TSUKUBA J. MATH.
Vol. 17 No. 2 (1993), 311–322

ON RULED REAL HYPERSURFACES IN A COMPLEX SPACE FORM

Dedicated to Professor Hisao NAKAGAWA on his sixtieth birthday

By

Seong-Soo AHN, Sung-Baik LEE* and Young-Jin SUH†

§0. Introduction.

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form consists of a complex projective space P_nC, a complex Euclidean space C^n or a complex hyperbolic space H_nC, according as $c>0$, $c=0$ or $c<0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g).

In his study of real hypersurfaces of a complex projective space P_nC, Takagi [10] classified all homogeneous real hypersurfaces and Cecil-Ryan [2] showed also that they are realized as the tubes of constant radius over Kaehlerian submanifolds if the structure vector field ξ is principal. On the other hand, real hypersurfaces of a complex hyperbolic space H_nC also investigated by Berndt [1], Montiel [7], Montiel and Romero [8] and so on. Berndt [1] classified all homogeneous real hypersurfaces of H_nC and showed that they are realized as the tubes of constant radius over certain submanifolds. According to Takagi's classification theorem and Berndt's one the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are all determined.

In particular, Maeda [6] and Okumura [9] (resp. Montiel [7] and Montiel-Romero [8]) considered real hypersurfaces of $M_n(c)$, $c>0$ (resp. $c<0$) whose second fundamental tensor A of M in $M_n(c)$, $c\neq 0$, satisfies

$$(0.1)\qquad (\nabla_X A)Y=\frac{c}{4}\{\eta(X)\phi Y-g(\phi X, Y)\xi\},$$

$$(0.2)\qquad A\phi-\phi A=0.$$

* The present studies were supported by the basic science research institute program, Korea ministry of Education, 1991-129.
† Partially supported by TGRC-KOSEF.
Received May 18, 1992, Revised July 9, 1992.

In fact, real hypersurfaces of type A are characterized by these properties. Namely, they proved the following.

THEOREM A. *Let M be a real hypersurface of P_nC, $n\geqq 3$. If it satisfies* (0.1) *or* (0.2), *then M is locally a tube of radius r over one of the following Kaehlerian submanifolds*:

(A_1) *a hyperplane $P_{n-1}C$, where $0<r<\pi/2$,*

(A_2) *a totally geodesic P_kC $(1\leqq k\leqq n-2)$, where $0<r<\pi/2$.*

THEOREM B. *Let M be a real hypersurface of H_nC, $n\geqq 3$. If it satisfies* (0.1) *or* (0.2), *then M is locally congruent to one of the following hypersurfaces*:

(A_0) *a horosphere in H_nC, i.e., a Montiel tube,*

(A_1) *a tube of a totally geodesic hyperplane H_kC $(k=0$ or $n-1)$,*

(A_2) *a tube of a totally geodesic H_kC $(1\leqq k\leqq n-2)$.*

Now let us define a distribution $T_0(x)=\{X\in T_xM : X\perp\xi_{(x)}\}$ of a real hypersurface M of $M_n(c)$, $c\neq 0$, which is holomorphic with respect to the structure tensor ϕ. If we restrict (0.1) and (0.2) to the distribution T_0, then the shape operator A of M satisfies $g((\nabla_XA)Y, Z)=0$ and $g((A\phi-\phi A)X, Y)=0$ for any vector fields X, Y and Z in T_0. Thus in this paper let us consider the following two conditions

$$g((\nabla_XA)Y, Z)=0 \tag{0.3}$$

and

$$g((A\phi-\phi A)X, Y)=0 \tag{0.4}$$

for any vector fields X, Y and Z in T_0.

About a ruled real hypersurface of P_nC some properties are investigated by Kimura [4], Kimura and Maeda [5]. Contrary to homogeneous real hypersurfaces of P_nC, it is known that any ruled real hypersurface of P_nC is not complete and its structure vector field ξ is not principal. In this paper we can assert that this fact is extended for any ruled real hypersuface of H_nC.

Also in [5] they gave a characterization for ruled real hypersurfaces of P_nC with an integrability condition of T_0 in addition to the condition (0.3). Replacing this integrability condition by the property (0.4) we give another characterization of ruled real hypersurfaces of $M_n(c)$, $c\neq 0$.

THEOREM C. *Let M be a connected real hypersurface of $M_n(c)$, $c\neq 0$ and $n\geqq 3$. If it satisfies* (0.3) *and* (0.4) *and the structure vector field ξ is not prin-*

cipal, then M is locally congruent to a ruled real hypersurface.

§ 1. Preliminaries.

We begin with recalling fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $(M_n(c), \bar{g})$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field defined on a neighborhood of a point x in M. We denote by J the almost complex structure of $M_n(c)$.

For a local vector field X on the neighbourhood of x in M, the images of X and C under the linear transformation J can be represented as

$$JX=\phi X+\eta(X)C, \qquad JC=-\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighbourhood of x in M, respectively. Then it is seen that $g(\xi, X)=\eta(X)$, where g denotes the Riemannian metric on M induced from the metric $\bar{g}$ on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called *an almost contact metric structure* M. They satisfy the following

$$\phi^2=-I+\eta\otimes\xi, \qquad \phi\xi=0, \qquad \eta(\phi X)=0, \qquad \eta(\xi)=1$$

for any vector field X, where I denotes the identity transformation. Furthermore the covariant derivatives of the structure tensors are given by

$$\nabla_X\xi=\phi AX, \qquad (\nabla_X\phi)Y=\eta(Y)AX-g(AX, Y)\xi \tag{1.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection of g and A denotes the shape operator in the direction of C on M.

Since the ambient space is of constant holomorphic sectional curvature c the equations of Gauss and Codazzi are respectively obtained:

$$\begin{aligned} R(X, Y)Z=\frac{c}{4}\{&g(Y, Z)X-g(X, Z)Y \\ &+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y-2g(\phi X, Y)\phi Z\} \\ &+g(AY, Z)AX-g(AX, Z)AY, \end{aligned} \tag{1.2}$$

$$(\nabla_X A)Y-(\nabla_Y A)X=\frac{c}{4}\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi\}, \tag{1.3}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

§2. The proof of Theorem C.

In this section we shall prove Theorem C. A characterization of a real hypersurface of type A of a complex space form $M_n(c)$, $c\neq 0$ is first recalled. It is seen by a theorem due to Okumura [9] and a theorem due to Montiel and Romero [8] that if the shape operator A and the structure vector ϕ commute each other, then a real hypersurface M is locally congruent to the real hypersurface of type A. Restricting this condition to the orthogonal distribution T_0 of ξ, we find the following.

LEMMA 2.1. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. If M satisfies the condition* (0.4), *then we have*

$$g((\nabla_X A)Y, Z)=\mathcal{S}g(AX, Y)g(Z, V), \tag{2.1}$$

where $\mathcal{S}$ denotes the cyclic sum with respect to X, Y and Z orthogonal to ξ and V stands for the vector field $\nabla_\xi \xi$.

PROOF. Differentiating the condition (0.4) covariantly in the direction of X, we get

$$g((\nabla_X A)\phi Y+A(\nabla_X\phi)Y+A\phi\nabla_X Y-(\nabla_X\phi)AY-\phi(\nabla_X A)Y-\phi A\nabla_X Y, Z)$$
$$+g((A\phi-\phi A)Y, \nabla_X Z)=0$$

for any vector fields X, Y and Z orthogonal to ξ. By taking account of (1.1) and by using the fact that $g(\nabla_X Y, \xi)=-g(Y, \phi AX)$, the above equation is reformed as

$$\begin{aligned} g((\nabla_X A)Y, \phi Z)+g((\nabla_X A)Z, \phi Y)&=\eta(AY)g(X, AZ)\\ &+\eta(AZ)g(Y, AX)+g(X, A\phi Y)g(Z, V)+g(X, A\phi Z)g(Y, V). \end{aligned} \tag{2.2}$$

Now let us denote by $f(X, Y, Z)$ the following equation

$$\begin{aligned} f(X, Y, Z)=&g((\nabla_X A)Y, \phi Z)+g((\nabla_X A)Z, \phi Y)\\ &-\eta(AY)g(X, AZ)-\eta(AZ)g(Y, AX)-g(X, A\phi Y)g(Z, V)\\ &-g(X, A\phi Z)g(Y, V). \end{aligned}$$

Then by (2.2) we have $f(X, Y, Z)=0$. Thus $f(X, Y, Z)+f(Y, Z, X)-f(Z, X, Y)=0$ implies

$$\begin{aligned} 2g((\nabla_X A)Y, \phi Z)=&2\eta(AZ)g(AX, Y)+g(X, V)\{g(Y, A\phi Z)-g(Z, A\phi Y)\}\\ &+g(Y, V)\{g(X, A\phi Z)-g(Z, A\phi X)\}, \end{aligned}$$

from which together with the condition (0.4) we get

$$g((\nabla_X A)Y,\ \phi Z)=\eta(AZ)g(AX,\ Y)+g(Y,\ V)g(X,\ A\phi Z)+g(X,\ V)g(Y,\ A\phi Z).$$

From this, replacing Z by ϕZ, we can get the above equation (2.1).

REMARK 2.1. Let us denote by S^{2n+1} (resp. H_1^{2n+1}) a $(2n+1)$-dimensional sphere (resp. anti-De Sitter space). Given a real hypersurface of $M_n(c)$, one can construct a (resp. Lorentzian) hypersurface N of S^{2n+1}(resp. H_1^{2n+1}) which is a principal S^1-bundle over M with (resp. time-like) totally geodesic fibers and the projection π: $N \to M$ in such a way that the diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & S^{2n+1}\ (H_1^{2n+1}) \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{i'} & P_nC\ (H_nC) \end{array}$$

is commutative (i, i' being the isometric immersions). Then it is seen (cf Yano and Kon [11]) that the second fundamental tensor A' of N is parallel if and only if the second fundamental tensor A of M satisfies the conditions (0.1) or (0.2).

Now, a ruled real hypersurface M of $M_n(c)$, $c \neq 0$, can be defined as follows: Let γ: $I \to M_n(c)$ be any regular curve. Then for any $t(\in I)$ let $M_{n-1}^{(t)}(c)$ be a totally geodesic complex hypersurface of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M=\{x \in M_{n-1}^{(t)}(c) : t \in I\}$. Then, by the construction, M becomes a real hypersurface of $M_n(c)$, which is called a ruled real hypersurface. This means that there are many ruled real hypersurfaces of $M_n(c)$. Let T_0 be a distribution defined by $T_0(x)=\{u \in T_xM : u \perp \xi(x)\}$ in the tangent space T_xM of M at any point x in M. Then it is seen in [4] that the shape operator A of a ruled real hypersurface M of P_nC satisfies

$$A\xi=\alpha\xi+\beta U\ (\beta \neq 0), \qquad AU=\beta\xi, \qquad AX=0$$

for any vector X orthogonal to ξ and U, where U is a unit vector orthogonal to ξ, and α and β are smooth functions on M. The second fundamental form is said to be η-parallel if the shape operator A satisfies $g((\nabla_X A)Y, Z)=0$ for any vector fields X, Y and Z. It is seen in [5] that the second fundamental form is η-parallel by (1.1) and (2.3). Thus the properties (0.1) and (0.2) hold along the distribution T_0 for a ruled real hypersurface of P_nC. Namely, it satisfies the conditions (0.3) and (0.4). By the similar argument to that in P_nC we can assert that a ruled real hypersurface of H_nC satisfies the conditions (0.3) and (0.4).

PROOF OF THEOREM C. Suppose that the shape operator A satisfies the conditions (0.3) and (0.4). Let us also suppose that the structure vector field ξ satisfies

$$A\xi=\alpha\xi+\beta U, \tag{2.4}$$

where U is a unit vector in the distribution T_0. Now let us denote V the vector field $\nabla_\xi \xi$. Then, from this definition together with (1.1) it follows

$$V=\beta\phi U. \tag{2.5}$$

Lemma 2.1 means that

$$\mathcal{S} g(AX, Y)g(Z, V)=0 \tag{2.6}$$

for any vector fields X, Y and Z in T_0, because the second fundamental form is η-parallel. When we put $Z=V$ in (2.6), it reduces to

$$g(AX, Y)g(V, V)+g(AY, V)g(X, V)+g(AV, X)g(Y, V)=0. \tag{2.7}$$

Furthermore, we put $Y=V$ in (2.7) and then $X=V$ in the obtained equation. Then the following equations

$$2g(AX, V)g(V, V)+g(AV, V)g(X, V)=0, \tag{2.8}$$

$$g(AV, V)g(V, V)=0 \tag{2.9}$$

are obtained.

Let M_0 be a set consisting of points x in M such that $V(x)\neq 0$. Since we assume that ξ is not principal, M_0 is not empty. On the subset M_0, from (2.8) and (2.9) we see $g(AV, V)=0$ and $g(AX, V)=0$ for any vector field on M_0 orthogonal to ξ, which implies $AV=g(AV, \xi)\xi$. Substituting (2.4) into the above equation, we see $AV=0$. Accordingly, (2.7) means that

$$g(AX, Y)=0$$

for any vector fields X and Y belonging to T_0. So, it follows from this and (2.4) we get $AX=g(AX, \xi)\xi=\beta g(X, U)\xi$ for any $X\in T_0$, which means that

$$AX=0, \qquad AU=\beta\xi \tag{2.10}$$

for any $X\in T_0$ orthogonal to U. Consequently we obtain

$$g(A\phi X, Y)=0, \qquad g(\phi AX, Y)=0$$

for any X and $Y\in T_0$. Accordingly we get $g(\nabla_X Y, \xi)=-g(\phi AX, Y)=0$ by (1.1), which means that $\nabla_X Y=\nabla_Y X$ is also contained in T_0, namely the distribution T_0 is integrable on the open set M_0.

Now we want to show that the open set M_0 coincides with the whole M. Thus let us suppose that the interior of $M-M_0$ is not empty. On the subset the vector field V vanishes identically and therefore ξ is principal. Thus we have

$$(2.11) \qquad (A\phi-\phi A)\xi=0.$$

This fact implies that T_0 is invariant by $A\phi-\phi A$. From this together with the condition (0.4) it follows $(A\phi-\phi A)X=0$ for any vector field X in T_0. From this and (2.11) we have

$$(2.12) \qquad A\phi-\phi A=0$$

on the interior of $M-M_0$. In the sequel, since the structure vector field ξ is principal with corresponding principal curvature α, it is seen in [3] and [7] that α is constant on the interior of $M-M_0$, because this is a local property. So it satisfies

$$(2.13) \qquad A\phi A=\frac{c}{2}\phi+\alpha(A\phi+\phi A).$$

Thus, if X is a principal vector field with corresponding principal curvature λ, then we have

$$(2.14) \qquad (2\lambda-\alpha)A\phi X=\left(\frac{c}{2}+\alpha\lambda\right)\phi X.$$

From (2.12) and (2.14) we get

$$2\lambda^2-2\alpha\lambda-\frac{c}{2}=0$$

and hence, from which it follows that all principal curvatures are constant on the interior of $M-M_0$.

From the assumption we know that the set M_0 is not empty. Thus by means of the continuity of principal curvatures, the interior of $M-M_0$ must be empty and therefore, by the continuity of principal curvatures again we see that M_0 concides with the whole M. Accordingly the distribution T_0 is integrable on M. Moreover the integral manifold of T_0 is totally geodesic in $M_n(c)$, because of $\bar{g}(D_XY, \xi)=g(\nabla_XY, \xi)=0$ and $\bar{g}(D_XC, Y)=-g(AX, Y)=0$ for any vector fields X and Y in T_0 by (1.1) and (2.10), where D denotes the Riemannian connection of $M_n(c)$. Since T_0 is J-invariant, its integral manifold is a complex manifold and therefore it is a complex space form $M_{n-1}(c)$. Thus M is locally congruent to a ruled real hypersurface.

Conversely, suppose that M is a ruled real hypersurface of $M_n(c)$. Then it satisfies (2.4) and (2.10). So we have

$$g(A\phi X, Y)=0, \qquad g(\phi AX, Y)=0$$

for any vector fields X and Y in T_0. Also it is seen by [5] that the second fundamental form of M in P_nC is η-parallel. In the complex hyperbolic space H_nC the property is derived from the same discussion. Thus they are equivalent to the conditions (0.3) and (0.4).

REMARK 2.2. It is proved by Kimura and Maeda [5] that for a real hypersurface M of P_nC if the distribution T_0 is integrable and if A is η-parallel, then M is locally congruent to a ruled real hypersurface.

§3. An example of minimal ruled real hypersurfaces of H_nC.

This section is concerned with an example of minimal ruled real hypersurfaces of H_nC. First of all, we recall about the fibration

$$\pi: H_1^{2n+1} \longrightarrow H_nC .$$

In a complex Euclidean space C^{n+1} with the standard basis, let F be a Hermitian form defined by

$$F(z, w)=-z_0\bar{w}_0+\sum_{k=1}^{n} z_k\bar{w}_k ,$$

where $z=(z_0, \cdots, z_n)$ and $w=(w_0, \cdots, w_n)$ are in C^{n+1}. Then (C^{n+1}, F) is a complex Minkowski space, which is simply denoted by C_1^{n+1}. The scalar product given by $\Re F(z, w)$ is an indefinite metric of index 2 in C_1^{n+1}, where $\Re z$ denotes the real part of the complex number z. Let H_1^{2n+1} be a real Lorentzian hypersurface of C_1^{n+1} defined by

$$H_1^{2n+1}=\{z\in C_1^{n+1}: F(z, z)=-1\} ,$$

and let G be a Lorentzian metric of H_1^{2n+1} induced from the Lorentzian metric $\Re F$. Then (H_1^{2n+1}, G) is the Lorentzian manifold of constant sectional curvature -1, which is called an anti-De Sitter space. For the anti-De Sitter space H_1^{2n+1} the tangent space $T_z(H_1^{2n+1})$ at each point z can be identified (through the parallel displacement in C_1^{n+1}) with $\{w\in C_1^{n+1} | \Re F(z, w)=0\}$. Let us denote by T_z' the orthogonal complement of the vector iz in $T_zH_1^{2n+1}$, that is,

$$T_z'=\{w\in C_1^{n+1}: \Re F(z, w)=0, \Re F(iz, w)=0\} .$$

Let S^1 be the multiplicative group of complex numbers of absolute value 1. Then H_1^{2n+1} can be regarded as a principal fiber bundle over a complex hyperbolic space H_nC with the group S^1 and the projection π. Furthermore, there is a connection such that T_z' is the horizontal subspace at z which is invariant under the S^1-action. The natural projection π of H_1^{2n+1} onto H_nC induces a linear isomorphism of T_z' onto $T_p(H_nC)$, where $p=\pi(z)$. The metric g of constant holomorphic sectional curvature -4 is given by $g_p(X, Y)=\Re F_z(X^*, Y^*)$ for tangent vectors X and Y in $T_p(H_nC)$, where z is any point in the fiber $\pi^{-1}(p)$ and, X^* and Y^* are vectors in T_z' such that $d\pi(X^*)=X$ and $d\pi(Y^*)=Y$, where $d\pi$ denotes the differential of the projection π.

On the other hand, a complex structure $\bar{J}$: $w\rightarrow iw$ in the subspace T_z' is compatible with the action of S^1 and induces an almost complex structure J on H_nC such that $d\pi\circ\bar{J}=J\circ d\pi$. Thus H_nC is a complex hyperbolic space of constant holomorphic sectional curvature -4 and it is seen that the principal S^1-bundle H_1^{2n+1} over H_nC with the projection π is a semi-Riemannian submersion with the fundamental tensor J and totally geodesic time-like fibers.

Now, let us denote by N a real Lorentzian hypersurface of H_1^{2n+1} given by

$$N=\{z=(re^{i\phi}\cosh\theta,\ re^{i\phi}\sinh\theta,\ (r^2-1)^{1/2}z_2, \cdots, (r^2-1)^{1/2}z_n)$$
$$\in C_1^{n+1}=C_1^2\times C^{n-1}:\ \textstyle\sum_{j=2}^{n}|z_j|^2=1,\ r>1,\ 0\leqq\phi<2\pi,\ \theta\in R\}\,.$$

Then $(r, \phi, \theta, z_2, \cdots, z_n)$ can be regarded as the coordinate system for N and tangent vectors at z for coordinates curves for coordinates r, ϕ and θ are given by

$$\left(\frac{\partial}{\partial r}\right)_z=(e^{i\phi}\cosh\theta,\ e^{i\phi}\sinh\theta,\ r(r^2-1)^{-1/2}z_2, \cdots, r(r^2-1)^{-1/2}z_n),$$
$$(3.1)\qquad \left(\frac{\partial}{\partial\phi}\right)_z=(ire^{i\phi}\cosh\theta,\ ire^{i\phi}\sinh\theta,\ 0, \cdots, 0),$$
$$\left(\frac{\partial}{\partial\theta}\right)_z=(re^{i\phi}\sinh\theta,\ re^{i\phi}\cosh\theta,\ 0, \cdots, 0),$$

where the vector $(\partial/\partial\phi)_z$ is time-like. We put at z.

$$(3.2)\qquad u=(0, 0, i(r^2-1)^{1/2}z_2, \cdots, i(r^2-1)^{1/2}z_n).$$

Let T_z'' be the subspace consisting of vectors $w=(0, 0, w_2, \cdots, w_n)$ orthogonal to vectors z and u through the parallel transformation in C_1^{n+1}. These vectors span the tangent space T_zN at the point z. A unit space-like normal vector $\bar{C}_z$ at z is given by

$$\bar{C}_z=(ie^{i\phi}\sinh\theta,\ ie^{i\phi}\cosh\theta,\ 0, \cdots, 0).$$

Let $\bar{\nabla}$ and $\bar{D}$ be the Levi-Civita connection of H_1^{2n+1} and C_1^{n+1}, respectively. Then, by (3.1) and the definition of $\bar{C}_z$ and by the choice of vectors u and w, it is easily seen that we get

$$\bar{\nabla}_{(\partial/\partial r)_z}\bar{C}=\bar{D}_{(\partial/\partial r)_z}=0,$$

$$\bar{\nabla}_{(\partial/\partial\phi)_z}\bar{C}=\bar{D}_{(\partial/\partial\phi)_z}\bar{C}=(-e^{i\phi}\sinh\theta,\ -e^{i\phi}\cosh\theta,\ \cdots,\ 0)$$
$$=-\frac{1}{r}\Big(\frac{\partial}{\partial\theta}\Big)_z,$$

(3.3) $$\bar{\nabla}_{(\partial/\partial\theta)_z}\bar{C}=\bar{D}_{(\partial/\partial\theta)_z}C=(ie^{i\phi}\cosh\theta,\ ie^{i\phi}\sinh\theta,\ 0,\ \cdots,\ 0)$$
$$=\frac{1}{r}\Big(\frac{\partial}{\partial\phi}\Big)_z,$$

$$\bar{\nabla}_u\bar{C}=\bar{D}_u\bar{C}=0,$$

$$\bar{\nabla}_w\bar{C}=\bar{D}_w\bar{C}=0,$$

for any vector w in T''_z. We denote by $\bar{A}$ the shape operator of N in H_1^{2n+1}. Then from (3.3), the Weingarten equation implies that

(3.4) $$\bar{A}\Big(\frac{\partial}{\partial r}\Big)_z=0,\quad \bar{A}\Big(\frac{\partial}{\partial\phi}\Big)_z=\frac{1}{r}\Big(\frac{\partial}{\partial\theta}\Big)_z,\quad \bar{A}\Big(\frac{\partial}{\partial\theta}\Big)_z=-\frac{1}{r}\Big(\frac{\partial}{\partial\phi}\Big)_z,$$
$$\bar{A}u=0,\qquad \bar{A}w=0.$$

for any vector w in T''_z.

On the other hand, for the vertical vector iz with respect to the submersion π at z we have $iz=(\partial/\partial\phi)_z+u$. We put

$$\bar{U}_z=(r^2-1)^{1/2}r^{-1}\Big(\frac{\partial}{\partial\phi}\Big)_z+r(r^2-1)^{-(1/2)}u\,.$$

Then $\bar{U}_z$ is a unit space-like horizontal vector in T_zN and again by (3.1) and (3.2) we get

$$J\bar{U}_z=i\bar{U}_z=-(r^2-1)^{1/2}\Big(\frac{\partial}{\partial r}\Big)_z.$$

Moreover, if we put $\bar{\xi}_z=-i\bar{C}_z$, then we get $\bar{\xi}_z=(1/r)(\partial/\partial\theta)_z$.

Given for the Lorentzian hypersurface N of a $(2n+1)$-dimensional anti-De Sitter space H_1^{2n+1} in C_1^{n+1}, a real hypersurface M of a complex hyperbolic space H_nC is given as follows: N is a principal S^1-bundle over M with time-like totally geodesic fibers and the projection $\pi: N\to M$. Since N is S^1-invariant, $C_{\pi(z)}=d\pi(\bar{C}_z)$ provides a unit vector normal to M. The tangent space T_xM of M at $x=\pi(z)$ is spanned by the vectors $\xi_x=d\pi(\bar{\xi}_z)$, $U_x=d\pi(\bar{U}_z)$, $\phi U_x=JU_x=d\pi(J\bar{U}_z)$ and $X_x=d\pi(\bar{X}_z)$ where any vector $\bar{X}_z\in T''_z$. In particular, ξ is the structure vector field on M. It is seen by Montiel and Romero [8] that the shape operator A of M satisfies $AX=d\pi(\bar{A}\bar{X})$, where $\bar{X}$ is the horizontal lift of the vector field X on M.

Now, because of $u=-(\partial/\partial\phi)_z+iz$, we get

$$\bar{U}_z = -r^{-1}(r^2-1)^{-1/2}\left(\frac{\partial}{\partial\phi}\right)_z + r(r^2-1)^{-1/2} iz$$

by means of the definition of u and $\bar{U}_z$. Accordingly, by (3.4) and the above equation we have

$$A\xi_x = d\pi\left(\bar{A}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)_z\right) = -\frac{1}{r^2}\, d\pi\left(\left(\frac{\partial}{\partial\phi}\right)_z\right) = \frac{1}{r^2}\, d\pi(r(r^2-1)^{1/2}\bar{U}_z)$$

and therefore we get

$$A\xi_x = (r^2-1)^{1/2} r^{-1} U_x\,, \tag{3.5}$$

because iz is vertical. By the similar calculation to that developed as above we have also the following relations:

$$\begin{aligned} \bar{A}\bar{U}_z &= \bar{A}((r^2-1)^{1/2}r^{-1}\left(\frac{\partial}{\partial\phi}\right)_z + r(r^2-1)^{-1/2}u) \\ &= (r^2-1)^{1/2}r^{-2}\left(\frac{\partial}{\partial\theta}\right)_z = (r^2-1)^{1/2}r^{-1}\bar{\xi}_z\,, \\ \bar{A}(J\bar{U}_z) &= \bar{A}(i\bar{U}_z) = -(r^2-1)^{1/2}\bar{A}\left(\frac{\partial}{\partial r}\right)_z = 0\,, \\ \bar{A}\bar{X}_z &= 0 \end{aligned} \tag{3.6}$$

for any vector $\bar{X}_z$ at z orthogonal to $\bar{\xi}_z$, $\bar{U}_z$ and $J\bar{U}_z$. Thus, it follows from (3.5) and (3.6) that

$$A\xi = (r^2-1)^{1/2}r^{-1}U, \qquad AU = (r^2-1)^{1/2}r^{-1}\xi, \qquad AX = 0 \tag{3.7}$$

for any vector field X orthogonal to the structure vector field ξ and U. By the similar discussion to that of the proof of the theorem the equations (3.7) mean that the real hypersurface M is minimal and the distribution T_0 is defined by $\{X(x)\in T_xM\colon X\perp\xi\}$ is integrable. Moreover the integral manifold is totally geodesic in H_nC. Since T_0 is J-invariant, its integral manifold is a complex hypersurface $H_{n-1}C$ and M is the ruled real hypersurface. However it is not complete. In fact, using (1.3) and (3.7), we have $(\nabla_\xi A)\phi U - (\nabla_{\phi U}A)\xi = U$ and

$$(\nabla_\xi A)\phi U - (\nabla_{\phi U}A)\xi = \beta A\xi - A\phi\nabla_\xi U - (\phi U\beta)U - \beta\nabla_{\phi U}U\,,$$

where $\beta = (r^2-1)^{1/2}/r$. Hence we have $-U + \beta A\xi - A\phi\nabla_\xi U - (\phi U\beta)U - \beta\nabla_{\phi U}U = 0$. This equation yields that $\nabla_{\phi U}U = 0$, because it is orthogonal to ξ and U. Thus we get

$$\phi U\beta = \beta^2 - 1\,,$$

which tells us that M is not complete.

References

[1] J. Berndt, Real hypersurfaces with constant principal curvatures in a complex hyperbolic space, J. Reine angew. Math. **395** (1989), 132-141.

[2] T.E. Cecil and P.J. Ryan, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481-499.

[3] U-H. Ki and Y.J. Suh, On real hypersurfaces of a complex space form, Math. J. Okayama **32** (1990), 207-221.

[4] M. Kimura, Sectional curvatures of a holomorphic plane in $\boldsymbol{P}_n(C)$, Math. Ann. **276** (1987), 487-497.

[5] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299-311.

[6] Y. Maeda, On real hypersurfrces of a complex projective space, J. Math. Soc. Japan **28** (1976), 529-540.

[7] S. Montiel, Real hypersurfaces of a complex hyperbolic space, J. Math. Soc. Japan **37** (1985), 515-535.

[8] S. Montiel and A. Romero, On some real hypersurfaces of a complex hyperbolic space, Geometriae Dedicata **20** (1986), 245-261.

[9] M. Okumura, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. **212** (1975), 355-364.

[10] R. Takagi, On homogeneous real hypersurfaces of a complex projective space, Osaka J. Math. **10** (1973), 495-506.

[11] K. Yano and M. Kon, "CR-submanifolds of Kaehlerian and Sasakian manifolds", Birkhäuser, Boston Basel Sturtgart, 1983.

Department of Mathematics
Chosun University
Kwangju, 501-759,
Korea

Department of Mathematics
Andong University
Andong, Kyung-Pook,
760-749, Korea