\xi_2)v(x)dx_2$.

PROOF. Assume that $\operatorname{supp}\tilde{v}(\cdot,\xi_2)\subset\{x_1\in\boldsymbol{R}\mid A(x_1)|\xi_2|^{1/2}\geqq 1/2\}$. Let $\varepsilon>0$. If $|\xi_2|>\varepsilon^{-1}$, then we have

$$\begin{aligned}\int\Gamma(x_1)(\log|\xi_2|)|\tilde{v}(x_1,\xi_2)|^2dx_1&\leqq C\int|\xi_2|^{1/2}|\tilde{v}(x_1,\xi_2)|^2dx_1\\&\leqq 2C\varepsilon\int A(x_1)|\xi_2|^2|\tilde{v}(x_1,\xi_2)|^2dx_1\\&\leqq 2C\varepsilon\int L_{\xi_2}\tilde{v}(x_1,\xi_2)\cdot\overline{\tilde{v}(x_1,\xi_2)}dx_1.\end{aligned}$$

Next assume that $\operatorname{supp}\tilde{v}(\cdot,\xi_2)\subset\{x_1\in\boldsymbol{R}\mid A(x_1)|\xi_2|^{1/2}\leqq 2\}$. Choose $a=a(\xi_2)$ so that $A(a)|\xi_2|^{1/2}=2$. Noting that $\tilde{v}(x_1,\xi_2)=-\int_{x_2}^a\partial\tilde{v}/\partial x_1(s,\xi_2)ds$, we have

$$\int_0^a\Gamma(x_1)|\tilde{v}(x_1,\xi_2)|^2dx_1\leqq\Gamma(a)a^22^{-1}\int L_{\xi_2}\tilde{v}(x_1,\xi_2)\cdot\overline{\tilde{v}(x_1,\xi_2)}dx_1.$$

By the assumption we can see that

$$\Gamma(a)a^2 \log|\xi_2| \leqq \varepsilon(\log|\xi_2|)(\log A(a))^{-1} \leqq C\varepsilon$$

if $\xi_2 \gg 1$. In fact, we have $\lim_{\xi_2\to 0} a(\xi_2)=0$. Therefore, we obtain

$$\int_0^a \Gamma(x_1)(\log|\xi_2|)|\tilde{v}(x_1, \xi_2)|^2 dx_1 \leqq C\varepsilon \int L_{\xi_2}\tilde{v}(x_1, \xi_2)\cdot\overline{\tilde{v}(x_1, \xi_2)}dx_1 .$$

Now we can prove the lemma for general $v\in C_0^\infty$, repeating the same argument as in the proof of Proposition 3.1 in [3].

Finally, if we prove

$$\text{(4.7)} \qquad |\mathrm{Re}\,(e_3(x, D)\,\mathrm{Im}\,\beta_1(x, 0, 1)(\log \lambda(D))v, v)|$$
$$\leqq 2\int \Gamma(x_1)(\log\langle\xi_2\rangle)|\tilde{v}(x_1, \xi_2)|^2 dx_1 d\xi_2 + C\{\|v\|^2 + (1-\chi(x, D)v\|_2^2\}$$

for $v\in C_0^\infty(U)$, then we will obtain (4.6) in view of Lemma 4.7. Note that $e_3(x, \xi)\equiv e_3(x)$ does not depend on ξ. Lət us prove (4.7).

Write

$$\mathrm{Re}\,(e_3(x)\,\mathrm{Im}\,\beta_1(x, 0, 1)(\log \lambda(D))v, v)$$
$$= \mathrm{Re}\,(e_3(x)\,\mathrm{Im}\,\beta_1(x, 0, 1)(\log \lambda(D))^{1/2}v, (\log \lambda(D))^{1/2}v)$$
$$+\mathrm{Re}\,([e_3(x)\,\mathrm{Im}\,\beta_1(x, 0, 1), (\log \lambda(D))^{1/2}](\log \lambda(D))^{1/2}v, v)$$
$$\equiv I_1 + I_2 .$$

Since $[e_3(x)\,\mathrm{Im}\,\beta_1(x, 0, 1), (\log \lambda(D))^{1/2}](\log \lambda(D))^{1/2} \in L_{1,0}^0$, we obtain

$$|I_2| \leqq C\|v\|^2 \qquad \text{for } v\in C_0^\infty(U) .$$

Next we shall consider I_1. For simplicity, set

$$(\log \lambda(D))^{1/2}v = (\log \lambda(D))^{1/2}v + \{(\log \lambda(D))^{1/2} - (\log\langle D_2\rangle)^{1/2}\}v$$
$$\equiv u_1 + u_2 ,$$

where $(\log\langle D_2\rangle)^{1/2}v = \int \exp(ix_2\xi_2)(\log\langle\xi_2\rangle)^{1/2}\tilde{v}(x_1, \xi_2)d\xi_2$. Recall that $|\mathrm{Im}\,\beta_1(x, 0, 1)| \leqq \Gamma(x_1)$. Therefore, we have

$$|I_1| \leqq \left|\int e_3(x)\,\mathrm{Im}\,\beta_1(x, 0, 1)|(\log \lambda(D))^{1/2}v|^2 dx\right|$$
$$\leqq C\sum_{j=1}^2 \int \Gamma(x_1)\left\{\int |u_j(x)|^2 dx_2\right\}dx_1 .$$

It is easy to see that

$$\int \Gamma(x_1)\left\{\int |u_1(x)|^2 dx_2\right\}dx_1$$

$$=\int \Gamma(x_1)\left\{\int (\log \langle\xi_2\rangle)|\tilde{v}(x_1, \xi_2)|^2 d\xi_2\right\}dx_1,$$

$$\int \Gamma(x_1)\left\{\int |u_2(x)|^2 dx_2\right\}dx_1$$

$$\leqq C\int |u_2(x)|^2 dx \leqq C'\int \langle\xi\rangle(1-\psi(\xi))|\hat{v}(\xi)|^2 d\xi,$$

where $\Psi(\xi)\in S^0_{1,0}$ is positively homogeneous of degree 0 for $|\xi|\geqq 3$, $0\leqq\Psi(\xi)\leqq 1$, $\Psi(\xi)=1$ if $\xi_2\geqq 2|\xi|/3\geqq 2$, and $\operatorname{supp}\Psi(\xi)\subset\{\xi\in \boldsymbol{R}^2|\xi_2>|\xi|/2>1\}$. Note that $\operatorname{supp}\chi\cap \boldsymbol{R}^2\times\operatorname{supp}(1-\Psi(\xi))=\emptyset$, thus we have

$$\int \langle\xi\rangle(1-\Psi(\xi))|\hat{v}(\xi)|^2 d\xi\leqq C\{\|(1-\chi(x, D))v\|_2^2+\|v\|^2\},$$

which proves (4.7).

Now take $\varphi(t)\in \mathscr{B}^\infty(\boldsymbol{R}^2)$ in $\Lambda_\delta(x, \xi)$ so that $\varphi(t)=t^2$. Then from the estimate (3.6), Lemma 4.1 and 4.6, we obtain (3.1) in the same manner as in the proof of Theorem 1.2. So we can apply Lemma 3.1, and prove that $(0, 0, 0, 1)\notin WF(u)$ if $u\in\mathscr{D}'$ and $(0, 0, 0, 1)\notin WF(P(x, D)u)$, applying the same argument with the origin replace by a point in $S\cap U\subset\{x\in \boldsymbol{R}^2|x_1=0\}$, we can prove the assertion (ii) of Theorem 1.3.

§ 5. Further remark.

In this section we consider the operator of the form $P_1(x), D)=D_1^2+\alpha(x)D_2^2+\beta(x)D_2$, in $\boldsymbol{R}^2$, where $\alpha(x)\in C^\infty(\boldsymbol{R}^2)$ is non-negative, $\alpha(0)=0$, and $\beta(x)\ni C^\infty(\boldsymbol{R}^2)$ is complex-valued. Put $S=\{x\in \boldsymbol{R}^2|\alpha(x)=0\}$. In what follows we consider the various types of S, and always assume that there exist a positive integer l and a constant $C>0$ so that

$$(\operatorname{Re}\beta(x))^2+(\operatorname{Im}\beta(x))^{2l}\leqq C\alpha(x).$$

EXAMPLE 1. Assume that $S=\{x|f(x)=0\}$, where $df(0)\neq 0$, $\partial f/\partial x_1(0)=0$, and $\partial f/\partial x_1(x)\neq 0$ if $x\neq 0$. Then P_1 is microhypoelliptic in $\boldsymbol{R}^2$. In fact, since $\tau^2(S)=\emptyset$, from Corollary 1 of Theorem 1.2 it follows that P_1 is microhypoelliptic in $\boldsymbol{R}^2$.

EXAMPLE 2. Assume that $S=S_1\cup S_2$, where $S_j=\{x|s_j(x)=0\}$, $\partial s_j/\partial x_1(x)\neq 0$ for $x\neq 0$ $(j=1, 2)$ and $S_1\cap S_2=\{0\}$. Then in the same manner as in Example

1, P_1 is microhypoelliptic in $\boldsymbol{R}^2$.

EXAMPLE 3. Assume that $S=\bigcup_{j=1}^{\infty}S_j\cup S_0$, where $S_j=\{x\,|\,s_j(x)=0\}$, $\partial s_j/\partial x_1(x)\neq 0$ for any $x\in\boldsymbol{R}^2(j=0, 1, 2, \cdots)$, $S_j\cap S_k=\emptyset$ if $j, k\geqq 1$ and $j\neq k$, and $S_0=\overline{\bigcup_{j=1}^{\infty}S_j}\setminus\bigcup_{j=1}^{\infty}S_j$. Then P_1 is microhypoelliptic in $\boldsymbol{R}^2$. In fact, since $\tau(S)\subset S_0$ and $\tau^2(S)\subset\tau(S_0)=\emptyset$, in view of Corollary 1 of Theorem 1.2, P_1 is microhypoelliptic in $\boldsymbol{R}^2$.

EXAMPLE 4. Assume that $S=\bigcup_{j=1}^{\infty}S_j\cup S_0\cup\bigcup_{k=1}^{\infty}T_k\cup T_0$, where $S_j=\{x\,|\,s_j(x)=0\}$, $\partial s_j/\partial x_1(x)\neq 0$ for any $x\in\boldsymbol{R}^2(j=1, 2, \cdots)$, $S_0=\}x\,|\,s_0(x)=0\}$, $\partial s_0/\partial x_1(x)\neq 0$ if $x\neq 0$, $s_0(0)=\partial s_0/\partial x_1(0)=0$, $T_k=\{x\,|\,t_k(x)=0\}$, $\partial t_k/\partial x_1(x)\neq 0$ for any $x\in\boldsymbol{R}^2(k=1, 2, \cdots)$. $T_0=\{x\,|\,t_0(x)=0\}$, $\partial t_0/\partial x_1(x)\neq 0$ if $x\neq 0$ and $t_0(0)=\partial t_0/\partial x_1(0)=0$, $S_j\cap S_{j'}=T_k\cap T_{k'}=\emptyset$ if $j\neq j'$, $k\neq k'$, $j,j',k.k'\geqq 1$, $S_0=\overline{\bigcup_{j=1}^{\infty}S_j}\setminus\bigcup_{j=1}^{\infty}S_j$, $T_0=\overline{\bigcup_{k=1}^{\infty}T_k}\setminus\bigcup_{k=1}^{\infty}T_k$ and $S_j\cap T_k=\{a_{j,k}\}$ $(j, k\geqq 1)$. Then P_1 is microhypoelliptic in $\boldsymbol{R}^2$. In fact since $\tau(S)\subset(S_0\cup T_0)\cup\{a_{j,k}\,|\,j, k\geqq 1\}$, $\tau^2(S)\subset\tau(S_0\cup T_0\cup\{a_{j,k}\,|\,j, k\geqq 1\})\subset S_0\cup T_0$, $\tau^3(S)\subset\tau^2(S_0\cup T_0\cup\{a_{j,k}\})\subset\tau(S_0\cup T_0)\subset\{0\}$ and $\tau^4(S)=\emptyset$, P_1 is microhypoelliptic in $\boldsymbol{R}^2$ in view of Corollary 1 or Theorem 1.2.

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