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ON A GENERALIZED DEGREE FOR CONTINUOUS MAPS BETWEEN MANIFOLDS

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Abstract. We present a generalized degree for maps between manifolds via framed bordism theory. We study when the degree of a map can be considered as an element of a homotopy group of spheres. Finally we apply this tools to prove a result concerning the generalized Hopf's invariant.

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1. Introduction and preliminary definitions.

In [G. M. V.] is presented a degree theory in euclidean spaces that generalizes the classical degree of Brouwer. In order to solve the additivity property problem, in [R2] we gave an alternative description of the generalized degree through differentiable methods. This tools are based on the Pontryagin's results of framed bordism (see [P]) and moreover can be used to extend the generalized degree to the contex of normed spaces ([R3]).

The aim of this paper is to extend the classical degree theory in manifolds for continuous maps $f: M^{n+k}\to M^n$ where M^{n+k} and M^n are compact connected oriented $(n+k)$ and n manifolds respectively. We will apply once again framed bordism theory.

In this paper, by M^{n+k} and M^n we shall mean compact Hausdorff oriented C^∞-manifolds with dimension $n+k$ and n respectively $\partial M^{n+k}=\partial M^n=\emptyset$, M^n will be supposed to be connected. All differentiable maps will be C^∞-maps.

In order to do this paper as selfcontained as possible we are going to point out the most important concepts and results that we will need.

A pair (M^k, F) is said to be a *framed submanifold of* M^{n+k} if M^k is a closed k-dimensional submanifold of M^{n+k} and $F=\{u_1, u_2, \cdots, u_n\}$ is a normal frame for M^k, i.e. F is a family of independent differentiable sections of the

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normal bundle (trivial) of M^k, denoted by $\nu(M^k)$.

Let $N.F.^k(M^{n+k})=\{(M^k, F): (M^k, F)$ is a framed submanifold of $M^{n+k}\}$. $(M_0^k, F_0), (M_1^k, F_1)\in N.F.^k(M^{n+k})$ are said to be *homologous*, written $(M_0^k, F_0)\cong(M_1^k, F_1)$, provided there exists a $k+1$-dimensional closed submanifold M^{k+1} of $M^{n+k}\times I$ such that $\partial M^{k+1}=(M_0^k\times\{0\})\cup(M_1^k\times\{1\})$, $M^{k+1}\cap(M^{n+k}\times\{t\})=M_0^k\times\{t\}$ for every $t\in[0, 1/3)$, $M^{k+1}\cap(M^{n+k}\times\{t\})=M_1^k\times\{t\}$ for every $t\in(2/3, 1]$ and there exists $G=\{G_1, G_2, \cdots, G_n\}$ normal frame for M^{k+1} verifying

$$G|_{M_0^k\times\{0\}}=F_0 \quad\text{and}\quad G|_{M_1^k\times\{1\}}=F_1.$$

It is easy to see that $\cong$ is a relation of equivalence, then we have the set $\mathfrak{F}^k(M^{n+k})=N.F.^k(M^{n+k})/\cong$.

Let us consider S^n to be the n-dimensional sphere and p, q the South and North poles of S^n respectively.

Now consider $f: M^{n+k}\to S^n$ to be a C^∞-map such that p is a regular value of f, written $p\in(r.v.)(f)$, one can assign to f an element $(M_f^k, F_f)\in N.F.^k(M^{n+k})$, where $M_f^k=f^{-1}(p)\subset M^{n+k}$ and the frame $F_f=\{u_1, u_2, \cdots, u_n\}$ satisfies that if $c'=(U, \varphi_n^{-1}, \boldsymbol{R}^n)$ denotes the local coordinates of S^n given by the projection $p\in U=S^n\setminus\{q\}\xrightarrow{\varphi_n^{-1}}\boldsymbol{R}_n$, then $T_xf(u_j(x))=\Theta_{c'}^p(e_j)$ for every $j=1, \cdots, n$ and every $x\in M^k$. ($\Theta_{c'}^p$ denotes the isomorphism between $\boldsymbol{R}^n$ and $T_p(S^n)$ induced by c').

Arguing as in [P] one can prove the following

THEOREM 1.1. *Let M^{n+k} be a manifold as above and $\Pi^n(M^{n+k})$ the n-th cohomotopy set (group if $n\geq k+2$) of M^{n+k}. Then, the function $\Pi^k: \Pi^n(M^{n+k})\to\mathfrak{F}^k(M^{n+k})$, defined by $\Pi_n^k([h])=[(M_{f'}^k, F_f)]$, where $f: M^{n+k}\to S^n$ is a C^∞-map homotopic to h such that $p\in(r.v.)(f)$, is bijective (isomorphism if $n\geq k+2$).*

The structure of the present work is the following: in the remaining part of this section by using Theorem 1.1 we introduce the generalized degree definition and prove the main properties. However, this definition poses some difficulties (of computation for example), then, it is interesting to find out conditions for making this definition easier. Note that when it is possible to identify $d(f)$ with an element of $\Pi_{n+k}(S^n)$, as in the case of the generalized degree in euclidean and normed spaces, we can use Pontryagin's theory to determinate in a reasonable way this degree and the structure of the sets $\Pi^n(M^{n+k})$. We devote section 2 to this task. Some ideas of section 2 are motivated by the work of Kervaire [K].

In section 3 we apply the results of section 2 to compute the degree of some functions that allows us to prove Theorem 3.1 that improves 6.1 of [K] and generalizes a classical theorem concerning the Hopf's invariant.

Applications of this results to the complementing maps theory can be found in [R4].

The reader is referred to the texts of [H] and [S] for information about differential and algebraic topology machinery.

DEFINITION 1.1. Let $f: M^{n+k} \to M^n$ be a C^∞-map and $x_0 \in (\mathrm{r.v.})\,(f)$. We define the generalized degree of f at x_0, written $d(f, x_0)$, by $d(f, x_0) = [(f^{-1}(x_0), F_f)] \in \mathfrak{F}^k(M^{n+k})$, where $F_f = \{u_1, \cdots, u_n\}$ is the normal frame for $f^{-1}(x_0)$ such that $T_x f(u_i(x)) = \Theta_c^{x_0}(e_i)$ for every $i=1, \cdots, n$ and every $x \in f^{-1}(x_0)$, and $c=(U, \varphi, \boldsymbol{R}^n)$ are local coordinates inducing the orientation of M^n, U containing x_0.

It is easy to check that the above definition is consistent, i.e. $d(f, x_0)$ depends neither on c nor on the choice of the positive basis of $\boldsymbol{R}^n$.

LEMMA 1.1. *Let* $H: M^{n+k} \times I \to M^n$ *be a* C^∞-*homotopy such that* $x_0 \in (r.v.)\,(H_0) \cap (r.v.)\,(H_1)$, *then* $d(H_0, x_0) = d(H_1, x_0)$.

The proof of Lemma 1.1 is straightforward if $x_0 \in (r.v.)(H)$, otherwise one can get another C^∞-homotopy $H': M^{n+k} \times I \to M^n$ such that $x_0 \in (r.v.)(H')$, and $H'_0 = H_0$ and $H'_1 = H_1$.

As a consequence of above lemma we obtain the next useful corollary

COROLLARY 1.1. *Let* $f: M^{n+k} \to M^n$ *be a* C^∞-*map and let* $x_0, x_1 \in (r.v.)(f)$, *then* $d(f, x_0) = d(f, x_1)$.

PROOF. Take a C^∞-isotopy $H: M^n \times I \to M^n$ such that $H_0 = \mathrm{Id}$ and $H_1(x_0) = x_1$. Now consider $G = H \circ (f \times \mathrm{Id}): M^{n+k} \times I \to M^n$. Corollary 1.1 implies that $d(G_0, x_1) = d(G_1, x_1)$. Since $G_0 = f$ and $G_1 = H_1 \circ f$, it follows that

$$\begin{aligned} d(f, x_1) = d(G_1, x_1) = d(H_1 \circ f, x_1) &= [(H_1 \circ f)^{-1}(x_1), F_{H_1 \circ f})] \\ &= [(f^{-1}(x_0), F_{H_1 \circ f})] = [(f^{-1}(x_0), F_f)] = d(f, x_0). \quad \square \end{aligned}$$

Corollary 1.1 allows us to state the following definition

DEFINITION 1.2. *a) Let* $f: M^{n+k} \to M^n$ *be a* C^∞-*map. We define the generalized degree of* f, *denoted by* $d(f)$, *by* $d(f) = d(f, x_0) \in \mathfrak{F}^k(M^{n+k})$, *where* x_0 *is any regular value of* f.

b) If $f: M^{n+k} \to M^n$ is a continuous map we define the degree of f by $d(f)=d(g)\in\mathfrak{F}^k(M^{n+k})$, where $g: M^{n+k}\to M^n$ is any C^∞-map such that f and g are homotopic.

In the sequel we shall refer to continuous maps shortly as maps.

REMARK 1. It is obvious that $d(f)=d(f')$ if f, f' are homotopic maps.

REMARK 2. If $M^n=S^n$ it follows that $d(f)=\Pi_n^k([f])$ for every map $f: M^{n+k}\to S^n$, then $d(f)$ characterizes the homotopy class of f.

From the previous definition one can check easily the following result

PROPOSITION 1.1. *Let $f: M^{n+k}\to M^n$ be a map such that $d(f)\neq 0$, then f is onto.*

2. Conditions for the generalized degree of a map to be an element of $\Pi_{n+k}(S^n)$.

In the above section we have defined the degree of a map $f: M^{n+k}\to M^n$ as an element of $\mathfrak{F}^k(M^{n+k})$. Those sets (groups if $n\geq k+2$) are in general very difficult to determinate, therefore, it is interesting to find out when it is possible to identify $d(f)$ with an element of $\Pi_{n+k}(S^n)$ as in the case of the generalized degree in euclidean and normed spaces ([G. M. V.], [R3]). Besides, in this situation $d(f)$ would take values in a common set for every M^{n+k} and M^n as in the classical degree theory.

First of all we recall the next lemma

LEMMA 2.1. *Let $n\in \boldsymbol{N}$ and $k\in \boldsymbol{N}\cup\{0\}$. The projection $\varphi_{n+k}: \boldsymbol{R}^{n+k}\to S^{n+k}\setminus\{q\}$ induces a bijection $\bar{\varphi}_{n+k}: \mathfrak{F}^k(\boldsymbol{R}^{n+k})\to\mathfrak{F}^k(S^{n+k})$ given by $\bar{\varphi}_{n+k}([(M^k, F)])=[(\varphi_{n+k}(M^k), T\varphi_{n+k}(F))]$, where $T\varphi_{n+k}$ denotes the map induced by φ_{n+k} between the corresponding normal bundles.*

Now let us suppose in this section M^{n+k} to be compact, connected, oriented without boundary and $c=(U, \phi^{-1}, \boldsymbol{R}^{n+k})$ be a chart of the orientation of M^{n+k}. By applying Lemma 2.1 one has $\mathfrak{F}^k(S^{n+k})\equiv\mathfrak{F}^k(\boldsymbol{R}^{n+k})\equiv\mathfrak{F}^k(B^{n+k}(0))$, $(B^{n+k}(0)=\{x\in\boldsymbol{R}^{n+k}$ such that $\|x\|<1\})$ then, we can define $\phi^*: \mathfrak{F}^k(S^{n+k})\to\mathfrak{F}^k(M^{n+k})$ by $\phi^*([(M^k, F)])=[(\phi(M^k), \overline{T\phi}(F))]$, where $\overline{T\phi}: \nu(M^k)\to\nu(\phi(M^k))$ is the map induced by

$$T\phi: T_{M^k}B^{n+k}(0) \longrightarrow T_{\phi(M^k)}M^{n+k}$$
$$(x, v)\longmapsto (\phi(x), T_x\phi(v))$$

between the normal bundles of M^k and $\phi(M^k)$. It is clear that $\overline{T\phi}$ is an isomorphism and ϕ^* is well defined. On the other hand, since ϕ transforms disjoint sets in disjoint sets, ϕ^* is a homomorphism if $n \geqq k+2$.

Now we state the next useful proposition

PROPOSITION 2.1 ([H], pag. 185). *Let $\phi', \phi: \bar{B}^{n+k}(0) \to M^{n+k}$ be orientation preserving diffeomorphisms onto its images, then there exists a C^∞-isotopy H: $M^{n+k} \times I \to M^{n+k}$ such that $H_0 = \mathrm{Id}$ and $H_1|_U = \phi' \circ \phi^{-1}$, where $U = \phi(B^{n+k}(0))$.*

As a consequence of Proposition 2.1 we have

PROPOSITION 2.2. *If $c = (U, \phi^{-1}, \boldsymbol{R}^{n+k})$ and $c' = (U', \phi'^{-1}, \boldsymbol{R}^{n+k})$ are charts inducing the orientation of M^{n+k} it follows that $\phi^* = \phi'^*$.*

PROOF. Consider H: $M^{n+k} \times I \to M^{n+k}$ as in Prop. 2.1 and let $\alpha: I \to I$ be a C^∞-map such that $\alpha|_{[0,1/3)} = 0$, $\alpha|_{(2/3,1]} = 1$ and $\alpha'(t) > 0$ for $t \in (1/3, 2/3)$. Let us define $\tilde{H}$: $M^{n+k} \times I \to M^{n+k} \times I$ by $\tilde{H}(x, t) = (H(x, \alpha(t)), t)$. Hence $\tilde{H}$ is a diffeomorphism. For every $[(M^k, F)] \in \mathfrak{F}^k(S^{n+k})$, $\tilde{H}(\phi(M^k) \times I)$ is a compact submanifold of $M^{n+k} \times I$, such that $\tilde{H}(\phi(M^k) \times I) \cap (M^{n+k} \times \{t\}) = \phi(M^k) \times \{t\}$ if $t \in [0, 1/3)$ and $\tilde{H}(\phi(M^k) \times I) \cap (M^{n+k} \times \{t\}) = \phi'(M^k) \times \{t\}$ if $t \in (2/3, 1]$. It is clear that $\overline{T\phi}(F)$ is a normal frame for $\phi(M^k) \times I$, therefore if $\overline{T\tilde{H}}$: $\nu(\phi(M^k) \times I) \to \nu(\tilde{H}(\phi(M^k) \times I))$ is the isomorphism induced by $T\tilde{H}$ it follows that the pair $(\tilde{H}(\phi(M^k) \times I), \overline{T\tilde{H}}(T\phi(F))$ achieves a homology between $(\phi(M^k), \overline{T\phi}(F))$ and $(\phi'(M^k), \overline{T\phi'}(F))$. This completes the proof of the proposition. □

In order to look into the main properties of ϕ^* we need the following

PROPOSITION 2.3 ([H], pag. 183). *Let M^{n+k} be as above. Assume that M^{n+k} is k-connected and $n \geqq k+2$. Let M^k and M^k_* be two k-dimensional diffeomorphic compact submanifolds of M^{n+k}. Then there exists a C^∞-isotopy H: $M^{n+k} \times I \to M^{n+k}$ such that $H_0 = \mathrm{Id}$ and $H_1(M^k_*) = M^k$.*

Now we are in a position of proving the following useful consequence

PROPOSITION 2.4. *If $n \geqq k+2$ and M^{n+k} is as in Prop. 2.3, then ϕ^* is onto.*

PROOF. Let $[(M^k, F)] \in \mathfrak{F}^k(M^{n+k})$ and let $f: M^{n+k} \to S^n$ be a C^∞-map such that $p \in (\text{r.v.})(f)$ and $\Pi^k_n([f]) = [(M^k, F)]$. Since $n \geqq k+2$, M^k can be embedded in U, where U is the domain of a chart $c=(U, \phi^{-1}, \boldsymbol{R}^{n+k})$ inducing the orientation of M^{n+k}. Let M_*^k be the image of a such embedding. From Prop. 2.3 there exists a C^∞-isotopy $H: M^{n+k}\times I\to M^{n+k}$ such that $H_0=\mathrm{Id}$ and $H_1(M_*^k)=M^k$. Now the map $f\circ H$ gives us a homotopy between f and $f\circ H_1=f'$.

Therefore $[(M^k F)]=\Pi_n^k([f])=\Pi_n^k([f'])=[(f'^{-1}(p), F_{f'})]$. Since $f'^{-1}(p)=M_*^k$ we have $\phi^*([(\phi^{-1}(M_*^k), \overline{T\phi}^{-1}(F_{f'}))])=[(M^k, F)]$ and ϕ^* is onto. □

A manifold M is said to be a *π-manifold* provided there exist $m\in \boldsymbol{N}$ and an embedding $f: M\to \boldsymbol{R}^m$ such that $\nu(f(M))$ is trivial. If M is a n-dimensional π-manifold there is $U=\{u_1, \cdots, u_{m-n}\}$ family of linearly independent sections of $\nu(f(M))$. By using f and U, M can be oriented following a standard way. In the sequel when we refer to a π-manifold M, as above, it will be supposed to have the orientation induced by f and U.

Now let M^{n+k} be a π-manifold, $f: M^{n+k}\to \boldsymbol{R}^{n+k+s}$ be an embedding and let $U=\{u_1, \cdots, u_s\}$ a normal frame for $f(M^{n+k})$. Let us define a function (homomorphism if $n\geq k+2$) $U_f^*: \mathfrak{F}^k(M^{n+k})\to \mathfrak{F}^k(\boldsymbol{R}^{n+k+s}) \overset{\bar{\varphi}_{n+k+s}}{\equiv} \mathfrak{F}^k(S^{n+k+s})$ by $U_f^*([M^k, F)])=[(f(M^k), (\overline{Tf}(F), U))]$ where $\overline{Tf}: \nu(M^k))\to \nu(f(M^k))$ is the map induced by Tf i.e. $\overline{Tf}(x, [v])=(f(x), [T_x f(v)])$ ($\nu(f(M^k))$ denotes the normal bundle of $f(M^k)$ in $f(M^{n+k})$). Obviously U_f^* is well defined.

If M^{n+k} is a π-manifold and $g: M^{n+k}\to M^n$ is a map one can consider the generalized degree of g, $d(g)\in \mathfrak{F}^k(M^{n+k})$, as $((\Pi_{n+s}^k)^{-1}\circ U_f^*)(d(g))\in \Pi_{n+k+s}(S^{n+s})$. This resultant alternative definition of the degree of a map presents the disadvantadge of depending on f and U and besides we may lose information because U_f^* is not bijective. In order to eliminate this gaps we are going to prove some propositions that moreover will determinate the relation between ϕ^* and U_f^*.

PROPOSITION 2.5. *Let M^{n+k} be a k-connected π-manifold, $n\geq k+2$. Let $f: M^{n+k}\to \boldsymbol{R}^{n+k+s}$ be a embedding such that $\nu(f(M^{n+k}))$ is trivial and $U=\{u_1, \cdots, u_s\}$, $V=\{v_1, \cdots, v_s\}$ be two normal frames for $f(M^{n+k})$ inducing the same orientation of M^{n+k}. Then $U_f^*=V_f^*: \mathfrak{F}^k(M^{n+k})\to \mathfrak{F}^k(\boldsymbol{R}^{n+k+s})\equiv \mathfrak{F}^k(S^{n+k+s})$*

PROOF. Let $[(M^k, F)[\in \mathfrak{F}^k(M^{n+k})$ and for every $j\in\{1, \cdots, s\}$ we write $u_j(x)=\sum_{i=1}^{t} a_{ij}(x)v_j(x)$ $(x\in f(M^{n+k}))$. Define $G: f(M^{n+k})\to GL_+(\boldsymbol{R}^s)$ by $G(x)=G_x=(a_{ij}(x))_{i,j=1}^s$. Now consider

$$G|_{f(M^k)}: f(M^k) \overset{i}{\hookrightarrow} f(M^{n+k}) \xrightarrow{G} GL_+(\boldsymbol{R}^s),$$

$G|_{f(M^k)}$ is homotopic to a constant map, then there exists a C^∞-map $H: f(M^k)\times I\to GL_+(\boldsymbol{R}^s)$ such that $H_0=G|_{f(M^k)}$ and $H_1=ct_{\mathrm{Id}}$ (the map of constant value the identity matrix).

It is clear that $(f(M^k)\times I, (\overline{Tf}(F), H(v_1, \cdots, v_s))$ achieves a homology between $(f(M^k), (\overline{Tf}(F), U))$ and $(f(M^k), (\overline{Tf}(F), V))$. □

PROPOSITION 2.6. *Let M^{n+k} be a π-manifold. Let $g, f: M^{n+k}\to\boldsymbol{R}^{n+k+s}$ be two embeddings such that $\nu(f(M^{n+k}))$ and $\nu(g(M^{n+k}))$ are trivial. Let U and U' be normal frames for $f(M^{n+k})$ and $g(M^{n+k})$ respectively and $c=(W, \phi^{-1}, \boldsymbol{R}^{n+k})$ be a chart of M^{n+k} where $\phi: \bar{B}^{n+k}(0)\to\overline{W}$. If $n\geq k+2$ and $s\geq n+k+2$ it follows that $U_g'^*|_{\mathrm{Im}\phi^*}=U_f^*|_{\mathrm{Im}\phi^*}: \mathrm{Im}\phi^*\to\mathfrak{F}^k(S^{n+k+s})$.*

PROOF. Let $[(M^k, F)]\in\mathfrak{F}^k(M^{n+k})$. We apply Prop. 2.3 to get a C^∞-isotopy $H: \boldsymbol{R}^{n+k+s}\times I\to\boldsymbol{R}^{n+k+s}$ such that $H_t=\mathrm{Id}$ for every $t\in[0, 1/3)$, $H_t=H_1$ if $t\in(2/3, 1]$ and $H_1|_{f(M^{n+k})}=g\circ f^{-1}$.

Define $\tilde{H}: \boldsymbol{R}^{n+k+s}\times I\to\boldsymbol{R}^{n+k+s}\times I$ by $\tilde{H}(x, t)=(H(x, t), t)$.

Since U can be considered as a frame normal frame for $f(M^{n+k})\times I$, we have that $U''=T\tilde{H}(U)$ is a frame for $\tilde{H}(f(M^{n+k})\times I)$. Then $(\tilde{H}(f(M^k)\times I), (\overline{T\tilde{H}}(Tf(F)), U''))$ achieves a homology between $(f(M^k), (\overline{Tf}(F), U))$ and $(g(M^k), (\overline{Tg}(F), U''|_{g(M^{n+k})}))$, therefore $U_f^*([(M^k, F)])=(U''|_{g(M^{n+k})})_g^*([(M^k, F)])$.

If $[(M^k, F)]\in\mathrm{Im}\phi^*$, there exists $[(N^k, G)]\in\mathfrak{F}^k(S^{n+k})$ such that $\phi^*([(N^k, G)])=[(M^k, F)]$. Hence it suffices to prove that $(U_g'^*\circ\phi^*)([(N^k, G)])=((U''|_{g(M^{n+k})})_g^*\circ\phi^*)([(N^k, G)])$. On the other hand

$$(U_g'^*\circ\phi^*)([(N^k, G)])=U_g'^*([(\phi(N^k), \overline{T\phi}(G)])$$

and

$$((U''|_{g(M^{n+k})})_g^*\circ\phi^*)([(N^k, G)])=(U''|_{g(M^{n+k})})_g^*([(\phi(N^k), \overline{T\phi}(G))]).$$

Now, since $\phi(N^k)\subset W$ and W is contractible arguing as in Prop. 2.5 we obtain $(U''|_{g(M^{n+k})})_g^*([(\phi(N^k), \overline{T\phi}(G))])=U_g'^*([(\phi(N^k), \overline{T\phi}(G))])$. This completes the proof. □

Now we state the next lemma, the proof is not difficult and we will omit it.

LEMMA 2.2. *Let $f: M^n\to\boldsymbol{R}^{n+s}$ be a diffeomorphic embedding and $a\in M^n$. Then there exist a diffeomorphic embedding $g: M^n\to\boldsymbol{R}^{n+s}$ and an open set U of M^n containing a such that $g(a)=0$ and $g(\bar{U})=\bar{B}^n_{(0)}\times\{0\}$.*

COROLLARY 2.1. *Let M^{n+k} be a π-manifold, $n\geq k+2$. Let $f: M^{n+k}\to\boldsymbol{R}^{n+k+s}$ be an embedding such that $\nu(f(M^{n+k}))$ trivial and let $U=\{u_1, \cdots, u_s\}$ be a normal* *frame for* $f(M^{n+k})$. *Let* $c=(U, \phi^{-1}, \boldsymbol{R}^{n+k})$ *be a chart inducing the orientation of* M^{n+k} *and* $\phi(\bar{B}^{n+k}(0))=\bar{U}$. *Then the following composition of homomorphisms*

$$\Pi_{n+k}(S^n) \xrightarrow{\Pi_n^k} \mathfrak{F}^k(S^{n+k}) \xrightarrow{\phi^*} \mathfrak{F}^k(M^{n+k}) \xrightarrow{U_f^*} \mathfrak{F}^k(S^{n+k+s}) \xrightarrow{(\Pi_{n+s}^k)^{-1}} \Pi_{n+k+s}(S^{n+s})$$

and Σ^s *coincide.*

($\Sigma: \Pi_{n+k}(S^n) \longrightarrow \Pi_{n+k+1}(S^{n+1})$ *denotes the suspension homomorphism*).

PROOF. Let $r>s$ such that $r \geqq n+k+2$.

It is clear that $\Sigma^{r-s} \circ U_f^* = U_f'^*$, where $U'=\{U, e_{n+k+s+1}, \cdots, e_{n+k+r}\}$ ([P]). In order to compute $U_f'^*|_{\mathrm{Im}\phi^*}$ we can assume f to be an embedding as in Lemma 2.2. Thus there exists an open set U such that $f(\bar{U})=\bar{B}^{n+k}(0) \times \{0\}$. On the other hand since ϕ does not depend on the chart, we can assume that $\phi^{-1}=f^{-1}|_U$.

Now take $[(M^k, F)] \in \mathfrak{F}^k(S^{n+k})$,

$$\begin{aligned}(U_f'^* \circ \phi^*)([(M^k, F)]) &= U_f'^*([(\phi(M^k), \overline{T\phi}(F))]) \\ &= [(M^k, (F, e_{n+k+s+1}, \cdots, e_{n+k+r}))].\end{aligned}$$

Since $M^k \subset B^{n+k}(0)$ and U' is defined for every $x \in B^{n+k}(0)$, arguing as in Prop. 2.5 we have

$$\begin{aligned}&[(M^k, (F, U, e_{n+k+s+1}, \cdots, e_{n+k+r}))] \\ &\quad = [(M^k, (F, e_{n+k+1}, \cdots, e_{n+k+r}))] = \Sigma^r([(M^k, F)]).\end{aligned}$$

Therefore $U_f'^* \circ \phi^* = \Sigma^r$ and $U_f^* \circ \phi^* = \Sigma^s$. □

COROLLARY 2.2. *Let* M^{n+k} *be the same one as in Corollary* 2.1. *We have the following consequences*

a) U_f^* *is onto and* ϕ^* *is injective.*

b) If M^{n+k} *is k-connected then from a) and Prop.* 2.4 *we obtain that* ϕ^* *is an isomorphism and* $U_f^* = \Sigma^s \circ \phi^{*-1}$, *thus* U_f^* *is an isomorphism which depends neither on* U *nor on* f.

Let $k \in \boldsymbol{N} \cup \{0\}$. It is well known that the suspension homomorphism Σ_n: $\Pi_{n+k}(S^n) \to \Pi_{n+k+1}(S^{n+1})$ is an isomorphism provided $n \geqq k+2$. One can consider the directed set $\boldsymbol{N}$, with the usual order $\leqq$, and the sequence of groups $\{\Pi_{k+n}(S^n)\}_{n \in N}$. For $i \leqq j$ there is a homomorphism $\Sigma_{i,j}$: $\Pi_{k+i}(S^i) \to \Pi_{k+j}(S^j)$ defined by $\Sigma_{i,j} = \Sigma_{j-1} \circ \Sigma_{j-2} \circ \cdots \circ \Sigma_i$. It follows that $\Sigma_{i,i} = \mathrm{Id}$, and for every $i \leqq j \leqq l$ one has $\Sigma_{i,l} = \Sigma_{j,l} \circ \Sigma_{i,j}$. Then, $\{\Pi_{k+n}(S^n), \Sigma_{i,j}\ i, j \in \boldsymbol{N}, \leqq\}_{n \in N}$ is the direct system of groups.

Denote by (Π_k, α_i), $\alpha_i : \Pi_{k+i}(S^i) \to \Pi_k$, the direct limit of the above system. Therefore the diagram

$$\begin{tikzcd} \Pi_{k+i}(S^i) \arrow[r, "\Sigma_{i,j}"] \arrow[dr, "\alpha_i"'] & \Pi_{k+j}(S^j) \arrow[d, "\alpha_j"] \\ & \Pi_k \end{tikzcd}$$

is commutative for every $i \leqq j$.

The previous results allow us to state next theorem

THEOREM 2.1. *Let M^{n+k} be a k-connected π-manifold, $n \geqq k+2$ Let $g: M^{n+k} \to M^n$ be a map. Thus the generalized degree of g, $d(g) \in \mathfrak{F}^k(M^{n+k})$ of Def. 1.2 can be identify with the element $(\alpha_{n+s} \circ (\Pi^k_{n+s})^{-1} \circ U_f^*)(d(g)) \in \Pi_k$, where $f: M^{n+k} \to \boldsymbol{R}^{n+k+s}$ is an embedding such that $\nu(f(M^{n+k}))$ is trivial, $U=\{u_1, \cdots, u_s\}$ is a normal frame for $f(M^{n+k})$ and $\alpha_{n+s}: \Pi_{n+k+s}(S^{n+s}) \to \Pi_k$. Besides $(\alpha_{n+s} \circ (\Pi^k_{n+s})^{-1} \circ U_f^*)(d(g))=0$ if and only if $d(g)=0$ and in particular if $M^n = S^n$, $(\alpha_{n+s} \circ (\Pi^k_{n+s})^{-1} \circ U_f^*)(d(g))$ characterizes the homotopy class of g.*

PROPOSITION 2.7. *Let M^{n+k} and M'^{n+k} be k-connected manifolds contained in $\boldsymbol{R}^{n+k+s}$, $n \geqq k+2$. Let F and F' be normal frames for M^{n+k} and M'^{n+k} respectively. If (M^{n+k}, F) and (M'^{n+k}, F') are homologous through (M^{n+k+1}, G) and $g: M^{n+k+1} \to M^n$ is a map, it follows that $d(g|_{M^{n+k}}) = d(g|_{M'^{n+k}})$.*

PROOF. There is no loss of generality in assuming $g: M^{n+k+1} \to M^n$ to be a C^∞-map. Let $r \in M^n$ be a regular value for g, $g|_{M^{n+k}}$ and $g|_{M'^{n+k}}$. Thus

$$d(g|_{M^{n+k}}) \equiv (\alpha_{n+s} \circ (\Pi^k_{n+s})^{-1} \circ F_i^*)([((g|_{M^{n+k}})^{-1}(r), F_{g|_{M^{n+k}}})]),$$

$$d(g|_{M'^{n+k}}) \equiv (\alpha_{n+s} \circ (\Pi^k_{n+s})^{-1} \circ F'^*_i)([((g|_{M'^{n+k}})^{-1}(r), F_{g|_{M'^{n+k}}})]),$$

(i denotes the inclusion).

Since $(g^{-1}(r), (F_g, G))$ achieves a homology between $((g|_{M^{n+k}})^{-1}(r), (F_{g|_{M^{n+k}}}, F))$ and $((g|_{M'^{n+k}})^{-1}(r), (F_{g|_{M'^{n+k}}}, F'))$ it follows that $F_i^*([(g|_{M^{n+k}})^{-1}(r), F_{g|_{M^{n+k}}})]) = F'^*_i([(g|_{M'^{n+k}})^{-1}(r), F_{g|_{M'^{n+k}}})])$ therefore $d(g|_{M^{n+k}}) = d(g|_{M'^{n+k}})$. $\square$

Even though M^{n+k} fails to be a k-connected π-manifold we have the following proposition.

PROPOSITION 2.8 [R1]. *Let M^{n+k+1} be a compact manifold and $M^{n+k} = \partial M^{n+k+1}$. Let us suppose that $\delta: \Pi^n(\partial M^{n+k+1}) \to \Pi^{n+1}(M^{n+k+1}, \partial M^{n+k+1})$ is injective. If $f: M^{n+k} \to M^n$ is a map admiting an extension to map $\bar{f}: M^{n+k+1} \to$*

M^n *then* $d(f)=0$.

3. A result concerning the generalized Hopf's invariant.

In [P] is presented a differentiable version of the Hopf's homomorphism $\gamma: \Pi_{2k+1}(S^{k+1}) \to \boldsymbol{Z}$ and it was pointed out that for every C^∞-map $f: S^{2k+1} \to S^{k+1}$, $\gamma(f)$ depends only on the position of $f^{-1}(a_0)$ and $f^{-1}(a_1)$ in $\boldsymbol{R}^{2k+1}$ where $a_0, a_1 \in$ (r. v.)(f). Several generalizations have been described, in particuiar those established by G. W. Whitehead [W] and M. A. Kervaire [K]. Kervaire's one extends Whitehead's one therefore and we wiil refer to the generalized Hopf's invariant as that of Kervaire.

Let $f: S^{n+d+1} \to S^{n+1}$ be a C^∞-map, $d \geqq n$. Let $a_0, a_1 \in$(r. v.)(f) such that $M^d = f^{-1}(a_0)$ and $M'^d = f^{-1}(a_1)$ are contained in $S^{n+d+1} \setminus \{q\}$. Let F_0 and F_1 the normal frames associated to M^d and M'^d by f. Then one can assume that $M^d \cup M'^d \subset \boldsymbol{R}^{n+d+1}$ and the map $\varphi: M^d \times M'^d \to S^{n+d} \varphi(x, y) = (y-x/\|y-x\|)$ is well defined. On the other hand $\{F_0 \times \{0\}, \{0\} \times F_1\}$ is a normal frame for $M^d \times M'^d$, as a submanifold of $\boldsymbol{R}^{n+d+1} \times \boldsymbol{R}^{n+d+1}$. It is defined the generalized Hopf's invariant of f, $h(f)$, by

$$h(f) = ((\Pi^{d-n}_{3n+d+2})^{-1} \circ (F_0 \times \{0\}, \{0\} \times F_1)^*_{i \times i} \circ \Pi^{d-n}_{2n+d})([\varphi]),$$

where $i: M^d \hookrightarrow \boldsymbol{R}^{n+d+1}$ is the inclusion, then h defines a homomorphism $h: \Pi_{d+n+1}(S^{n+1}) \to \Pi_{2d+2n+2}(S^{3n+d+2})$.

Let $[f] \in \Pi_{d+n+1}(S^{n+1})$ and $\Pi^d_{n+1}([f]) = [(N^d, F)]$, $[(N^d, F)] \in \mathfrak{F}^d(S^{n+d+1}) \equiv \mathfrak{F}^d(\boldsymbol{R}^{n+d+1})$. There is no loss of generality in assuming N^d to be connected and F to be an orthonormal frame [P] (pags. 56 and 77).

We are going to limit ourselves to the case of N^d to be $(d-n)$-connected and $2n-2 \geqq d \geqq n \geqq 1$. Let us note that $h(f)$ can be considered the degree of φ, $d(\varphi)$. Write $F = \{v_1, \cdots, v_{n+1}\}$ and for every $\bar{c} = (c_1, \cdots, c_{n+1}) \in \boldsymbol{R}^{n+1}$, such that $\|\bar{c}\|$ is small enough we define $\bar{C}: N^d \to \boldsymbol{R}^{n+d+1}$ by $\bar{C}(x) = x + c_1 v_1(x) + \cdots + c_{n+1} v_{n+1}(x)$. From Prop. 2.7 it follows that $h(f) = d(L)$, where $L: N^d \times N^d \to S^{n+d}$ is the C^∞-map defined by $L(y, x) = (x - \bar{C}(y)/\|x - \bar{C}(y)\|)$ (see [P] for the classical case).

In order to obtain the most important theorem of this section we state

LEMMA 3.1. *Let* $[(M^k, F)] \in \mathfrak{F}^k(S^{n+k})$ *where* $F = \{v_1, \cdots, v_n\}$. *Let* σ *be an arbitrary permutation of the set* $\{1, \cdots, n\}$. *Let* $\varepsilon_i \in \{-1, 1\}$ *for each* $i \in \{1, \cdots, n\}$. *Consider the frame* $F' = \{\varepsilon_1 v_{\sigma(1)}, \cdots, \varepsilon_n v_{\sigma(n)}\}$. *It follows that* $[(M^k, F)] = \pm [(M^k, F')]$.

The proof of the Lemma 3.1 is standard and we omit it.

LEMMA 3.2 ([S], pag. 405). *Let $f: X \to Y$ be a n-equivalence* ([S] *pag.* 404). *For every CW-complex P such that $\dim P \leqq n$ (resp. $\dim P \leqq n-1$) it follows that the map $f_*: [P, X] \to [P, Y]$ is onto (resp. f_* is injective).*

LEMMA 3.3 [F. R.], pag. 438-439). *The map*

$$i: SO(n+d) \longrightarrow SO(n+d+1) \text{ defined by } i(A)=\left[\begin{array}{c:c} A & 0 \\ \hdashline 0 & 1 \end{array}\right]$$

is a $(n+d-1)$-equivalence.

Now we are in a position of proving the next result. Notice that the next theorem involves the tools developped in section 2. Theorem 3.1 generalizes a useful result of the classical theory case and improves 6.1 of [K].

THEOREM 3.1. *Let $[f] \in \Pi_{n+d+1}(S^{n+1})$. Let us suppose that $2n-2 \geqq d \geqq n \geqq 1$ and $\Pi_{n+1}^{d}([f])=[(M^d, F)]$ such that M^d is connected and $F=\{v_1, \cdots, v_{n+1}\}$ is a C^∞ orthonormal frame for M^d. Let us also assume that M^d is a $(d-n)$-connected manifold contained in $\boldsymbol{R}^{n+d}$ ($q \notin M^d$, $\boldsymbol{R}^{n+d} \subset \boldsymbol{R}^{n+d+1}$). Consider $\bar{t}=e_{n+d+1} \in \boldsymbol{R}^{n+d+1}$ and let us write for every $x \in M^d$ $e_{n+d+1}=\alpha_1(x)v_1(x)+\cdots+\alpha_{n+1}(x)v_{n+1}(x)$. Then the map $\phi: M^d \to S^n$ defined by $\phi(x)=(\alpha_1(x), \cdots, \alpha_{n+1}(x))$ satisfies that $h(f)= \pm d(\phi)$.*

PROOF. Let δ be a small enough positive number and $L: M^d \times M^d \to S^{n+d}$ be the map defined by $L(y, x)=(x-y-\delta v_{n+1}(y)/\|x-y-\delta v_{n+1}(y)\|)$.

It is sufficient to prove that

$$((F \times \{0\}, \{0\} \times F)^*_{i \times i} \circ \Pi^{d-n}_{n+d})([L])=(\Sigma^{n+d+1} \circ F^*_i \circ \Pi^{d-n}_{n})([\phi])$$
$$=((Fe_{n+d+2}, \cdots, e_{2n+2d+2})^*_i \circ \Pi^{d-n}_{n})([\phi]) \in \Pi_{2n+2d+2}(S^{3n+d+2})$$

up to sign.

One can assume $e_{n+1} \in S^n$ to be a regular value of ϕ. Let $V^{d-n}=\phi^{-1}(e_{n+1})$, submanifold of M^d, contained in the open subset $\phi^{-1}(E^n_+)(E^n_+=\{(x_1, \cdots, x_{n+1}) \in S^n: x>0\})$. Using the same arguments discussed in [P] (page 71) we have $L^{-1}(e_{n+d+1})=\Delta(V^{d-n})$, where $\Delta: M^d \to M^d \times M^d$ is the diagonal map, $\Delta(x)=(x, x)$.

Let $x \in \phi^{-1}(E^n_+)$, then $\alpha_{n+1}(x)>0$ and

$$v_{n+1}(x)=\frac{\bar{t}}{\alpha_{n+1}(x)}-\frac{\alpha_1(x)}{\alpha_{n+1}(x)}v_1(x)-\cdots\cdots-\frac{\alpha_n(x)}{\alpha_{n+1}(x)}v_n(x).$$

Consequently $\{v_1(x), \cdots, v_n(x), \bar{t}\}$ is a basis of the normal space at x to M^d in $\boldsymbol{R}^{n+d+1}$ for every $x \in \phi^{-1}(E_+^n)$. Let $p: \boldsymbol{R}^{n+d+1} \to \boldsymbol{R}^{n+d}$ be the natural projection. For each $j \in \{1, 2, \cdots, n\}$ denote $w_j = p(v_j)$. Then $\{w_1(x), \cdots, w_n(x)\}$ is a basis of the normal space at x to M^d in $\boldsymbol{R}^{n+d}$ for every $x \in \phi^{-1}(E_+^n)$.

Since M^d is $(d-n)$-connected, there exists a chart $c=(U, \varphi, \boldsymbol{R}^d)$, of M^d, such that U is a contractible open subset of M^d containing V^{d-n}. Let us denote $V = U \cap \phi^{-1}(E_+^n)$.

Define a C^∞-map $E = B^n(0) \times V \to \boldsymbol{R}^{n+d}$ by $E((\lambda_1, \cdots, \lambda_n), x) = x + \delta(\lambda_1 w_1(x) + \cdots + \lambda_n w_n(x))$. It is obvious that for a small enough δ, E is a diffeomorphism onto its image.

For each $x \in V^{d-n}$ we have that $\alpha_i(x)=0$ for every $i \in \{1, \cdots, n\}$ and $\alpha_{n+1}(x) = 1$. Then if we write ϕ and L using local coordinates, and keep the same notation, $\phi: V \to \boldsymbol{R}^n$ will be given by $\phi(x) = (\alpha_1(x), \cdots, \alpha_n(x))$ and $L: V \times V \to \boldsymbol{R}^{n+d}$ by

$$L(y, x) = x - y - \delta\Big(-\frac{\alpha_1(y)}{\alpha_{n+1}(y)} w_1(y) - \cdots\cdots - \frac{\alpha_n(y)}{\alpha_{n+1}(y)} w_n(y)\Big),$$

then L can be expressed as the following composition:

$$L: V \times V \xrightarrow{N} (B_1^n(0) \times V) \times (B_1^n(0) \times V) \xrightarrow{E \times E} \boldsymbol{R}^{n+d} \times \boldsymbol{R}^{n+d} \xrightarrow{M} \boldsymbol{R}^{n+d},$$

where

$$N(y, x) = \Big(\Big(-\frac{\alpha_1(y)}{\alpha_{n+1}(y)}, \cdots\cdots, -\frac{\alpha_n(y)}{\alpha_{n+1}(y)}, y\Big), (0, 0, \cdots, 0, x)\Big)$$

and $M(a, b) = a - b$.

Therefore for every $x \in V^{d-n}$,

$$D\phi(x) = \begin{pmatrix} \dfrac{\partial \alpha_1}{\partial x_1} & \cdots\cdots & \dfrac{\partial \alpha_1}{\partial x_d} \\ \vdots & & \vdots \\ \dfrac{\partial \alpha_n}{\partial x_1} & \cdots\cdots & \dfrac{\partial \alpha_n}{\partial x_d} \end{pmatrix}$$

and for every $(x, x) \in \Delta(V^{d-n})$

$$DL(x, x) = DL(0, x) \begin{array}{c} \left(\begin{array}{ccc:ccc} & D\phi(x) & & & 0 & \\ \hdashline -1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & 1 \end{array}\right) \\ \underbrace{\qquad\quad}_{d} \quad \underbrace{\qquad\quad}_{d} \end{array} \begin{array}{l} \}\, n \\ \\ \}\, d \end{array}, \text{ up to sign}$$

in each column (Lemma 3.1 implies that there is no problem in changing the sign all columns).

For every $x \in V^{d-n}$ we will denote by A_x

$$A_x = \left(\begin{array}{ccc:ccc} & D\phi(x) & & & 0 & \\ \hdashline -1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & -1 & & & 1 \end{array} \right).$$

Let $\{z_1, \cdots, z_n\}$ be the normal frame for V^{d-n} in $\boldsymbol{R}^d$, such that $D\phi(x)(z_j(x)) = e_j$ for every $j \in \{1, \cdots, n\}$. Then $\{\Theta_c^x(z_1(x)), \cdots, \Theta_c^x(z_n(x))\}$ is the normal frame associated to ϕ, F_ϕ, at $x \in V^{d-n}$.

For all $j \in \{1, \cdots, n\}$ and $x \in V^{d-n}$ we define $S_j(x, x) = (z_j(x), z_j(x)) \in \boldsymbol{R}^{2d}$ and $S_{n+i}(x, x) = (-e_i, e_i) \in \boldsymbol{R}^{2d}$ for each $i \in \{1, \cdots, d\}$. It is clear that $A_x(S_j(x, x)) = e_j$ for $x \in V^{d-n}$ and $j \in \{1, \cdots, n\}$ and

$$A_x(S_{n+i}(x, x)) = \left(\begin{array}{c} -\dfrac{\partial \alpha_1}{\partial x_i} \\ \vdots \\ -\dfrac{\partial \alpha_n}{\partial x_i} \\ 0 \\ 0 \\ \vdots \\ 2 \\ \vdots \\ 0 \end{array} \right) \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ (n+i)\text{-th line} \\ \\ \\ \end{array}.$$

$\Theta_{c \times c}^{(x, x)}(\{S_1(x, x), \cdots, S_n(x, x), S_{n+1}(x, x), \cdots, S_{n+d}(x, x)\})$ is a basis of the normal space at (x, x) to $\Delta(V^{d-n})$ in $M^d \times M^d$ satisfying $A_x(S_1(x, x), \cdots, S_{n+d}(x, x)) =$

$$= \left(\begin{array}{ccc:ccc} 1 & & & -\dfrac{\partial \alpha_1}{\partial x_1} & \cdots & -\dfrac{\partial \alpha_1}{\partial x_d} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & -\dfrac{\partial \alpha_n}{\partial x_1} & \cdots & -\dfrac{\partial \alpha_n}{\partial x_d} \\ \hdashline & & & 2 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & 2 \end{array} \right) = B_x =$$

$=B_x(e_1, \cdots, e_{n+d})$.

Consequently, $DL(x, x)((S_1(x, x), \cdots, S_{n+d}(x, x))B_x^{-1}DE(0, x)^{-1})=(e_1, \cdots, e_{n+d})$ and $\Theta_{c\times c}^{(x,x)}((S_1(x, x), \cdots, S_{n+d}(x, x))B_x^{-1}DE(0, x)^{-1})$ is the normal frame associated to $\Delta(V^{d-n})$ at (x, x). We will show at the end of the proof that the map ε: $V^{d-n}\to GL(\boldsymbol{R}^{n+d})$ defined by $\varepsilon(x)=B_x^{-1}DE(0, x)^{-1}$ is either homotopic to the constant identity map or to the constant map

$$I'=\left(\begin{array}{c:c} -1 & 0 \\ \hdashline 0 & \mathrm{Id} \end{array}\right).$$

Then $[(\Delta(V^{d-n}), (\Theta_{c\times c}^{\cdot}(S_1), \cdots, \Theta_{c\times c}^{\cdot}(S_{n+d}), F\times\{0\}, \{0\}\times F))]=\pm[(\Delta(V^{d-n}), ((\Theta_{c\times c}^{\cdot}(S_1), \cdots, \Theta_{c\times c}^{\cdot}(S_{n+d}))B_{\cdot}^{-1}DE(0, \cdot)^{-1}, F\times\{0\}, \{0\}\times F))]$ and therefore $(F\times\{0\}, \{0\}\times F)_{i\times i}^*(\Pi_{n+d}^{d-n}([L]))=[(\Delta(V^{d-n}), (\Theta_{c\times c}^{\cdot}(S_1), \cdots, \Theta_{c\times c}^{\cdot}(S_{n+d}), F\times\{0\}, \{0\}\times F))]$.

On the other hand, since $(E^{n+d+1}\circ F_i^*\circ\Pi_n^{d-n})([\phi])$ depends neither on the embedding nor on the frame F, one can choose the embedding Δ: $M^d\to M^d\times M^d \xrightarrow{i\times i} \boldsymbol{R}^{n+d+1}\times\boldsymbol{R}^{n+d+1}$. Let H be the normal frame obtained (using an isotopy for example) carrying $(F, e_{n+d+1}, \cdots, e_{2n+2d+2})$ to $\Delta(M^d)$.

Then one has that $(E^{n+d+1}\circ F_i^*\circ\Pi_n^{d-n})([\phi])=(H_\Delta^*\circ\Pi_n^{d-n})([\phi])=[(\Delta(V^{d-n}), (\Delta(\Theta_c^{\cdot}(z_1)), \cdots, \Delta(\Theta_c^{\cdot}(z_n)), H))]$.

Observe that $\Delta(\Theta_c^x(z_j(x)))=\Theta_{c\times c}^{(x,x)}(S_j(x, x))$ for every $x\in V^{d-n}$ and $j\in\{1, \cdots, n\}$. Now, since $(\Theta^{(x,x)}(S_{n+1}(x, x), \cdots, S_{n+d}(x, x)), F\times\{0\}, \{0\}\times F)$ and $H|_{\Delta(U)}$ are defined for every $x\in U$, and U is contractible it follows that $((F\times\{0\}, \{0\}\times F)_{i\times i}^*\circ\Pi_{n+d}^{d-n})([L])=(H_\Delta^*\circ\Pi_n^{d-n})([\phi])$ up to sign and then $h(f)=\pm d(\phi)$.

In order to complete the proof we only have to show that ε: $V^{d-n}\to GL(\boldsymbol{R}^{n+d})$ is homotopic to a constant map. It is clear that the map $x\mapsto B_x$ is homotopic to a constant map. Then it suffices to work with the map ε': $V^{d-n}\to GL(\boldsymbol{R}^{n+d})$ defined by $\varepsilon'(x)=DE(0, x)$. Let us define $\mathcal{H}$: $B^n(0)\times M^d\times(-1, 1)\to\boldsymbol{R}^{n+d+1}$ by $\mathcal{H}(\lambda_1, \cdots, \lambda_n, x, \lambda_{n+1})=x+\delta(\lambda_1 v_1(x)+\cdots+\lambda_{n+1}v_{n+1}(x))$. $\mathcal{H}$ is a diffeomorphism onto its image then $\mathcal{H}|_{B^n(0)\times U\times(-1,1)}$: $B^n(0)\times U\times(-1, 1)\to\boldsymbol{R}^{n+d+1}$ so is. One can assume that $|D\mathcal{H}(y)|>0$ for every $y\in B^n(0)\times U\times(-1, 1)$.

Consider $x\in U$.

$$D\mathscr{H}(0,\ x,\ 0)=\left(\begin{array}{ccc|ccc|c}
\delta v_1^1(x) & \cdots & \delta v_n^1(x) & \cdots & & \cdots & \delta v_{n+1}^1(x) \\
\vdots & & \vdots & & Q_x & & \vdots \\
\delta v_1^{n+d+1}(x) & \cdots & \delta v_n^{n+d+1}(x) & 0 & \cdots & 0 & \delta v_{n+1}^{n+d+1}(x)
\end{array}\right)$$

where $v_i^j(x)$ denotes the j-th coordinate of $v_i(x)$.

Since U is contractible, the following composition $V^{d-n}\xrightarrow{i}U\xrightarrow{D\mathscr{H}(0,\cdot,0)}GL(\boldsymbol{R}^{n+d+1})$ is homotopic to the constant identity map. Now for each $x\in V^{d-n}$ one has that

$$D\mathscr{H}(0,\ x,\ 0)=\left(\begin{array}{ccc|ccc|c}
\delta w_1^1(x) & \cdots & \delta w_n^1(x) & \cdots & & \cdots & 0 \\
\vdots & & \vdots & & Q_x & & \vdots \\
\delta w_1^{n+d}(x) & \cdots & \delta w_n^{n+d}(x) & \cdots & & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \delta
\end{array}\right)$$

because $v_j^{n+d+1}(x)=(v_j(x),\ \bar{t})=(v_j(x),\ v_{n+1}(x))=0$.

Then $D\mathscr{H}(0,\ x,\ 0)=\left[\begin{array}{c|c} DE(0,\ x) & 0 \\ \hline 0 & \delta \end{array}\right]$, applying a simple homotopy one can suppose that $\delta=1$. Therefore one has the following commutative diagram

$$\begin{tikzcd}
 & GL_+(\boldsymbol{R}^{n+d+1}) \arrow[r, "r"] & SO(n+d+1) \\
V^{d-n} \arrow[ur, "D\mathscr{H}(0,\ \cdot,\ 0)"] \arrow[dr, "DE(0,\ \cdot)"'] & & \\
 & GL_+(\boldsymbol{R}^{n+d}) \arrow[r, "r"] & SO(n+d) \arrow[uu, "i"']
\end{tikzcd}$$

where r is the canonical homotopy equivalence and i: $SO(n+d)\rightarrow SO(n+d+1)$ has been given in Lemma 3.2.

The map η : $V^{d-n}\rightarrow SO(n+d+1)$ defined by $\eta(x)=r(D\mathscr{H}(0,\ x,\ 0))$ is homotopic to a constant map. Then it is enough to prove that

$$\begin{aligned}\eta'\colon V^{d-n} &\longrightarrow SO(n+d)\\ x &\longmapsto r(DE(0,\ x))\end{aligned}$$

also is nulhomotopic.

Lemma 3.2 asserts that i is a $(n+d-1)$-equivalence and using Lemma 3.3 one has that i_* : $[V^{d-n},\ SO(n+d)]\rightarrow[V^{d-n},\ SO(n+d+1)]$ is injective if $d-n\leqq(n+d+1)-1$. Since $d-n\leqq n+d-2$ iff $2n-2\geqq 0$ it follows that i_* is injective.

Since $i_*([\eta'])=[i\circ\eta']=[\eta]=0$ then $[\eta']=0$ and the proof is complete. □

Let $\Sigma: \Pi_{n+d}(S^n)\to\Pi_{n+d+1}(S^{n+1})$ be the suspension homomorphism. It is well known that $h(f)=0$ for every $[f]\in\mathrm{Im}\Sigma$ ([W] pag 192). However the converse is much more difficult to solve. One can use the above theorem to obtain the next corollary.

COROLLARY 3.1. *Let* $[f]\in\Pi_{n+d+1}(S^{n+1})$. *Assume that* $2n-2\geqq d\geqq n\geqq 1$ *and* $\Pi^d_{n+1}([f])=[(M^d, F)]$, *where* M^d *is a* $(d-n)$*-connected manifold contained in* $\boldsymbol{R}^{n+d}$ *and* $F=\{v_1, \cdots, v_{n+1}\}$ *is an orthonormal frame for* M^d. *Then if* $h(f)=0$ *one has that* $[f]\in\mathrm{Im}\Sigma$.

PROOF. Since $d(\phi)=h(f)=0$, it follows that ϕ is homotopic to a constant map. Now [P] implies that $[f]\in\mathrm{Im}\Sigma$. □

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References

[F.R.] D. B. Fuks and V. A. Rokhlin, Beginner's course in topology, Springer series in Soviet mathematics, Springer Verlag, 1984.

[G.M.V.] K. Geba, I. Massabo and A. Vignoli, Generalized degree and bifurcation, Nonl. Funct. Anal. and its Appl. (Edited by S. P. Singh), Nato Asi Series, Reidel Publishing, 1986, 55-73.

[H] M. W. Hirsch, Differential Topology, Springer Verlag, 1976.

[K] M. A. Kervaire, An Interpretation of G. Whitehead's generalization of Hopf's invariant, Annals of Math. 1959, 69, n°-2, 345-365.

[P] L. S. Pontryagin, Smooth manifold and their applications in homotopy Theory, Amer. Math. Soc. Trans., 1959, 11, 1-114.

[R1] F. R. Ruiz del Portal, Teoría del grado topológico generalizado y aplicaciones, Publicaciones de la U. C. M. 1992, (thesis).

[R2] F. R. Ruiz del Portal, On the additivity property of the generalized degree, Math. Japonica 37, No. 4 (1992), 657-664.

[R3] F. R. Ruiz del Portal, Generalized degree in normed spaces, Publ. Matemàtiques, 36 (1992), 157-166.

[R4] F. R. Ruiz del Portal, Generalized complementing maps, Preprint.

[S] E. Spanier, Algebraic Topology, McGraw Hill 1966.

[W] G. W. Whitehead, A generalization of the Hopf invariant, Annals of Math. 1950, n°-1, 192-237.

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