TSUKUBA J. MATH.
Vol. 17 No. 1 (1993), 286—267

ON TUBES OF NONCONSTANT RADIUS

By

Naoyuki KOIKE

Abstract. One purpose of this paper is to obtain formulae for the shape operators of tubes, whose radius is not necessarily constant, over Riemannian submanifolds. Another purpose of this paper is to define the notion of the natural lift of a distribution on a Riemannian submanifold to a tube over the submanifold and investigate its integrability. Also, we shall construct models of certain kind of Riemannian hypersuface in terms of the formulae for the shape operators of tubes.

Introduction.

Tubes of constant radius over Riemannian submanifolds have been studied by many geometricians. For example, T. E. Cecil and P. J. Ryan obtained formulae for their shape operators (cf. [2]). In this paper, we will investigate tubes of which radius is not necessarily constant. For its purpose, in § 1, we will define the notion of the natural lift of a tangent vector field on a Riemannian submanifold to a tube over the submanifold and investigate its properties. In § 2, we will obtain formulae for the shape operators of tubes over a Riemannian submanifold in an Euclidean space, which are generalizations of some of those given by T. E. Cecil and P. J. Ryan. In § 3, we will define the notion of the natural lift of a distribution on a Riemannian submanifold to a tube over the submanifold and investigate its integrability. As an application of results in § 2, 3, we will construct soft models of Riemannian hypersurfaces in an Euclidean space of which the number of mutually distinct principal curvatures is constant on the hypersurface or a dense subset of the hypersurface (see § 4).

§ 1. Preliminaries.

Throughout this paper, unless otherwise mentioned, we assume that all objects are smooth and all manifolds are connected. Let M^n be an n-dimensional

Received December 12, 1991. Revised June 30, 1992.

Riemannian manifold isometrically immersed by an immersion f into an $(n+r)$-dimensional Riemannian manifold $\tilde{M}^{n+r}$. Then we call (M^n, f) a *Riemannian submanifold* in $\tilde{M}^{n+r}$. Especially, in case of $r=1$, we call it a *Riemannian hypersurface* in $\tilde{M}^{n+r}$. Denote by p the bundle projection of the normal bundle $T^{\perp}M$ of (M, f). Set $W:=\{\xi\in T^{\perp}M \mid {}^{\exists}\widetilde{\exp}(\iota^{\perp}(\xi))\}$, where $\widetilde{\exp}$ is the exponential map of $\tilde{M}$ and $\iota^{\perp}$ is the natural imbedding of $T^{\perp}M$ into $T\tilde{M}$. It is clear that W is a neighbourhood of 0-section in $T^{\perp}M$. Define a map $\exp^{\perp}: W \to \tilde{M}$ by $\exp^{\perp}:=\widetilde{\exp}\circ\iota^{\perp}$. For a positive function ε on M, set $N_{\varepsilon}(M):=\{\xi\in T^{\perp}M \mid |\xi|\leqq\varepsilon(p(\xi))\}$ and $t_{\varepsilon}(M):=\partial N_{\varepsilon}(M)$, where $|\xi|$ is the norm of ξ and $\partial N_{\varepsilon}(M)$ is the boundary of $N_{\varepsilon}(M)$. If $t_{\varepsilon}(M)\subset W$, then we set $f_{\varepsilon}:=\exp^{\perp}|_{t_{\varepsilon}(M)}$. Denote by $F(M, f)$ the focal set of (M, f). If ε satisfies $\exp^{\perp}(N_{\varepsilon}(M))\cap F(M, f)=\emptyset$, then $(t_{\varepsilon}(M), f_{\varepsilon})$ is a Riemannian hypersurface in $\tilde{M}^{n+r}$, where we give $t_{\varepsilon}(M)$ the metric induced from that of $\tilde{M}$ by f_{ε}. This hypersurface is called the *tube of radius* ε over (M, f). In the sequel, we shall call it an *ε-tube* for simplicity. We shall suppress f_*, $f_{\varepsilon *}$, $\iota^{\perp}$ and the natural imbedding $\iota_{\varepsilon}^{\perp}$ of $T^{\perp}t_{\varepsilon}(M)$ into $T\tilde{M}$. Denote by ∇ (resp. $\tilde{\nabla}$) the Levi-Civita connection of M (resp. $\tilde{M}$) and $\nabla^{\perp}$ the normal connection of (M, f). Also, denote by A and A^{ε} the shape operators of (M, f) and $(t_{\varepsilon}(M), f_{\varepsilon})$, respectively. Let Q be a horizontal distribution on $T^{\perp}M$ induced from $\nabla^{\perp}$ and denote by X_{ξ}^{L} the horizontal lift of $X\in T_{p(\xi)}M$ to ξ with respect to Q.

LEMMA 1.1. *For $X\in T_{p(\xi)}M$ $(\xi\in t_{\varepsilon}(M))$, $X_{\xi}^{L}+\dfrac{X\varepsilon}{\varepsilon(p(\xi))}\xi(\in T_{\xi}(T^{\perp}M))$ is tangent to $t_{\varepsilon}(M)$.*

PROOF. Let $x(t)$ $(t\in[0, 1])$ be an integral curve of $X\in T_{p(\xi)}M$, $\tilde{\xi}(t)$ $(t\in[0, 1])$ a normal vector field along the curve $x(t)$ given by parallel translating ξ along the curve $x(t)$ with respect to $\nabla^{\perp}$ and $\tilde{x}(t)$ $(t\in[0, 1])$ a curve in $t_{\varepsilon}(M)$ defined by $\tilde{x}(t):=\dfrac{\varepsilon(p(\tilde{\xi}(t)))}{\varepsilon(p(\xi))}\tilde{\xi}(t)$. We shall show $\dot{x}(0)=X_{\xi}^{L}+\dfrac{X\varepsilon}{\varepsilon(p(\xi))}\tilde{\xi}$. Denote by $\dot{x}(0)_Q$ (resp. $\dot{x}(0)_V$) the Q-component (resp. the V-component) of $\dot{x}(0)$ with respect to the decomposition $T(T^{\perp}M)=Q\oplus V$, where V is the tangent bundle of fibres of $T^{\perp}M$. We have only to show $\dot{x}(0)_Q=X_{\xi}^{L}$ and $\dot{x}(0)_V=\dfrac{X\varepsilon}{\varepsilon(p(\xi))}\xi$. Easily we have $p_*\dot{x}(0)=X$ and hence $\dot{x}(0)_Q=X_{\xi}^{L}$. Let $\delta:[0, 1]\times[0, 1]\to T^{\perp}M$ be the rectangle with respect to Q and V of which diagonal is $\tilde{x}(t)$, that is, for each $s\in[0, 1]$, $\delta_{\cdot s}(t\to\delta(t, s))$ is a Q-curve, for each $t\in[0, 1]$, $\delta_{t\cdot}(s\to\delta(t, s))$ is a V-curve and $\delta(t, t)=\tilde{x}(t)$ $(t\in[0, 1])$. It is clear that $\dot{x}(0)_V=\dot{\delta}_{0\cdot}(0)$ and $\delta_{0\cdot}(s)=\dfrac{\varepsilon(p(\hat{\xi}(s)))}{\varepsilon(p(\xi))}\xi$. Therefore, we have $\dot{x}(0)_V=\dfrac{X\varepsilon}{\varepsilon(p(\xi))}\xi$. This completes the proof. ■

Now we shall define new terminologies.

DEFINITION 1.1. For $X\in T_{p(\xi)}M$, we call $X_\xi^L+\frac{X\varepsilon}{\varepsilon(p(\xi))}\xi$ $(\in T_\xi(t_\varepsilon(M)))$ the *natural lift* of X to ξ and denote it by $\tilde{X}_\xi$. Also, for $Y\in\Gamma(TM)$, we call $\tilde{Y}(\in\Gamma(T(t_\varepsilon(M)))$ defined by $\tilde{Y}(\xi):=\widetilde{Y(p(\xi))}_\xi$ for $\xi\in t_\varepsilon(M)$ the *natural lift* of Y.

Denote by $\tilde{P}_\alpha$ the parallel translation from $\alpha(0)$ to $\alpha(1)$ along a curve α in $\tilde{M}$ with respect to $\tilde{\nabla}$, and $P_\beta^\perp$ the parallel translation from $\beta(0)$ to $\beta(1)$ along a curve β in M with respect to $\nabla^\perp$. For $\xi\in T^\perp M$ and $X\in T_{p(\xi)}M$, let $J_{\xi,X}$ be the Jacobi field along γ_ξ with $J_{\xi,X}(0)=X$ and $J'_{\xi,X}(0)=-A_\xi X$, where γ_ξ is a geodesic in $\tilde{M}$ with $\gamma_\xi(0)=p(\xi)$ and $\dot{\gamma}_\xi(0)=\xi$, and $J'_{\xi,X}=\tilde{\nabla}_{\dot{\gamma}_\xi}J_{\xi,X}$.

LEMMA 1.2. *Let $\xi\in t_\varepsilon(M)$ and $X\in T_{p(\xi)}M$. Then*

$$f_{\varepsilon*}\tilde{X}_\xi=J_{\xi,X}(1)+\frac{X\varepsilon}{\varepsilon(p(\xi))}\dot{\gamma}_\xi(1)$$

holds.

PROOF. From the definition of $\tilde{X}_\xi$ and f_ε, we have

$$\begin{aligned}f_{\varepsilon*}\tilde{X}_\xi&=\exp^\perp_*(X_\xi^L)+\frac{X\varepsilon}{\varepsilon(p(\xi))}\exp^\perp_*(\xi)\\&=\exp^\perp_*(X_\xi^L)+\frac{X\varepsilon}{\varepsilon(p(\xi))}\dot{\gamma}_\xi(1).\end{aligned}$$

Hence we have only to show $\exp^\perp_*(X_\xi^L)=J_{\xi,X}(1)$. Let $\alpha(t)$ $(t\in[0,1])$ be a sufficiently small curve in M with $\dot{\alpha}(0)=X$, $\beta(t)$ $(t\in[0,1])$ a curve in $T^\perp M$ defined by $\beta(t)=P^\perp_{\alpha|[0,t]}\xi$ and $\delta:[0,1]\times[0,1]\to\tilde{M}$ a map defined by $\delta(t,s):=\exp^\perp(s\beta(t))$. Since $\dot{\beta}(0)=X_\xi^L$, we have $\dot{\delta}_{\cdot 1}(0)=\exp^\perp_*(\dot{\beta}(0))=\exp^\perp_*(X_\xi^L)$. On the other hand, since each curve $\delta_{t\cdot}$ is a geodesic in $\tilde{M}$, $J(s):=\dot{\delta}_{\cdot s}(0)$ is a Jacobi field along γ_ξ. It is clear that $J(0)=X$. Moreover, we have

$$\begin{aligned}J'(0)=\tilde{\nabla}_\xi J&=\tilde{\nabla}_{\partial/\partial s}\frac{\partial}{\partial t}\Big|_{(0,0)}\\&=\tilde{\nabla}_{\partial/\partial s}\frac{\partial}{\partial s}\Big|_{(0,0)}=\tilde{\nabla}_X\frac{\partial}{\partial s}=-A_\xi X\end{aligned}$$

because of $\nabla^\perp_X\frac{\partial}{\partial s}=0$. Thus it follows from the uniqueness of the Jacobi field that $J=J_{\xi,X}$. Therefore, we obtain $\exp^\perp_*(X_\xi^L)=\dot{\delta}_{\cdot 1}(0)=J(1)=J_{\xi,X}(1)$. This completes the proof. ■

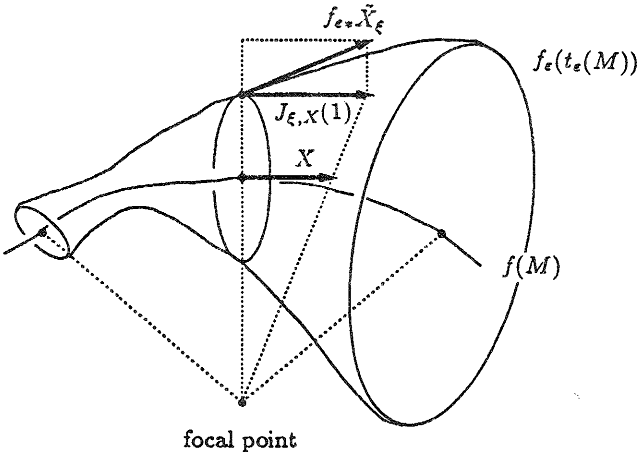

Fig. 1.1.

Easily we have the following lemma.

LEMMA 1.3. *For* $X, Y\in T_{p(\xi)}M$ $(\xi\in t_\varepsilon(M))$, $\langle\tilde{X}_\xi, \tilde{Y}_\xi\rangle=\langle J_{\xi,X}(1), J_{\xi,Y}(1)\rangle+(X\varepsilon)(Y\varepsilon)$ *holds.*

PROOF. By Lemma 1.2, we have

$$\begin{aligned}\langle\tilde{X}_\xi, \tilde{Y}_\xi\rangle&=\langle f_{\varepsilon*}\tilde{X}_\xi, f_{\varepsilon*}\tilde{Y}_\xi\rangle\\&=\left\langle J_{\xi,X}(1)+\frac{X\varepsilon}{\varepsilon(p(\xi))}\dot{\gamma}_\xi(1), J_{\xi,Y}(1)+\frac{Y\varepsilon}{\varepsilon(p(\xi))}\dot{\gamma}_\xi(1)\right\rangle.\end{aligned}$$

Since both $J_{\xi,X}(0)$ and $J'_{\xi,X}(0)$ are orthogonal to $\dot{\gamma}_\xi(0)$, $J_{\xi,X}(1)$ is orthogonal to $\dot{\gamma}_\xi(1)$. Similarly, $J_{\xi,Y}(1)$ is orthogonal to $\dot{\gamma}_\xi(1)$. Hence we obtain the desired equality. ■

In the case where $\tilde{M}$ is flat, we have the following lemma.

LEMMA 1.4. *For* $X\in T_{p(\xi)}M$ $(\xi\in t_\varepsilon(M))$, $J_{\xi,X}(1)=\tilde{P}_{\gamma_\xi}(X-A_\xi X)$ *holds.*

PROOF. Since $\tilde{M}$ is flat, $J''_{\xi,X}=0$ holds. By solving $J''_{\xi,X}=0$ under the initial conditions $J_{\xi,X}(0)=X$ and $J'_{\xi,X}(0)=-A_\xi X$, we have $J_{\xi,X}(t)=\tilde{P}_{\gamma_\xi|_{[0,t]}}(X-tA_\xi X)$ and hence $J_{\xi,X}(1)=\tilde{P}_{\gamma_\xi}(X-A_\xi X)$. ■

From this lemma, we have the following result.

PROPOSITION 1.5. *Assume that* $\tilde{M}$ *is flat. For* $X\in T_{p(\xi)}M$ *and* $Y\in T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$ $(\xi\in t_\varepsilon(M))$, $\langle\tilde{X}_\xi, Y\rangle=0$ *holds.*

From Lemma 1.3 and 1.4, we obtain the following result.

PROPOSITION 1.6. *Assume that* $\tilde{M}$ *is flat. Let* $X, Y \in T_{p(\xi)}M$ *such that* $\langle X, Y\rangle=0$. *If either* X *or* Y *is a eigenvector of* A_ξ *and either* X *or* Y *is orthogonal to grad* ε, *then* $\langle \tilde{X}_\xi, \tilde{Y}_\xi\rangle=0$ *holds.*

§ 2. Shape operators of tubes.

In this section, we shall investigate the shape operator of the ε-tube over an n-dimensional Riemannian submanifold M in an $(n+r)$-dimensional Euclidean space R^{n+r}. For its purpose, we shall calculate the inward unit normal vector field of the ε-tube. In the sequal, we identify T_xR^{n+r} with R^{n+r} under the natural correspondence for every $x\in R^{n+r}$. Define $\bar{E}\in\Gamma(f_\varepsilon^*TR^{n+r})$ by $\bar{E}(\xi):=-\frac{\xi}{|\xi|}$ for $\xi\in t_\varepsilon(M)$, where $f_\varepsilon^*TR^{n+r}$ is the bundle induced from TR^{n+r} by f_ε. Let $\{\lambda_1>\cdots>\lambda_g\}$ be the set of all the mutually distinct eigenvalues of A_ξ $(\xi\in t_\varepsilon(M))$. Note that $\varepsilon(p(\xi))<\frac{1}{|\lambda_i|/|\xi|}$, that is, $|\lambda_i|<1$ $(i=1,\cdots,g)$ because $\exp^\perp(N_\varepsilon(M))\cap F(M,f)=\emptyset$ and the ambient space is R^{n+r}. Then we denote by X_{λ_i} the $\mathrm{Ker}(A_\xi-\lambda_iI)$-component of $X(\in T_{p(\xi)}M)$ with respect to the decomposition $T_{p(\xi)}M=\mathrm{Ker}(A_\xi-\lambda_1I)\oplus\cdots\oplus\mathrm{Ker}(A_\xi-\lambda_gI)$ $(i=1,\cdots,g)$.

LEMMA 2.1. *The inward unit normal vector field* E *of* $(t_\varepsilon(M), f_\varepsilon)$ *is given by*

$$E(\xi)=\frac{\sum_{i=1}^{g}\frac{\mathrm{grad}\,\varepsilon(p(\xi))_{\lambda_i}}{1-\lambda_i}+\bar{E}(\xi)}{\sqrt{1+\sum_{i=1}^{g}\frac{|\mathrm{grad}\,\varepsilon(p(\xi))_{\lambda_i}|^2}{(1-\lambda_i)^2}}}$$

for $\xi\in t_\varepsilon(M)$.

PROOF. It is clear that $T_{f_\varepsilon(\xi)}R^{n+r}=T_{p(\xi)}M\oplus T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))\oplus\langle\bar{E}(\xi)\rangle$ (orthogonal direct sum), where $\langle\bar{E}(\xi)\rangle$ is the 1-dimensional subspace spanned by $\bar{E}(\xi)$. Hence $E(\xi)$ can be expressed as $E(\xi)=\frac{Y+Z+\bar{E}(\xi)}{|Y+Z+\bar{E}(\xi)|}$ for $Y\in T_{p(\xi)}M$ and $Z\in T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$. It follows from $T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))\subset T_\xi(t_\varepsilon(M))$ that

$$\langle E(\xi), Z\rangle=\frac{\langle Z, Z\rangle}{|Y+Z+\bar{E}(\xi)|}=0,$$

that is, $Z=0$. Take any $X\in T_{p(\xi)}M$. It follows from Lemma 1.2 and 1.4 that

$$f_{\varepsilon*}\tilde{X}_\xi=X-A_\xi X-(X\varepsilon)\bar{E}(\xi). \tag{2.1}$$

Hence we have

$$\begin{aligned}\langle E(\xi),\ f_{\varepsilon*}\tilde{X}_\xi\rangle &= \frac{\langle Y+\bar{E}(\xi),\ X-A_\xi X-(X\varepsilon)\bar{E}(\xi)\rangle}{|Y+\bar{E}(\xi)|}\\ &= \frac{\langle Y, X-A_\xi X\rangle - X\varepsilon}{|Y+\bar{E}(\xi)|}\\ &= \frac{\langle Y,\ X-A_\xi X\rangle-\langle \operatorname{grad}\varepsilon(p(\xi)),\ X\rangle}{|Y+\bar{E}(\xi)|}\\ &= \frac{\langle Y-A_\xi Y-\operatorname{grad}\varepsilon(p(\xi)),\ X\rangle}{|Y+\bar{E}(\xi)|}\\ &=0\,.\end{aligned}$$

From the arbitrarity of X, $Y-A_\xi Y-\operatorname{grad}\varepsilon(p(\xi))=0$ is deduced. Therefore, we obtain $Y_{\lambda_i}=\frac{\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i}}{1-\lambda_i}$, that is, $Y=\sum_{i=1}^{g}\frac{\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i}}{1-\lambda_i}$. After all we obtain

$$E(\xi)=\frac{\sum_{i=1}^{g}\frac{\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i}}{1-\lambda_i}+\bar{E}(\xi)}{\sqrt{1+\sum_{i=1}^{g}\frac{|\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i}|^2}{(1-\lambda_i)^2}}}\,.$$

■

For $X\in R^{n+r}$, denote by X_{T_x} (resp. $X_{\perp_x}$) the T_xM-component (resp. the $T_x^\perp M$-component) of X with respect to the decomposition $R^{n+r}=T_xM\oplus T_x^\perp M$ and X_{T_ξ} (resp. $X_{\perp_\xi}$) the $T_\xi(t_\varepsilon(M))$-component (resp. the $T_\xi^\perp(t_\varepsilon(M))$-component) of X with respect to the decomposition $R^{n+r}=T_\xi(t_\varepsilon(M))\oplus T_\xi^\perp(t_\varepsilon(M))$. Define $\hat{E}\in f_\varepsilon^*(TR^{n+r})$ by $\hat{E}(\xi):=\frac{E(\xi)_{T_{p(\xi)}}}{|E(\xi)_{\perp_{p(\xi)}}|}$ for $\xi\in t_\varepsilon(M)$.

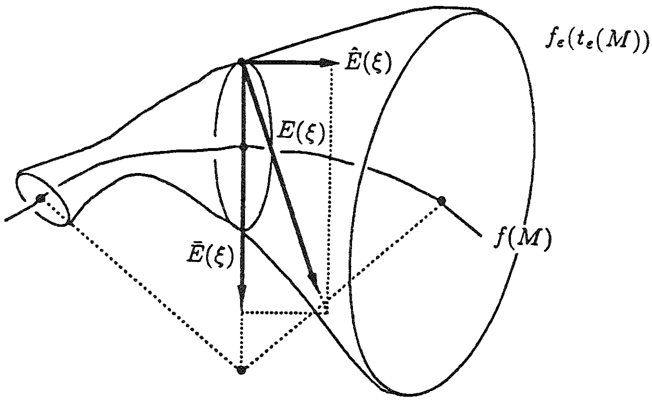

Fig. 2.1.

For the shape operator A^ε of the ε-tube, we have the following formulae.

THEOREM 2.2. (i) *For* $X\in T_{\xi}(t_{\varepsilon}(M)\cap p^{-1}(p(\xi)))$, *the following equality holds:*

$$A^{\varepsilon}_{E(\xi)}X=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{1}{\varepsilon(p(\xi))}X-(\tilde{\nabla}_X\hat{E})_{T_\xi}\right\}. \tag{2.2}$$

(ii) *Let* λ *be an eigenvalue of* A_ξ $(\xi\in t_{\varepsilon}(M))$. *Then, for* $X\in\operatorname{Ker}(A_\xi-\lambda I)$, *the following equality holds:*

$$A^{\varepsilon}_{E(\xi)}\tilde{X}_\xi=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{\lambda}{\varepsilon(p(\xi))(\lambda-1)}(\tilde{X}_\xi+(X\varepsilon)\bar{E}(\xi)_{T_\xi})-(\tilde{\nabla}_{\tilde{X}_\xi}\hat{E})_{T_\xi}\right\}. \tag{2.3}$$

PROOF. (i) Let $\xi(t)$ $(t\in[0,1])$ be a curve in $t_{\varepsilon}(M)$ with $\dot{\xi}(0)=X$. Then we have

$$\begin{aligned}A^{\varepsilon}_{E(\xi)}X&=-\tilde{\nabla}_X E=\left.\frac{dE(\xi(t))}{dt}\right|_{t=0}\\
&=-\frac{d}{dt}\left\{\frac{1}{\sqrt{1+|\hat{E}(\xi(t))|^2}}(\bar{E}(\xi(t))+\hat{E}(\xi(t)))\right\}\Big|_{t=0}\\
&=\left.\frac{d|\hat{E}(\xi(t))|^2}{dt}\right|_{t=0}\frac{\bar{E}(\xi)+\hat{E}(\xi)}{2\sqrt{(1+|\hat{E}(\xi)|^2)^3}}\\
&\quad-\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\frac{d}{dt}(\bar{E}(\xi(t))+\hat{E}(\xi(t)))|_{t=0}\\
&=\left.\frac{d|\hat{E}(\xi(t))|^2}{dt}\right|_{t=0}\frac{\bar{E}(\xi)+\hat{E}(\xi)}{2\sqrt{(1+|\hat{E}(\xi)^2)^3}}\\
&\quad-\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}(\tilde{\nabla}_X\bar{E}+\tilde{\nabla}_X\hat{E}).\end{aligned}$$

It follows from this equality and $\bar{E}(\xi)+\hat{E}(\xi)\in T^{\perp}_{\xi}(t_{\varepsilon}(M))$ that

$$A^{\varepsilon}_{E(\xi)}X=-\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}(\tilde{\nabla}_X\bar{E}+\tilde{\nabla}_X\hat{E})_{T_\xi}. \tag{2.4}$$

On the other hand, we have

$$\begin{aligned}\tilde{\nabla}_X\bar{E}&=\left.\frac{d\bar{E}(\xi(t))}{dt}\right|_{t=0}=-\frac{d}{dt}\left(\frac{\xi(t)}{|\xi(t)|}\right)\Big|_{t=0}\\
&=-\frac{1}{|\xi|}\left.\frac{d\xi(t)}{dt}\right|_{t=0}-\left(\frac{d}{dt}\frac{1}{|\xi(t)|}\right)\Big|_{t=0}\xi\\
&=-\frac{1}{|\xi|}\frac{d}{dt}\{f_{\varepsilon}(\xi(t))-p(\xi(t))\}|_{t=0}+\frac{1}{|\xi|^2}\left.\frac{d|\xi(t)|}{dt}\right|_{t=0}\xi\\
&=-\frac{1}{\varepsilon(p(\xi))}(X-p_*X)-\frac{(p_*X)\varepsilon}{\varepsilon(p(\xi))}\bar{E}(\xi)\\
&=-\frac{1}{\varepsilon(p(\xi))}\{X-p_*X+\{(p_*X)\varepsilon\}\bar{E}(\xi)\}.\end{aligned} \tag{2.5}$$

Since $p_*X=0$, $\tilde{\nabla}_X\bar{E}=-\frac{1}{\varepsilon(p(\xi))}X$ holds. Therefore, we obtain

$$A^{\varepsilon}_{E(\xi)}X=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{1}{\varepsilon(p(\xi))}X-(\tilde{\nabla}_X\hat{E})_{T\xi}\right\}.$$

(ii) It is clear that the same equalities as (2.4) and (2.5) hold:

$$A^{\varepsilon}_{E(\xi)}\tilde{X}_\xi=-\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}(\tilde{\nabla}_{\tilde{X}_\xi}\bar{E}+\tilde{\nabla}_{\tilde{X}_\xi}\hat{E})_{T\xi}$$

and

$$\tilde{\nabla}_{\tilde{X}_\xi}\bar{E}=-\frac{1}{\varepsilon(p(\xi))}\{\tilde{X}_\xi-p_*\tilde{X}_\xi+\{(p_*\tilde{X}_\xi)\varepsilon\}\bar{E}(\xi)\}.$$

Hence, since $p_*\tilde{X}_\xi=X$, we have

$$A^{\varepsilon}_{E(\xi)}\tilde{X}_\xi=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{1}{\varepsilon(p(\xi))}(\tilde{X}_\xi-X+(X\varepsilon)\bar{E}(\xi))-\tilde{\nabla}_{\tilde{X}_\xi}\hat{E}\right\}_{T\xi}.$$

Moreover, since $X=\frac{1}{1-\lambda}\{\tilde{X}_\xi+(X\varepsilon)\bar{E}(\xi)\}$ by (2.1) and $A_\xi X=\lambda X$, we obtain

$$\begin{aligned}A^{\varepsilon}_{E(\xi)}\tilde{X}_\xi&=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{\lambda}{\varepsilon(p(\xi))(\lambda-1)}(\tilde{X}_\xi+(X\varepsilon)\bar{E}(\xi))-\tilde{\nabla}_{\tilde{X}_\xi}\hat{E}\right\}_{T\xi}\\&=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{\lambda}{\varepsilon(p(\xi))(\lambda-1)}(\tilde{X}_\xi+(X\varepsilon)\bar{E}(\xi)_{T\xi})-(\tilde{\nabla}_{\tilde{X}_\xi}\hat{E})_{T\xi}\right\}.\end{aligned}$$

■

In the case where ε is constant, we have the following result.

COROLLARY 2.3. (i) *For $X\in T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$, the following equality holds:*

$$A^{\varepsilon}_{E(\xi)}X=\frac{1}{\varepsilon}X.$$

(ii) *Let λ be an eigenvalue of A_ξ ($\xi\in t_\varepsilon(M)$). Then, for $X\in\mathrm{Ker}\,(A_\xi-\lambda I)$, the following equality holds:*

$$A^{\varepsilon}_{E(\xi)}\tilde{X}_\xi=\frac{\lambda}{\varepsilon(\lambda-1)}\tilde{X}_\xi.$$

PROOF. Since ε is constant, $\hat{E}=0$ holds. Hence, the statements (i) and (ii) are deduced from the previous theorem. ■

REMARK. The results in this corollary are stated in [2, Theorem 3.2 of P 131].

Let τ be a function on $t_\varepsilon(M)$ defined by assigning the number of the mutually distinct eigenvalues of A_ξ to each $\xi\in t_\varepsilon(M)$ and $G_m(M)$ the Grassmann bundle of m-planes over M. It is easy to show the following lemma.

LEMMA 2.4. *Assume that τ is constant on an open subset U of $t_\varepsilon(M)$.*

(i) *Let λ_i $(i=1, \cdots, g)$ be functions on U such that $\{\lambda_1(\xi)> \cdots >\lambda_g(\xi)\}$ is the set of all mutually distinct eigenvalues of A_ξ for every $\xi\in U$. Then λ_i is a smooth function and the multiplicity of the eigenvalue $\lambda_i(\xi)$ of A_ξ is independent of $\xi\in U$ $(i=1, \cdots, g)$,*

(ii) *Let D_{λ_i} be a section of $(p|_U)^*G_{m_i}(M)$ defined by $D_{\lambda_i}(\xi):=\mathrm{Ker}(A_\xi-\lambda_i(\xi)I)$ for $\xi\in U$ $(i=1, \cdots, g)$, where $m_i=\dim\mathrm{Ker}(A_\xi-\lambda_i(\xi)I)$. Then D_{λ_i} is a smooth section of $(p|_U)^*G_{m_i}(M)$ $(i=1, \cdots, g)$.*

Let $U\subset t_\varepsilon(M)$. For $\Phi\in\Gamma((p|_U)^*(T^*M\otimes TM))$, denote by $\bar{\Phi}$ an element of $\Gamma((p|_U)^*(T^*R^{n+r}\otimes TR^{n+r}))$ defined by

$$\bar{\Phi}(\xi)X=\begin{cases}\Phi(\xi)X & (X\in T_{p(\xi)}M)\\ 0 & (X\in T^{\perp}_{p(\xi)}M)\end{cases}$$

for $\xi\in U$. Note that $\bar{\Phi}$ is regarded as a $(R^{n+r})^*\otimes R^{n+r}$-valued function on U. Denote by the same symbol $\tilde{\nabla}$ the connection on $(p|_U)^*(T^*R^{n+r}\otimes TR^{n+r})$ induced from the Levi-Civita connection $\tilde{\nabla}$ of R^{n+r}. Then it is clear that $\tilde{\nabla}_X\bar{\Phi}=\frac{d\bar{\Phi}(\xi(t))}{dt}|_{t=0}$ holds for every $X\in TU$, where $\xi(t)$ $(t\in[0, 1])$ is a curve in U with $\dot{\xi}(0)=X$. If τ is constant on a neighbourhood of $\xi\in t_\varepsilon(M)$, then the equalities (2.2) and (2.3) in Proposition 2.2 are written in more detail as follows.

THEOREM 2.5. *Assume that τ is constant on an open subset U of $t_\varepsilon(M)$. Let λ_i and D_{λ_i} $(i=1, \cdots, g)$ be as in Lemma 2.4 and P_{λ_i} an element of $\Gamma((p|_U)^*(T^*M\otimes TM))$ defined by assigning the orthogonal projection of $T_{p(\xi)}M$ onto $D_{\lambda_i}(\xi)$ to each $\xi\in U$.*

(i) *For $X\in T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$ $(\xi\in U)$, the following equality holds:*

$$A^{\varepsilon}_{E(\xi)}X=\frac{1}{\sqrt{B(\xi)}}\Big(\frac{1}{\varepsilon(p(\xi))}X-\tilde{Y}_\xi\Big). \tag{2.6}$$

Here $B(\xi)=1+\sum_{i=1}^{g}\frac{|\mathrm{grad}\,\varepsilon(p(\xi))_{\lambda_i(\xi)}|^2}{(1-\lambda_i(\xi))^2}$ *and Y is an element of $T_{p(\xi)}M$ given by*

$$Y=\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^3}\Big\{X\lambda_i-\frac{1-\lambda_i(\xi)}{B(\xi)}\Big(\sum_{j=1}^{g}\frac{(X\lambda_j)|V_{\lambda_j(\xi)}|^2}{(1-\lambda_j(\xi))^3}+\sum_{j,l=1}^{g}\frac{\langle(\tilde{\nabla}_X\bar{P}_{\lambda_j})V,\ V_{\lambda_l(\xi)}\rangle}{(1-\lambda_j(\xi))(1-\lambda_l(\xi))}\Big)\Big\}V_{\lambda_i(\xi)}+\sum_{i,j=1}^{g}\frac{\{(\tilde{\nabla}_X\bar{P}_{\lambda_j})V\}_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))(1-\lambda_j(\xi))},$$

where $V:=\mathrm{grad}\,\varepsilon(p(\xi))$.

(ii) *For* $X\in D_{\lambda_{i_0}}(\xi)$ $(\xi\in U)$, *the following equality holds:*

$$A^{\varepsilon}_{\bar E(\xi)}\tilde X_\xi=\frac{1}{\sqrt{B(\xi)}}\Big\{\frac{\lambda_{i_0}(\xi)}{\varepsilon(p(\xi))(\lambda_{i_0}(\xi)-1)}\tilde X_\xi+\frac{\lambda_{i_0}(\xi)(X\varepsilon)}{\varepsilon(p(\xi))(\lambda_{i_0}(\xi)-1)}\bar E(\xi)_{T_\xi}+\tilde Z_\xi-\sum_{i=1}^{g}\frac{\{(\tilde\nabla_{\tilde X_\xi}\bar P_{\lambda_i})(\operatorname{grad}\varepsilon(p(\xi)))\}_{F_\xi}}{1-\lambda_i(\xi)}\Big\}. \tag{2.7}$$

Here $\{(\tilde\nabla_{\tilde X_\xi}\bar P_{\lambda_i})(\operatorname{grad}\varepsilon(p(\xi)))\}_{F_\xi}$ *is the* $T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$*-component of* $(\tilde\nabla_{\tilde X_\xi}\bar P_{\lambda_i})$ $(\operatorname{grad}\varepsilon(p(\xi)))$ *with respect to the decomposition* $R^{n+r}=T_{p(\xi)}M\oplus T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$ $\oplus\langle\bar E(\xi)\rangle$, *and* Z *is an element of* $T_{p(\xi)}M$ *given by*

$$\begin{aligned}Z=&-\sum_{i,j=1}^{g}\frac{\{\{(\tilde\nabla_{\tilde X_\xi}\bar P_{\lambda_j})V\}_{T_{p(\xi)}}\}_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))(1-\lambda_j(\xi))}-\sum_{i=1}^{g}\frac{(\nabla_X\operatorname{grad}\varepsilon)_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))^2}\\&-\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^3}\Big\{\tilde X_\xi\lambda_i-\frac{1-\lambda_i(\xi)}{B(\xi)}\Big(\sum_{j=1}^{g}\frac{(\tilde X_\xi\lambda_j)|V_{\lambda_j(\xi)}|^2}{(1-\lambda_j(\xi))^3}\\&+\sum_{j,k=1}^{g}\frac{\langle(\tilde\nabla_{\tilde X_\xi}\bar P_{\lambda_j})V,V_{\lambda_k(\xi)}\rangle}{(1-\lambda_j(\xi))(1-\lambda_k(\xi))}+\sum_{j=1}^{g}\frac{\langle\nabla_X\operatorname{grad}\varepsilon,V_{\lambda_j(\xi)}\rangle}{(1-\lambda_j(\xi))^2}\\&+\sum_{j=1}^{g}\frac{\langle(\tilde\nabla_{\tilde X_\xi}\bar P_{\lambda_j})V,\bar E(\xi)\rangle}{1-\lambda_j(\xi)}\Big)\Big\}V_{\lambda_i(\xi)},\end{aligned}$$

where $V:=\operatorname{grad}\varepsilon(p(\xi))$.

PROOF. (i) By Theorem 2.2, we have

$$A^{\varepsilon}_{\hat E(\xi)}X=\frac{1}{\sqrt{1+|\hat E(\xi)|^2}}\Big\{\frac{1}{\varepsilon(p(\xi))}X-(\tilde\nabla_X\hat E)_{T_\xi}\Big\}.$$

It follows the definition of $\hat E(\xi)$ and Lemma 2.1 that $1+|\hat E(\xi)|^2=B(\xi)$. Let $\xi(t)$ $(t\in[0,1])$ be a curve in $t_\varepsilon(M)\cap p^{-1}(p(\xi))$ with $\dot\xi(0)=X$. Since $\hat E(\xi(t))=\sum_{i=1}^{g}\frac{\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i(\xi(t))}}{1-\lambda_i(\xi(t))}\in T_{p(\xi)}M$, we have

$$\begin{aligned}\tilde\nabla_X\hat E&=\sum_{i=1}^{g}\Big\{\frac{(X\lambda_i)\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))^2}+\frac{\frac{d}{dt}\operatorname{grad}\varepsilon(p(\xi))_{\lambda_i(\xi(t))}|_{t=0}}{1-\lambda_i(\xi)}\Big\}\\&=\sum_{i=1}^{g}\Big\{\frac{(X\lambda_i)V_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))^2}+\frac{(\tilde\nabla_X\bar P_{\lambda_i})V}{1-\lambda_i(\xi)}\Big\}\in T_{p(\xi)}M.\end{aligned} \tag{2.8}$$

Therefore, $(\tilde\nabla_X\hat E)_{T_\xi}$ is orthogonal to $t_\varepsilon(M)\cap p^{-1}(p(\xi))$ and hence there exists $\bar Y\in T_{p(\xi)}M$ such that $\tilde{\bar Y}_\xi=(\tilde\nabla_X\hat E)_{T_\xi}$ by Proposition 1.5. We have only to show $\bar Y=Y$. From (2.8), we can obtain

$$(2.9)\qquad (\tilde{\nabla}_X\hat{E})_{T\xi}=\tilde{\nabla}_X\hat{E}-\langle\tilde{\nabla}_X\hat{E},\ E(\xi)\rangle E(\xi)$$
$$=\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^2}\left\{X\lambda_i-\frac{C(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}$$
$$+\sum_{i=1}^{g}\frac{(\tilde{\nabla}_X\bar{P}_{\lambda_i})V}{1-\lambda_i(\xi)}-\frac{C(\xi)}{B(\xi)}\bar{E}(\xi),$$

where $C(\xi)$ is defined by

$$C(\xi)=\sum_{i=1}^{g}\frac{(X\lambda_i)|V_{\lambda_i(\xi)}|^2}{(1-\lambda_i(\xi))^3}+\sum_{i,j=1}^{g}\frac{\langle(\tilde{\nabla}_X\bar{P}_{\lambda_i})V,V_{\lambda_j(\xi)}\rangle}{(1-\lambda_i(\xi))(1-\lambda_j(\xi))}.$$

On the other hand, by Lemma 1.2 and 1.4, we have

$$(2.10)\qquad \tilde{\bar{Y}}_\xi=\bar{Y}-A_\xi\bar{Y}-(\bar{Y}\varepsilon)\bar{E}(\xi).$$

By comparing the $T_{p(\xi)}M$-component of (2.9) and (2.10), we have

$$\bar{Y}-A_\xi\bar{Y}=\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^2}\left\{X\lambda_i-\frac{C(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}$$
$$+\sum_{i=1}^{g}\frac{(\tilde{\nabla}_X\bar{P}_{\lambda_i})V}{1-\lambda_i(\xi)},$$

this is,

$$(1-\lambda_i(\xi))\bar{Y}_{\lambda_i(\xi)}=\frac{1}{(1-\lambda_i(\xi))^2}\left\{X\lambda_i-\frac{C(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}$$
$$+\sum_{j=1}^{g}\frac{\{(\tilde{\nabla}_X\bar{P}_{\lambda_j})V\}_{\lambda_i(\xi)}}{1-\lambda_j(\xi)}.$$

Therefore, we can obtain $\bar{Y}=\sum_{i=1}^{g}\bar{Y}_{\lambda_i(\xi)}=Y$.

(ii) By Theorem 2.2, we have

$$(2.11)\qquad A^{\xi}_{E(\xi)}\tilde{X}_\xi=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{\lambda_{i_0}(\xi)(\tilde{X}_\xi+(X\varepsilon)\bar{E}(\xi)_{T\xi})}{\varepsilon(p(\xi))(\lambda_{i_0}(\xi)-1)}-(\tilde{\nabla}_{\tilde{X}_\xi}\hat{E})_{T\xi}\right\}.$$

It follows from the definition of $\hat{E}(\xi)$ and Lemma 2.1 that $1+|\hat{E}(\xi)|^2=B(\xi)$. Let $\xi(t)$ ($t\in[0,1]$) be a curve in $t_\varepsilon(M)$ with $\dot{\xi}(0)=\tilde{X}_\xi$. Since $\hat{E}(\xi(t))=\sum_{i=1}^{g}\frac{\mathrm{grad}\,\varepsilon(p(\xi(t)))_{\lambda_i(\xi(t))}}{1-\lambda_i(\xi(t))}$, we have

$$\tilde{\nabla}_{\tilde{X}_\xi}\hat{E}=\sum_{i=1}^{g}\left\{\frac{(\tilde{X}_\xi\lambda_i)\,\mathrm{grad}\,\varepsilon(p(\xi))_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))^2}+\frac{\frac{d}{dt}\mathrm{grad}\,\varepsilon(p(\xi(t)))_{\lambda_i(\xi(t))}|_{t=0}}{1-\lambda_i(\xi)}\right\}$$
$$=\sum_{i=1}^{g}\left\{\frac{(\tilde{X}_\xi\lambda_i)V_{\lambda_i(\xi)}}{(1-\lambda_i(\xi))^2}+\frac{(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V+(\nabla_X\,\mathrm{grad}\,\varepsilon)_{\lambda_i(\xi)}}{1-\lambda_i(\xi)}\right\}$$

and hence

$$(\tilde{\nabla}_{\tilde{X}_\xi}\hat{E})_{T\xi}=\tilde{\nabla}_{\tilde{X}_\xi}\hat{E}-\langle\tilde{\nabla}_{\tilde{X}_\xi}\hat{E},\ E(\xi)\rangle E(\xi)$$
$$=\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^2}\left\{\tilde{X}_\xi\lambda_i-\frac{\bar{C}(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}$$
$$+\sum_{i=1}^{g}\frac{(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V+(\nabla_X\operatorname{grad}\varepsilon)_{\lambda_i(\xi)}}{1-\lambda_i(\xi)}-\frac{\bar{C}(\xi)}{B(\xi)}\bar{E}(\xi),$$

where $\bar{C}(\xi)$ is defined by

$$\bar{C}(\xi)=\sum_{i=1}^{g}\frac{(\tilde{X}_\xi\lambda_i)|V_{\lambda_i(\xi)}|^2}{(1-\lambda_i(\xi))^3}+\sum_{i,j=1}^{g}\frac{\langle(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V,\ V_{\lambda_j(\xi)}\rangle}{(1-\lambda_i(\xi))(1-\lambda_j(\xi))}$$
$$+\sum_{i=1}^{g}\frac{\langle\nabla_X\operatorname{grad}\varepsilon,\ V_{\lambda_i(\xi)}\rangle}{(1-\lambda_i(\xi))^2}+\sum_{i=1}^{g}\frac{\langle(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V,\ \bar{E}(\xi)\rangle}{1-\lambda_i(\xi)}.$$

Therefore, we have

$$(\tilde{\nabla}_{\tilde{X}_\xi}\hat{E})_{T\xi}=\sum_{i=1}^{g}\frac{\{(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V\}_{F\xi}}{1-\lambda_i(\xi)}+W(\xi),\tag{2.12}$$

where $W(\xi)$ is defined by

$$W(\xi)=\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^2}\left\{\tilde{X}_\xi\lambda_i-\frac{\bar{C}(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}\tag{2.13}$$
$$+\sum_{i=1}^{g}\frac{\{(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V\}_{T_{p(\xi)}}+(\nabla_X\operatorname{grad}\varepsilon)_{\lambda_i(\xi)}}{1-\lambda_i(\xi)}$$
$$+\sum_{i=1}^{g}\frac{\langle(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V,\bar{E}(\xi)\rangle\bar{E}(\xi)}{1-\lambda_i(\xi)}-\frac{\bar{C}(\xi)}{B(\xi)}\bar{E}(\xi).$$

Since $W(\xi)\in T_\xi(t_\varepsilon(M))\cap T_\xi^\perp(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$, there exists $\bar{Z}\in T_{p(\xi)}M$ such that $\tilde{\bar{Z}}_\xi=W(\xi)$ by Proposition 1.5. According to (2.11) and (2.12), we have only to show $\bar{Z}=-Z$. By Lemma 1.2 and 1.4, we have

$$\tilde{\bar{Z}}_\xi=\bar{Z}-A_\xi\bar{Z}-(\bar{Z}\varepsilon)\bar{E}(\xi).\tag{2.14}$$

By comparing the $T_{p(\xi)}M$-component of (2.13) and (2.14), we have

$$\bar{Z}-A_\xi\bar{Z}=\sum_{i=1}^{g}\frac{1}{(1-\lambda_i(\xi))^2}\left\{\tilde{X}_\xi\lambda_i-\frac{\bar{C}(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}$$
$$+\sum_{i=1}^{g}\frac{\{(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_i})V\}_{T(\xi)}+(\nabla_X\operatorname{grad}\varepsilon)_{\lambda_i(\xi)}}{1-\lambda_i(\xi)},$$

that is,

$$(1-\lambda_i(\xi))\bar{Z}_{\lambda_i(\xi)}=-\frac{1}{(1-\lambda_i(\xi))^2}\left\{\tilde{X}_\xi\lambda_i-\frac{\bar{C}(\xi)(1-\lambda_i(\xi))}{B(\xi)}\right\}V_{\lambda_i(\xi)}$$
$$-\sum_{j=1}^{g}\frac{\{\{(\tilde{\nabla}_{\tilde{X}_\xi}\bar{P}_{\lambda_j})V\}_{T_{p(\xi)}}\}_{\lambda_i(\xi)}}{1-\lambda_j(\xi)}-\frac{(\nabla_X\operatorname{grad}\varepsilon)_{\lambda_i(\xi)}}{1-\lambda_i(\xi)}.$$

Therefore, we can obtain $\bar{Z}=\sum_{i=1}^{g}\bar{Z}_{\lambda_i(\xi)}=-Z$. ■

Let (M', f') be an n'-dimensional Riemannian submanifold in $R^{n'+r'}$ and (R^l, ι) an l-dimensional totally geodesic Riemannian submanifold in R^{l+s}. Then $(M'\times R^l, f'\times\iota)$ is an n-dimensional Riemannian submanifold in R^{n+r} called a cylinder over (M', f'), where $n=n'+l$ and $r=r'+s$. Assume that the set of all principal curvatures of (M', f') is bounded.

THEOREM 2.6. *Let* $(M, f)=(M'\times R^l, f'\times\iota)$ *and* ε *a sufficiently small positive function on* M *such that* $\operatorname{grad}\varepsilon\in\Gamma(TR^l)$, *where* TR^l *is a foliation on* M *of which leaves are* $\{\cdot\}\times R^l$. *Then*

(i) *For* $X\in T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$, *the following equality holds:*

$$A^\varepsilon_{E(\xi)}X=\frac{1}{\sqrt{1+|\operatorname{grad}\varepsilon(p(\xi))|^2}\,\varepsilon(p(\xi))}X\,,$$

(ii) *Let* λ *be a nonzero eigenvalue of* A_ξ $(\xi\in t_\varepsilon(M))$. *Then, for* $X\in\operatorname{Ker}(A_\xi-\lambda I)$, *the following equality holds:*

$$A^\varepsilon_{E(\xi)}\tilde{X}_\xi=\frac{\lambda}{\sqrt{1+|\operatorname{grad}\varepsilon(p(\xi))|^2}\,\varepsilon(p(\xi))(\lambda-1)}\tilde{X}_\xi\,,$$

(iii) *For* $X\in\operatorname{Ker}A_\xi\cap TM'$, *the following equality holds:*

$$A^\varepsilon_{E(\xi)}\tilde{X}_\xi=0\,,$$

(iv) *For* $X\in TR^l$ $(\subset\operatorname{Ker}A_\xi)$, *the following equality holds:*

$$A^\varepsilon_{E(\xi)}\tilde{X}_\xi=\frac{1}{2\sqrt{1+|\operatorname{grad}\varepsilon(p(\xi))|^2}}\{X\{\log(1+|\operatorname{grad}\varepsilon|^2)\}\widetilde{\operatorname{grad}\varepsilon(p(\xi))}_\xi-2\widetilde{\nabla_X\operatorname{grad}}\,\varepsilon_\xi\}\,.$$

PROOF. (i) According to (i) of Theorem 2.2, the following equality holds:

$$A^\varepsilon_{E(\xi)}X=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{1}{\varepsilon(p(\xi))}X-(\tilde{\nabla}_X\hat{E})_{T\xi}\right\}.$$

Since $\operatorname{grad}\varepsilon\in TR^l\subset\operatorname{Ker}A_\xi$, we have $\hat{E}=\operatorname{grad}\varepsilon\circ p$ and hence $\tilde{\nabla}_X\hat{E}=0$. Therefore, we obtain the desired equality.

(ii) According to (ii) of Theorem 2.2, the following equality holds:

$$A^\varepsilon_{E(\xi)}\tilde{X}_\xi=\frac{1}{\sqrt{1+|\hat{E}(\xi)|^2}}\left\{\frac{\lambda}{\varepsilon(p(\xi))(\lambda-1)}(\tilde{X}_\xi+(X\varepsilon)\bar{E}(\xi)_{T\xi})-(\tilde{\nabla}_{\tilde{X}}\hat{E})_{T\xi}\right\}.$$

Since $\operatorname{grad}\varepsilon\in TR^l\subset\operatorname{Ker}A_\xi$ and $X\in T^\perp R^l$, we have $X\varepsilon=0$ and

$$\tilde{\nabla}_{\tilde{X}_\xi}\hat{E}=\tilde{\nabla}_{\tilde{X}_\xi}(\operatorname{grad}\varepsilon\circ p)=\tilde{\nabla}_X\operatorname{grad}\varepsilon=0\,.$$

Therefore, we obtain the desired equality.

(iii) According to (ii) of Theorem 2.2, the following equality holds:

$$A^{\xi}_{\hat{E}(\xi)}\tilde{X}_{\xi}=\frac{-1}{\sqrt{1+|\hat{E}(\xi)|^2}}(\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E})_{T_{\xi}}. \tag{2.15}$$

Since $\mathrm{grad}\,\varepsilon\in TR^l\subset\mathrm{Ker}\,A_{\xi}$ and $X\in T^{\perp}R^l$, we have $\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E}=0$. Therefore, we obtain $A^{\xi}_{\hat{E}(\xi)}\tilde{X}_{\xi}=0$.

(iv) According to (ii) of Theorem 2.2, the above equality (2.15) holds. Let h be the second fundamental form of (M, f). It follows from $h(X, \mathrm{grad}\,\varepsilon(p(\xi)))=0$, $\hat{E}=\mathrm{grad}\,\varepsilon\circ p$ and $E(\xi)=\frac{1}{\sqrt{1+|\mathrm{grad}\,\varepsilon(p(\xi))|^2}}\{\mathrm{grad}\,\varepsilon(p(\xi))+\bar{E}(\xi)\}$ that

$$\begin{aligned}(\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E})_{T_{\xi}}&=(\tilde{\nabla}_X\,\mathrm{grad}\,\varepsilon)_{T_{\xi}}=(\nabla_X\,\mathrm{grad}\,\varepsilon)_{T_{\xi}}\\&=\nabla_X\,\mathrm{grad}\,\varepsilon-\langle\nabla_X\,\mathrm{grad}\,\varepsilon,\,E(\xi)\rangle E(\xi)\\&=\nabla_X\,\mathrm{grad}\,\varepsilon-\frac{\langle\nabla_X\,\mathrm{grad}\,\varepsilon,\,\mathrm{grad}\,\varepsilon(p(\xi))\rangle}{1+|\mathrm{grad}\,\varepsilon(p(\xi))|^2}\{\mathrm{grad}\,\varepsilon(p(\xi))+\bar{E}(\xi)\}\\&=\nabla_X\,\mathrm{grad}\,\varepsilon-\frac{1}{2}X\{\log(1+|\mathrm{grad}\,\varepsilon|^2)\}\{\mathrm{grad}\,\varepsilon(p(\xi))+\bar{E}(\xi)\}.\end{aligned}$$

Thus since $(\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E})_{T_{\xi}}\in\mathrm{Ker}\,A_{\xi}\oplus\langle\bar{E}(\xi)\rangle$, $(\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E})_{T_{\xi}}$ is the natural lift of $\{(\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E})_{T_{\xi}}\}_{T_{p(\xi)}}$ to ξ, this is,

$$(\tilde{\nabla}_{\tilde{X}_{\xi}}\hat{E})_{T_{\xi}}=\widetilde{\nabla_X\,\mathrm{grad}}\,\varepsilon_{\xi}-\frac{1}{2}X\{\log(1+|\mathrm{grad}\,\varepsilon|^2)\}\,\widetilde{\mathrm{grad}\,\varepsilon(p(\xi))}_{\xi}.$$

Therefore, we obtain the desired equality. ∎

§3. The natural lifts of distributions.

Let (M^n, f) be a Riemannian submanifold in $\tilde{M}^{n+r}$ and D a distribution on M. Set $\tilde{D}:=\bigcup_{\xi\in t_{\varepsilon}(M)}\{\tilde{X}_{\xi}|X\in D_{p(\xi)}\}$. It is shown that $\tilde{D}$ is a (smooth) distribution on $t_{\varepsilon}(M)$. In fact, for a local base $(X_1, \cdots, X_m)$ of D on an open subset U of M, $(\tilde{X}_1, \cdots, \tilde{X}_m)$ is a local base of $\tilde{D}$ on $p^{-1}(U)$, when $m=\dim D$. We shall call this distribution $\tilde{D}$ the *natural lift* of D to $t_{\varepsilon}(M)$. We have the following result with respect to the integrability of the natural lift.

THEOREM 3.1. *Let F be a foliation on M. If the normal connection of (M, f) is flat along each leaf of F, then the natural lift $\tilde{F}$ of F to $t_{\varepsilon}(M)$ is a foliation on $t_{\varepsilon}(M)$.*

PROOF. Let $X, Y\in\Gamma(F|_U)$, where U is an open subset of M. We shall show $[\tilde{X}, \tilde{Y}]=\widetilde{[X, Y]}$. Fix $\xi_0\in p^{-1}(U)$. Let $V(\subset U)$ be a simply connected neighbourhood about $p(\xi_0)$ in a leaf of F through $p(\xi_0)$ and $\tilde{\xi}$ a $\nabla^{\perp}$-parallel normal vector field on V with $\tilde{\xi}(p(\xi_0))=\xi_0$. Note that the existence of $\tilde{\xi}$ is assured by the flatness of $\nabla^{\perp}$ along V. Define a normal vector field $\hat{\xi}$ on V by $\hat{\xi}(x):=\frac{\varepsilon(x)}{\varepsilon(p(\xi_0))}\tilde{\xi}(x)$ for $x\in V$. Let $V^*:=\bigcup_{x\in V}\gamma_{\hat{\xi}(x)}([0,1])$. It is clear that V^* is a submanifold (with boundary) in $\tilde{M}^{n+r}$. Let $\hat{X}$ (resp. $\hat{Y}$) be a vector field on V^* such that $\hat{X}(\gamma_{\hat{\xi}}(0))=X(\gamma_{\hat{\xi}}(0))$ and $[\hat{X},\dot{\gamma}_{\hat{\xi}}]=0$ (resp. $\hat{Y}(\gamma_{\hat{\xi}}(0))=Y(\gamma_{\hat{\xi}}(0))$ and $[\hat{Y},\dot{\gamma}_{\hat{\xi}}]=0$). We see that $\hat{X}(\gamma_{\hat{\xi}(x)}(1))=\tilde{X}_{\hat{\xi}(x)}$ and $\hat{Y}(\gamma_{\hat{\xi}(x)}(1))=\tilde{Y}_{\hat{\xi}(x)}$ for every $x\in V$ (cf. Proof of Lemma 1.1). Hence $[\hat{X},\hat{Y}](\gamma_{\hat{\xi}(x)}(1))=[\tilde{X},\tilde{Y}](\gamma_{\hat{\xi}(x)}(1))$ holds for every $x\in V$. It is clear that $[\hat{X},\hat{Y}](\gamma_{\hat{\xi}}(0))=[X,Y](\gamma_{\hat{\xi}}(0))$. By the Jacobi identity, we have

$$[[\hat{X},\hat{Y}],\dot{\gamma}_{\hat{\xi}}]=-[[\hat{Y},\dot{\gamma}_{\hat{\xi}}],\hat{X}]-[[\dot{\gamma}_{\hat{\xi}},\hat{X}],\hat{Y}]=0\,.$$

Hence we see that $[\hat{X},\hat{Y}](\gamma_{\hat{\xi}(x)}(1))=\widetilde{[X,Y]}_{\hat{\xi}(x)}$ for every $x\in V$. As a consequence, $[\tilde{X},\tilde{Y}](\gamma_{\hat{\xi}(x)}(1))=\widetilde{[X,Y]}_{\hat{\xi}(x)}$ is deduced for every $x\in V$. Especially, we obtain that $[\tilde{X},\tilde{Y}]=\widetilde{[X,Y]}$ at ξ_0. This completes the proof. ∎

Also, we can prove the following result.

THEOREM 3.2. *Let F be a foliation on M and F_S a foliation on $t_\varepsilon(M)$ of which leaves are $t_\varepsilon(M)\cap p^{-1}(p(\xi))$ $(\xi\in t_\varepsilon(M))$. If the normal connection of (M,f) is flat along each leaf of F, then $\tilde{F}\oplus F_S$ is a foliation on $t_\varepsilon(M)$.*

PROOF. Take $X\in\Gamma(\tilde{F}|_U)$ and $Y\in\Gamma(F_S|_U)$, where U is a sufficiently small open subset of $t_\varepsilon(M)$. We have only to show $[X,Y]\in\Gamma(\tilde{F}\oplus F_S|_U)$ because $\tilde{F}$ and F_S are integrable, respectively. Let $(\tilde{X}_1,\cdots,\tilde{X}_m)$ be a local base of $\tilde{F}$ on U consisting of the natural lifts of vector fields on $p(U)$, where $m=\dim F$. Let $X=\sum_{i=1}^{m}a_i\tilde{X}_i$. Then we have

$$\begin{aligned}[X,Y]&=\sum_{i=1}^{m}[a_i\tilde{X}_i,Y]\\&=\sum_{i=1}^{m}\{-(Ya_i)\tilde{X}_i+a_i[\tilde{X}_i,Y]\}.\end{aligned}$$

It is clear that $\tilde{X}_i$ is a foliated vector field with respect to F_S, that is, $[\tilde{X}_i,Y]\in\Gamma(F_S|_U)$. Therefore, we obtain $[X,Y]\in\Gamma(\tilde{F}\oplus F_S|_U)$. ∎

§ 4. Applications.

We expect that various arguments for tubes of nonconstant radius contribute to constructing various models of Riemannian hypersurfaces. For example, in this section, we shall construct soft models of Riemannian hypersurfaces of which the number of mutually distinct principal curvatures is constant on the hypersurface or a dense subset of the hypersurface in terms of Theorem 2.6 mainly. First we shall prepare the following two lemmas.

LEMMA 4.1. *Let* (M, f) *be an open portion* $(M'\times U, f'\times\iota|_U)$ *of a Riemannian hypersurface* $(M'\times R^l, f'\times\iota)$ *as in Theorem* 2.6 *and* $\varepsilon=\bar{\varepsilon}\circ q$. *Here* U *is an open subset of* R^l *and* q *is the natural projection of* $M=M'\times U$ *onto* U *and* $\bar{\varepsilon}$ *is a positive function on* U *such that* $\exp^{\perp}(N_{\varepsilon}(M))\cap F(M, f)=\emptyset$ *and the graph of* $\bar{\varepsilon}$ *is a Riemannian hypersurface which has exactly* g *mutually distinct principal curvatures* $\lambda_1>\cdots>\lambda_g$. *Then the statements* (i), (ii) *and* (iii) *in Theorem* 2.6 *hold and* $A^{\varepsilon}_{E(\xi)}|_{\tilde{TU}_\xi}$ $(\tilde{TU}_\xi\to\tilde{TU}_\xi)$ *has exactly* g *mutually distinct eigenvalues* $\lambda_1(q(p(\xi)), \bar{\varepsilon}(q(p(\xi))))>\cdots>\lambda_g(q(p(\xi)), \bar{\varepsilon}(q(p(\xi))))$, *where* TU *is a foliation on* $M=M'\times U$ *given by assigning* $T(\{x_1\}\times U)_{(x_1, x_2)}$ *to* $(x_1, x_2)\in M'\times U$.

PROOF. It is clear that $\operatorname{grad}\varepsilon\in\Gamma(TU)$. Hence (i), (ii) and (iii) in Theorem 2.6 hold. Fix $\xi_0\in t_{\varepsilon}(M)$. Since the normal connection $\nabla^{\perp}$ of (M, f) is flat along each leaf of TU, the natural lift $\tilde{TU}$ is a foliation on $t_{\varepsilon}(M)$ by Theorem 3.1. Let L be a leaf of $\tilde{TU}$ through ξ_0. It is easy to show that a $\nabla^{\perp}$-parallel normal vector field along a curve in $p(L)$ is parallel with respect to $\tilde{\nabla}$, where $\tilde{\nabla}$ is the Levi-Civita connection of R^{n+r}. Hence we see that L is congruent to the graph of $\bar{\varepsilon}$ and $E|_L$ corresponds to the unit normal vector field of the graph of $\bar{\varepsilon}$ (cf. Proof of Lemma 1.1). Therefore. we see that $A^{\varepsilon}_{E(\xi_0)}|_{\tilde{TU}_{\xi_0}}$ has exactly g mutually distinct eigenvalues $\lambda_1(q(p(\xi_0)), \bar{\varepsilon}(q(p(\xi_0))))>\cdots>\lambda_g(q(p(\xi_0)), \bar{\varepsilon}(q(p(\xi_0))))$. ∎

LEMMA 4.2. *Let* $(M, f)=(M'\times R^1, f'\times\iota)$ *and* $\varepsilon=\bar{\varepsilon}\circ q$, *where* $\bar{\varepsilon}$ *is a positive function on* R^1 *such that* $\exp^{\perp}(N_{\varepsilon}(M))\cap F(M, f)=\emptyset$ *and* q *is the natural projection of* $M=M'\times R^1$ *onto* R^1. *Then the statements* (i), (ii) *and* (iii) *in Theorem* 2.6 *hold and, for* $X\in T_{p(\xi)}R^1$ $(\subset\operatorname{Ker} A_\xi)$, *the following equality holds:*

$$A^{\varepsilon}_{E(\xi)}\tilde{X}_\xi=\frac{-\varepsilon''(p(\xi))}{\sqrt{\{1+\varepsilon'(p(\xi))^2\}^3}}\tilde{X}_\xi\,. \tag{4.4}$$

Here $\varepsilon'=\dfrac{d\varepsilon}{dt}$ *and* $\varepsilon''=\dfrac{d^2\varepsilon}{dt^2}$, *where* (t) *is the natural coordinate of* R^1.

PROOF. It is clear that $\operatorname{grad}\varepsilon\in\Gamma(TR^1)$. Hence (i), (ii), (iii) and (iv) in Theorem 2.6 hold. According to (iv) of Theorem 2.6, the following equality holds:

$$A^{\varepsilon}_{E(\xi)}\left(\tilde{\frac{\partial}{\partial t}}\right)_\xi=\frac{1}{2\sqrt{1+|\operatorname{grad}\varepsilon(p(\xi))|^2}}\cdot\left\{\frac{\partial}{\partial t}\log(1+|\operatorname{grad}\varepsilon|^2)\}\,\widetilde{\operatorname{grad}\varepsilon(p(\xi))}_\xi-2\widetilde{\nabla_{\partial/\partial t}\operatorname{grad}\varepsilon}_\xi\right\}. \tag{4.5}$$

Since $\operatorname{grad}\varepsilon=\varepsilon'\dfrac{\partial}{\partial t}$, we have

$$\frac{\partial}{\partial t}\{\log(1+|\operatorname{grad}\varepsilon|^2)\}\operatorname{grad}\varepsilon(p(\xi))-2\nabla_{\partial/\partial t}\operatorname{grad}\varepsilon=\frac{-2\varepsilon''(p(\xi))}{1+\varepsilon'(p(\xi))^2}\frac{\partial}{\partial t}. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$A^{\varepsilon}_{E(\xi)}\left(\frac{\tilde{\partial}}{\partial t}\right)_{\xi}=\frac{-\varepsilon''(p(\xi))}{\sqrt{\{1+\varepsilon'(p(\xi))^2\}^3}}\left(\frac{\tilde{\partial}}{\partial t}\right)_{\xi}.$$

Thus the equality (4.4) holds. ■

First, by using Lemma 4.1, we shall construct soft models of Riemannian hypersurfaces stated in the beginning of this section. Let (M', f') be a Riemannian hypersurface in R^{n_1+1} with exactly g_1 mutually distinct principal curvatures $\lambda_1>\cdots>\lambda_{g_1}$ and $\bar{\varepsilon}$ a positive function on an open subset U of R^{n_2}, of which graph is a Riemannian hypersurface with exactly g_2 mutually distinct principal curvatures $\mu_1>\cdots\mu_{g_2}$. Here we assume that λ_i $(1\leqq i\leqq g_1)$ and μ_j $(1\leqq j\leqq g_2)$ have no zero point. Let m_i $(i=1,\cdots,g_1)$ be the multiplicity of λ_i and m'_j $(j=1,\cdots,g_2)$ that of μ_j. Set $(M, f):=(M'\times U, f'\times(\iota|_U))$ and $\varepsilon:=\bar{\varepsilon}\circ q_U$, where ι is the inclusion mapping of R^{n_2} into R^{n_2+r-1} and q_U is the natural projection of $M=M'\times U$ onto U. By letting a Riemannian hypersurface given by homothetic transforming with a sufficiently large coefficient (M', f') be (M', f') newly, letting $\bar{\varepsilon}-c$ (c is a positive constant with $c<\inf_{x\in U}\bar{\varepsilon}(x)$) be $\bar{\varepsilon}$ newly and shrinking M' and U if necessary, we may assume that

$$\exp^{\perp}(N_{\varepsilon}(M))\cap F(M, f)=\emptyset,\quad \max_{1\leq i\leq g_1}\sup\varepsilon|\lambda_i|<\frac{1}{2}, \tag{4.7}$$

$$\max_{1\leq i\leq g_1}\sup\frac{2|\lambda_i|}{\sqrt{1+|\operatorname{grad}\varepsilon|^2}}<\min_{1\leq j\leq g_2}\inf|\mu_j|,$$

$$\max_{1\leq j\leq g_2}\sup|\mu_j|<\inf\frac{1}{\sqrt{1+|\operatorname{grad}\varepsilon|^2}\varepsilon}.$$

Denote by E' the unit normal vector field of (M', f') and set $W_\delta:=\{\xi\in t_\varepsilon(M)\mid \frac{1}{\varepsilon(p(\xi))}\langle\xi, E'\rangle>\delta\}$, where $\delta\in(0, 1)$. For $\xi\in W_\delta$, the set of all the mutually disinct eigenvalues of A_ξ is $\{0\}\cup\{\langle\xi, E'(q_{M'}(p(\xi)))\rangle\lambda_i(q_{M'}(p(\xi)))\mid i=1,\cdots,g_1\}$, where A is the shape operator of (M, f) and $q_{M'}$ is the natural projection of $M'\times U$ onto M'. According to Lemma 4.1, the shape operator A^{ε} of $(t_\varepsilon(M), f_\varepsilon)$ satisfies the following three conditions (i)∼(iii):

(i) for $X\in T_\xi(t_\varepsilon(M)\cap p^{-1}(p(\xi)))$,

$$A^{\varepsilon}_{E(\xi)}X=\frac{1}{\sqrt{1+|\operatorname{grad}\varepsilon(p(\xi))|^2}\varepsilon(p(\xi))}X,$$

(ii) for $X\in\mathrm{Ker}\,(A_\xi-\langle\xi, E'(q_{M'}(p(\xi)))\rangle\lambda_i(q_{M'}(p(\xi)))I)$ $(\xi\in W_\delta)$,

$$A^\varepsilon_{E(\xi)}\tilde{X}_\xi=\frac{\langle\xi, E'(q_{M'}(p(\xi)))\rangle\lambda_i(q_{M'}(p(\xi)))\tilde{X}_\xi}{\sqrt{1+|\mathrm{grad}\,\varepsilon(p(\xi))|^2}\varepsilon(p(\xi))(\langle\xi, E'(q_{M'}(p(\xi)))\rangle\lambda_i(q_{M'}(p(\xi)))-1)},$$

(iii) $A^\varepsilon_{E(\xi)}|_{\widetilde{TU}_\xi}$ has exactly g_2 mutually distinct eigenvalues $\mu_1(q_U(p(\xi)), \bar{\varepsilon}(q_U(p(\xi))))>\cdots>\mu_{g_2}(q_U(p(\xi)), \bar{\varepsilon}(q_U(p(\xi))))$. Define functions $\nu_1, \cdots, \nu_{g_1+g_2+1}$ on W_δ by

$$\nu_i(\xi)=\frac{\langle\xi, E'(q_{M'}(p(\xi)))\rangle\lambda_i(q_{M'}(p(\xi)))}{\sqrt{1+|\mathrm{grad}\,\varepsilon(p(\xi))|^2}\varepsilon(p(\xi))(\langle\xi, E'(q_{M'}(p(\xi)))\rangle\lambda_i(q_{M'}(p(\xi)))-1}$$

$$(i=1, \cdots, g_1),$$

$$\nu_{g_1+1}(\xi)=\frac{1}{\sqrt{1+|\mathrm{grad}\,\varepsilon(p(\xi))|^2}\varepsilon(p(\xi))}$$

and

$$\nu_j(\xi)=\mu_{j-g_1-1}(q_U(p(\xi)), \bar{\varepsilon}(q_U(p(\xi))))\quad (j=g_1+2, \cdots, g_1+g_2+1).$$

It is clear that $\nu_{i_1}\neq\nu_{i_2}$ $(1\leqq i_1\neq i_2\leqq g_1)$ and $\nu_{j_1}\neq\nu_{j_2}$ $(g_1+2\leqq j_1\neq j_2\leqq g_1+g_2+1)$ at each point of W_δ. It follows from (4.7) that $|\nu_i|<|\nu_j|<|\nu_{g_1+1}|$ $(1\leqq i\leqq g_1,\ g_1+2\leqq j\leqq g_1+g_2+1)$. Thus $\nu_1, \cdots, \nu_{g_1+g_2+1}$ are mutually distinct at each point of W_δ. Therefore $(W_\delta, f_\varepsilon|_{W_\delta})$ is a Riemannian hypersurface with mutually distinct principal curvatures $\nu_1, \cdots, \nu_{g_1+g_2+1}$. Note that ν_i $(i=1, \cdots, g_1+g_2+1)$ have no zero point and the multiplicities of $\nu_1, \cdots, \nu_{g_1+g_2+1}$ are $m_1, \cdots, m_{g_1}, r-1, m'_1, \cdots, m'_{g_2}$, respectively. By the way, the existence of the above Riemannian hypersurface (M', f') for $g_1=1, 2$ (any multiplicity) and that of the above function $\bar{\varepsilon}$ for $g_2=1$ (any multiplicity) are well-known. Hence, by the above construction, for every set $\{m_1, \cdots, m_g\}$ of positive integers, we can obtain soft models of Riemannian hypersurfaces in an Euclidean space which has exactly g mutually distinct principal curvatures, which have multiplicities $m_1, \cdots, m_g$ and are nonzero, at each point.

Next, by using Lemma 4.2, we shall construct soft models of complete Riemannian hypersurfaces stated in the beginning of this section. Let (M', f') be a complete Riemannian hypersurface in R^{n+1} with exactly g mutually distinct principal curvatures $\lambda_1>\cdots>\lambda_g$ on a dense subset W' of M', where we assume $\sup|\lambda_i|<\infty$ $(i=1, \cdots, g)$. Let $\bar{\varepsilon}$ be a positive function on R^1 such that $\bar{\varepsilon}<\min_{1\leq i\leq g}\inf\frac{1}{2|\lambda_i|}$, $\sup\frac{|\bar{\varepsilon}''|}{\sqrt{(1+\bar{\varepsilon}'^2)^3}}<\infty$ and the set of all solutions of the equation $\bar{\varepsilon}''(\bar{\varepsilon}a-1)+a(1+\bar{\varepsilon}'^2)=0$ is discrete for every constant a. It is clear that such a function $\bar{\varepsilon}$ exists innumerably. Set $(M, f)=(M'\times R^1, f'\times\mathrm{id})$ and $\varepsilon:=\bar{\varepsilon}\circ q_1$, where id is the identity map of R^1 and q_1 is the natural projection of $M=M'\times R^1$ onto R^1. Since $\varepsilon<\min_{1\leq i\leq g}\inf\frac{1}{2|\lambda_i|}$, $\exp^\perp(N_\varepsilon(M))\cap F(M, f)=\emptyset$, that is, the ε-tube $(t_\varepsilon(M), f_\varepsilon)$ is defined. According to Lemma 4.2, the shape operator A_ε of $(t_\varepsilon(M), f_\varepsilon)$ satisfies the following conditions (i) and (ii):

(i) for $X \in \operatorname{Ker}(A_\xi - \varepsilon(p(\xi))\lambda_i(q_{M'}(p(\xi)))I)$,

$$A^\xi_{E(\xi)}\tilde{X}_\xi = \frac{\lambda_i(q_{M'}(p(\xi)))}{\sqrt{1+\varepsilon'(p(\xi))^2(\varepsilon(p(\xi))\lambda_i(q_{M'}(p(\xi)))-1)}}\tilde{X}_\xi$$

and

(ii) for $X \in T_{p(\xi)}R^1$ $(\subset T_{p(\xi)}M)$,

$$A^\xi_{E(\xi)}\tilde{X}_\xi = \frac{-\varepsilon''(p(\xi))}{\sqrt{\{1+\varepsilon'(p(\xi))^2\}^3}}\tilde{X}_\xi,$$

where $q_{M'}$ is the natural projection of $M=M'\times R^1$ onto M'. Define functions $\mu_1, \cdots, \mu_{g+1}$ on $t_\varepsilon(M)$ by

$$\mu_i(\xi) = \frac{\lambda_i(q_{M'}(p(\xi)))}{\sqrt{1+\varepsilon'(p(\xi))^2(\varepsilon(p(\xi))\lambda_i(q_{M'}(p(\xi)))-1)}} \quad (i=1, \cdots, g)$$

and

$$\mu_{g+1}(\xi) = \frac{-\varepsilon''(p(\xi))}{\sqrt{\{1+\varepsilon'(p(\xi))^2\}^3}}.$$

It is clear that $\mu_1, \cdots, \mu_g$ are mutually distinct at each point of $W'\times R^1$. Since the set of all solutions of the equation $\bar{\varepsilon}''(\bar{\varepsilon}a-1)+a(1+\bar{\varepsilon}'^2)=0$ is discrete for every constant a, μ_i and μ_{g+1} are mutually distinct at each point of a dense subset of $t_\varepsilon(M)$ $(i=1, \cdots, g)$. Therefore, $\mu_1, \cdots, \mu_{g+1}$ are mutually distinct at each point of a dense subset W of $t_\varepsilon(M)$. Thus $(t_\varepsilon(M), f_\varepsilon)$ is a complete Riemannian hypersurface in R^{n+2} with exactly $g+1$ mutually distinct principal curvatures $\mu_1, \cdots, \mu_{g+1}$ on W. Moreover, since $\sup|\lambda_i|<\infty$, $\varepsilon<\min_{1\le i\le g}\inf\frac{1}{2|\lambda_i|}$ and $\sup\frac{|\bar{\varepsilon}''|}{\sqrt{(1+\bar{\varepsilon}'^2)^3}}<\infty$, we have $\sup|\mu_i|<\infty$ $(i=1, \cdots, g+1)$. By the way, the existence of the above complete Riemannian hypersurface (M', f') for $g=1$ is well-known. Hence, by the above construction, for every positive integer g, we can obtain soft models of complete Riemannian hypersurfaces in an Euclidean space with exactly g mutually distinct principal curvatures at each point of a dense subset.

References

[1] Abe, N., Koike, N. and Yamaguchi, S., Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form, Yokohama Math. J. 35 (1987), 123-136.

[2] Cecil, T.E. and Ryan, P.J., Tight and taut immersions of manifolds, Pitman Advanced Pub., Boston, 1985.

[3] Gray, A., Tubes, Addison Wesley, Redwood City, California, 1990.
[4] Ryan, P. J., Homogeneity and some curvature conditions for hypersurfaces, Tôhoku Math. J. 21 (1969), 363-388.

Department of Mathematics
Faculty of Science
Science University of Tokyo
26 Wakamiya Shinjuku-ku,
Tokyo 162 Japan