

PARTIAL COXETER FUNCTORS AND STABLE EQUIVALENCES FOR SELF-INJECTIVE ALGEBRAS

By

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(Dedicated to Professor Hisao Tominaga on his 60th birthday)

Introduction.

The important notion of reflection functors was introduced into the representation theory of algebras by Bernstein-Gelfand-Ponomarev [8]. Those functors were defined only for hereditary tensor algebras given by quivers and species [12]. Then Auslander-Platzek-Reiten [7] arranged the notion by non-diagrammatic treatment so that it is possible to apply the concept for any algebras. Brenner-Butler [10] extended the Auslander-Platzek-Reiten partial Coxeter functor and defined the tilting theory. Further, Happel-Ringel [15] generalized the Brenner-Butler tilting theory and studied tilted algebras.

We regard the tilting theory as a powerful method of deforming algebras and their module categories. A tilting functor, however, is nothing but a Morita equivalence, for any self-injective algebra. Hence, it is natural to search for a way of applying the tilting theory to the study of self-injective algebras.

Let A be a basic indecomposable artin algebra. Denote by $\text{mod-}A$ (resp. $A\text{-mod}$) the category of all finitely generated right (resp. left) A -modules. Let $D: \text{mod-}A \rightarrow A\text{-mod}$ be the ordinary duality functor. In the following, we shall consider the trivial extension self-injective algebra $R = A \ltimes DA$ defined as follows: R is $A \oplus DA$ as an additive group and its multiplication is given by $(a, q) \cdot (a', q') = (a \cdot a', a \cdot q' + q \cdot a')$ for any $(a, q), (a', q') \in A \oplus DA = R$.

In the paper [19], Tachikawa started in the study of self-injective algebras R , and in [20], he has proved that $\underline{\text{mod-}}R$ is equivalent to $\underline{\text{mod-}}S$ ($S = B \ltimes DB$) if A is hereditary tensor algebra and B is given by reflection procedure from A . Here $\underline{\text{mod-}}R$ is the projectively (=injectively) stable category of $\text{mod-}R$ in the sense of Auslander.

Let $e \in A$ be a primitive idempotent such that eA is simple non-injective and $\tau^{-1}eA \otimes_A DA = 0$, where τ^{-1} (resp. τ) denotes the Auslander-Reiten translation TrD (resp. DTr). By putting $T_A = (1-e)A \oplus \tau^{-1}eA$ and $B = \text{End}(T_A)$, the Auslander-Platzek-Reiten partial Coxeter functor is defined to be the functor $\text{Hom}_A(T, ?)$:

$\text{mod-}A \rightarrow \text{mod-}B$.

In the present paper, we shall generalize the above result of Tachikawa and show the following

THEOREM. There is an equivalence $\tilde{F}: \underline{\text{mod-}}R \rightarrow \underline{\text{mod-}}S$ such that $\tilde{F}(X) \cong \text{Hom}_A(T, X)$ for any A -module X not possessing eA as a direct summand.

As an application of our Theorem, making use of the Assem-Happel's characterization of generalized tilted algebras [1], we shall prove that the trivial extension self-injective algebra R of a generalized tilted algebra A of Dynkin class A_n is always stably equivalent to a serial self-injective algebra.

Gabriel-Riedtmann [14] proved that every DJK -algebra of a Brauer quiver is stably equivalent to a serial self-injective algebra. It is easy to check that any DJK -algebra of a Brauer quiver of multiplicity 1 is the trivial extension algebra of a generalized tilted algebra of Dynkin class A_n . Therefore, our argument can be seen as another proof of the result of Gabriel-Riedtmann in the case of DJK -algebras of multiplicity 1. Notice that another case of DJK -algebras of multiplicity greater than 1 can be reduced to our case.

Throughout this paper, we fix a commutative artin ring K and all algebras are artin K -algebras and modules are finitely generated. D always denotes the ordinary duality functor.

1. Preliminaries

Throughout this paper, we shall freely use the results on tilting theory proved in [7], [9], [10] and [15] and also the facts about modules over the trivial extension self-injective algebras given by [13], [19] and [20]. But it is convenient to remember some of them which will be used frequently. In this section, we shall recall the basic properties of partial Coxeter functors and trivial extension self-injective algebras and fix the notations.

Let A be an artin algebra and eA a simple non-injective module with the property $\tau^{-1}eA \otimes_A DA = 0$. We assume that A is self-basic and indecomposable. Let us put $T_A = (1-e)A \oplus \tau^{-1}eA$ and $B = \text{End}(T_A)$. It is easy to see that the module T_A becomes a tilting module, that is, it satisfies the the following three conditions:

- (1) $\text{proj. dim. } T_A \leq 1$,
- (2) $\text{Ext}^1(T_A, T_A) = 0$ and
- (3) There is a short exact sequence $0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$ with T' and T'' are direct summands of direct sums of copies of T_A .

Let $F = \text{Hom}_A(T, ?)$, $F' = \text{Ext}^1(T_A, ?)$ be the functors from $\text{mod-}A$ to $\text{mod-}B$ and

$G=(? \otimes_B T)$, $G'=\text{Tor}_1(? , {}_B T)$ those from $\text{mod-}B$ to $\text{mod-}A$. Let $\mathcal{T}=\{X|F'(X)=0\}$, $\mathcal{F}=\{X|F(X)=0\}$ be the full subcategories of $\text{mod-}A$ and $\mathcal{X}=\{Y|G(Y)=0\}$, $\mathcal{Y}=\{Y|G'(Y)=0\}$ those of $\text{mod-}B$. Then, by Brenner-Butler [10] or Happel-Ringel [15], the following facts are known.

(a) $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion theories in the categories $\text{mod-}A$ and $\text{mod-}B$, respectively.

(b) The left module ${}_B T$ is again a tilting module with $\text{End}({}_B T)=A$.

(c) \mathcal{T} and \mathcal{Y} (resp. \mathcal{F} and \mathcal{X}) are category equivalent under the restrictions of F and G (resp. F' and G') which are mutually inverses.

We call our special kind of tilting functor F the partial Coxeter functor with respect to a simple module eA , following Auslander-Platzek-Reiten [7].

Since ${}_B T_A$ is balanced, it is proved that ${}_A D A_A \cong {}_A D T \otimes_B T_A$ and ${}_B D B_B \cong {}_B T \otimes_A D T_B$ as bimodules. Sometimes, we shall identify DA (resp. DB) with $DT \otimes_B T$ (resp. $T \otimes_A DT$) by these isomorphisms. We denote by κ the composite: $DB \otimes_B T \cong T \otimes_A DT \otimes_B T \cong T \otimes_A DA$.

It is well known [3] that the AR -sequence starting from eA is at the same time the minimal projective resolution of $\tau^{-1}eA$:

$$(*) \quad 0 \rightarrow eA \xrightarrow{\alpha} P \xrightarrow{\beta} \tau^{-1}eA \rightarrow 0$$

From this sequence, by tensoring DA , we have the minimal injective resolution of eA (note that $\tau^{-1}eA \otimes_A DA = 0$):

$$(*\dagger) \quad 0 \rightarrow eA \rightarrow eA \otimes_A DA \xrightarrow{\alpha \otimes DA} F \otimes_A DA \rightarrow 0$$

Applying the partial Coxeter functor F to the sequence $(*)$, we have the minimal projective resolution of the B -module $F'(eA)$:

$$0 \rightarrow F(P) \xrightarrow{F(\beta)} F(\tau^{-1}eA) \rightarrow F'(eA) \rightarrow 0$$

Since $(*)$ is an AR -sequence, denoting by \hat{e} the primitive idempotent element of B corresponding to the direct summand $\tau^{-1}eA$ of T_A , we see that $F(P) = \text{rad } \hat{e}B$ and $F'(eA) = \text{top } \hat{e}B$. Further, tensoring DB to the above sequence, we have the minimal injective resolution of the B -module $F'(eA)$:

$$\begin{array}{ccccccc} 0 \rightarrow & \tau F'(eA) & \longrightarrow & F(P) \otimes_B DB & \xrightarrow{F(\beta) \otimes DB} & F(\tau^{-1}eA) \otimes_B DB & \rightarrow 0 \\ & \parallel & & \parallel & & \parallel & \\ 0 \rightarrow & \text{DHom}_A(eA, T) & \xrightarrow{\text{DHom}(\alpha, T)} & \text{DHom}_A(P, T) & \xrightarrow{\text{DHom}(\beta, T)} & \text{DHom}_A(\tau^{-1}eA, T) & \rightarrow 0 \end{array}$$

On the other hand, applying the functor F to the sequence $(*\dagger)$, we have

$$\begin{array}{ccccccc} 0 \rightarrow & F(eA \otimes_A DA) & \xrightarrow{F(\alpha \otimes DA)} & F(P \otimes_A DA) & \longrightarrow & F'(eA) & \rightarrow 0 \\ & \parallel & & \parallel & & \parallel & \\ 0 \rightarrow & \text{DHom}_A(eA, T) & \xrightarrow{\text{DHom}(\alpha, T)} & \text{DHom}_A(P, T) & \xrightarrow{\text{DHom}(\beta, T)} & \text{DHom}_A(\tau^{-1}eA, T) & \rightarrow 0 \end{array}$$

Thus we have the following commutative diagram with exact rows:

LEMMA 1.1.

$$\begin{array}{ccccccc}
 0 \rightarrow & F(eA \otimes_A DA) & \xrightarrow{F(\alpha \otimes DA)} & F(P \otimes_A DA) & \longrightarrow & F'(eA) & \rightarrow 0 \\
 & \parallel & & \parallel & & \parallel & \\
 0 \rightarrow & F'(eA) & \longrightarrow & F(P) \otimes_B DB & \xrightarrow{F(\beta) \otimes DB} & F(\tau^{-1}eA) \otimes_B DB & \rightarrow 0
 \end{array}$$

By the above diagram, we see that $\hat{e}B = F'(eA) \cong F(\tau^{-1}eA) \otimes_B DB \cong \hat{e}DB$, i. e., $B\hat{e}$ is simple projective. Moreover, we have the following isomorphism:

$$\begin{aligned}
 DB \otimes_B \tau^{-1}B\hat{e} &\cong DB \otimes_B D\tau F'(eA) \\
 &\cong \text{DHom}_B(D\tau F'(eA), B) \\
 &\cong \text{DHom}_B(T \otimes_A Ae, B) \\
 &\cong \text{DHom}_B(Te, B) \cong \text{DHom}_A(T, eA) = 0
 \end{aligned}$$

Therefore, $\text{Hom}_B(T, -) : B\text{-mod} \rightarrow A\text{-mod}$ is again a partial Coxeter functor. We shall use this A - B -symmetry to define the desired functors.

For a given bimodule ${}_B U_A$, there is always an adjoint pair of functors $\text{Hom}_A(U, -)$ and $(- \otimes_B U)$. Denote by $\eta^U : 1_{\text{mod-}B} \rightarrow \text{Hom}_A(U, (- \otimes_B U))$ (resp. $\varepsilon^U : \text{Hom}_A(U, -) \otimes_B U \rightarrow 1_{\text{mod-}A}$) the unit (resp. the counit) of this adjunction.

In the later part of the paper, the natural transformation $\eta_{(-) \otimes_{DT} \cdot}^T \cdot \varepsilon_{(-) \otimes}^T \otimes DT : F(-) \otimes_B DB \rightarrow F(- \otimes_A DA)$ will appear frequently. We shall denote by θ this natural transformation. It should be noted that θ_P is an isomorphism for any projective A -module P not possessing eA as a direct summand.

Denote by $T(A)$ (resp. $T(B)$) the trivial extension self-injective algebra $A| \times DA$ (resp. $B| \times DB$). See Introduction for definition. Since there is an algebra epimorphism $T(A) \rightarrow A \rightarrow 0$, each A -module can be seen as a $T(A)$ -module. We call such $T(A)$ -modules being of 1st kind and others 2nd kind, following Tachikawa [19].

On the other hand, since A is a subalgebra of $T(A)$, any $T(A)$ -module can be seen as an A -module and we call it the underlying A -module. Hence, a $T(A)$ -module $X_{T(A)}$ is given by its underlying A -module X_A with its multiplication by $DA : X \otimes_A DA \xrightarrow{\phi} X$. Note that $(X \cdot DA) \cdot DA = 0$, i. e., $\phi \cdot (\phi \otimes DA) = 0$. We shall indicate this structure of $X_{T(A)}$ by the pair (X_A, ϕ) . For a $T(A)$ -morphism f from $X_{T(A)} = (X_A, \phi)$ to $X'_{T(A)} = (X'_A, \phi')$, it can be seen as an A -morphism between underlying A -modules such that the following diagram commutes:

$$\begin{array}{ccc}
 X \otimes_A DA & \xrightarrow{\phi} & X \\
 f \otimes DA \downarrow & & \downarrow f \\
 X' \otimes_A DA & \xrightarrow{\phi'} & X'
 \end{array}$$

Since the torsion theory $(\mathcal{T}, \mathcal{F})$ is splitting, each A -module X_A can be decomposed as $X=L \oplus V$ with $L_A \in \mathcal{F}$ and $V_A \in \mathcal{T}$. Further, DA is in \mathcal{T} and so is $X \otimes_A DA$, thus $\text{Hom}_A(X \otimes_A DA, L) = 0$. Therefore, any $T(A)$ -module has to be of the form $X_{T(A)} = \left(L \oplus V, \begin{pmatrix} 0 & 0 \\ \lambda & \phi \end{pmatrix} \right)$ with $L_A \in \mathcal{F}$ and $V_A \in \mathcal{T}$.

In the above argument, we considered that the operation $X \times DA \rightarrow X$ of a $T(A)$ -module $X_{T(A)}$ is given by $X \otimes_A DA \rightarrow X$. But by the adjunction isomorphism $\text{Hom}_A(X \otimes_A DA, X) \cong \text{Hom}_A(X, \text{Hom}_A(DA, X))$, we may consider that the operation is given by $X \rightarrow \text{Hom}_A(DA, X)$. We shall say $X_{T(A)} = (X_A, \phi)$ to be T -form (resp. H -form) if $\phi \in \text{Hom}_A(X \otimes_A DA, X)$ (resp. $\phi \in \text{Hom}_A(X, \text{Hom}_A(DA, X))$). In this paper, we shall study modules only by T -forms.

For any object of \mathcal{F} , as L_A is a direct sum of copies of eA , we can define the sequence:

$$0 \rightarrow L \xrightarrow{\alpha_L} P(L) \xrightarrow{\beta_L} T(L) \rightarrow 0$$

by taking the direct sum of the sequence (*). We shall use this sequence frequently.

By the A - B -symmetry, in the category of left B -modules, we may consider the sequence similarly above. Applying the duality functor D to this sequence, we have the following sequence in the category of right B -modules $\text{mod-}B$:

$$0 \rightarrow W(K) \xrightarrow{\delta_K} I(K) \xrightarrow{\gamma_K} K \rightarrow 0,$$

where K is an object in \mathcal{X} , i. e., K_B is a direct sum of copies of $\hat{e}DB = \text{Ext}_A^1(T, eA) = F'(eA)$ and $I(K)$ is a direct sum of copies of $F(P \otimes_A DA) = F(P) \otimes_B DB$, γ_K is isomorphic to $\oplus F(\beta) \otimes DB$ and δ_K is isomorphic to $\oplus F(\alpha \otimes DA)$. These sequences also appear frequently.

Note that any module of the category $\text{mod-}B$ has to be of the form $W \oplus K$ with $W \in \mathcal{Q}$ and $K \in \mathcal{X}$, since the torsion theory $(\mathcal{X}, \mathcal{Q})$ is also splitting.

2. Stable functor $F: \text{mod-}T(A) \rightarrow \text{mod-}T(B)$

At first, we shall construct the correspondence $\tilde{F}: \text{mod}^e\text{-}T(A) \rightarrow \text{mod}^e\text{-}T(B)$ which will be used to define the stable functor $\hat{F}: \text{mod-}T(A) \rightarrow \text{mod-}T(B)$, where $\text{mod}^e\text{-}T(A)$ and $\text{mod}^e\text{-}T(B)$ are the full subcategories of $\text{mod-}T(A)$ and $\text{mod-}T(B)$ whose classes of objects consisting of modules not possessing as direct summands $eT(A)$ and $\hat{e}T(B)$, respectively.

LEMMA 2.1. Let $X_{T(A)} = \left(L \oplus V, \begin{pmatrix} 0 & 0 \\ \lambda & \phi \end{pmatrix} \right)$ be a $T(A)$ -module with $V \in \mathcal{T}$ and $L \in \mathcal{F}$. If $X_{T(A)} \in \text{mod}^e\text{-}T(A)$ then there is a morphism $\hat{\lambda}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & L \otimes_A DA & \xrightarrow{\alpha_L \otimes DA} & P(L) \otimes_A DA \rightarrow 0 \\
& & & & \lambda \downarrow & \nearrow \hat{\lambda} & \\
& & & & V & &
\end{array}$$

PROOF. Denote by $\nu_i: eA \rightarrow L$ the i -th injection of $L = \bigoplus_i eA$. If λ does not factor through $\alpha_L \otimes DA$, $\lambda \cdot \nu_i \otimes DA$ is a monomorphism for some i , hence there is a $T(A)$ -monomorphism:

$$\begin{pmatrix} \nu_i & 0 \\ 0 & \lambda \cdot \nu_i \otimes DA \end{pmatrix}: eT(A) = eA \oplus eA \otimes_A DA \rightarrow X = L \oplus V.$$

However, this is a contradiction since $eT(A)_{T(A)}$ is injective and $X_{T(A)}$ has not $eT(A)$ as a direct summand.

Using the extended morphism $\hat{\lambda}$, we shall construct a $T(B)$ -module $\tilde{F}(X)$ as follows:

$$\tilde{F}(X) = \left(F(P(L)) \oplus F(V) \oplus F(T(L)) \otimes_B DB, \begin{pmatrix} 0 & 0 & 0 \\ \lambda^* & \phi^* & 0 \\ F(\beta_L) \otimes DB & 0 & 0 \end{pmatrix} \right),$$

$$\lambda^* = (F(P(L)) \otimes_B DB \xrightarrow{\theta_{P(L)}} F(P(L) \otimes_A DA) \xrightarrow{F(\alpha)} F(V)) \text{ and}$$

$$\phi^* = (F(V) \otimes_B DB \xrightarrow{\theta_V} F(V \otimes_A DA) \xrightarrow{F(\phi)} F(V)).$$

LEMMA 2.2. $\tilde{F}(X)$ is a $T(B)$ -module.

PROOF. It is sufficient to verify that $\phi^* \cdot \lambda^* \otimes DB = 0$ and $\phi^* \cdot \phi^* \otimes DB = 0$. For the first equality, consider the following commutative diagram:

$$\begin{array}{ccccc}
F(P(L)) \otimes_B DB \otimes_B DB & \xrightarrow{\theta_{P(L)} \otimes DB} & F(P(L) \otimes_A DA) \otimes_B DB & \xrightarrow{F(\alpha) \otimes DB} & F(V) \otimes_B DB \\
& & \theta_{P(L)} \otimes DA \downarrow & & \downarrow \theta_V \\
& & F(P(L) \otimes_A DA \otimes_A DA) & \xrightarrow{F(\alpha \otimes DA)} & F(V \otimes_A DA) \xrightarrow{F(\phi)} F(V).
\end{array}$$

By the definition, $\phi^* \cdot \lambda^* \otimes DB = F(\phi) \cdot \theta_V \cdot F(\hat{\lambda}) \otimes DB \cdot \theta_{P(L)} \otimes DB = F(\phi) \cdot F(\hat{\lambda} \otimes DA) \cdot \theta_{P(L)} \otimes DB = 0$ since $\phi \cdot \hat{\lambda} \otimes DA = 0$. The second equality can be proved similarly.

LEMMA 2.3. $\tilde{F}(X)_{T(B)} \in \text{mod}^e\text{-}T(B)$.

PROOF. If $eT(B)$ is a direct summand of $\tilde{F}(X)$ as a $T(B)$ -module then so is as a B -module, therefore $eB \otimes_B DB = F(\tau^{-1}eA) \otimes_B DB$ is a summand of $F(T(L)) \otimes_B DB$ by Krull-Remar-Schmidt theorem. Thus there is an A -morphism f such that $F(\beta_L) \otimes DB \cdot \lambda^* \cdot F(f) \otimes DB$ coincides with this injection map. This is a contradiction since $\text{Hom}_A(\tau^{-1}eA, P) = 0$.

Thus we have defined the correspondence $\tilde{F}: \text{mod}^e\text{-}T(A) \rightarrow \text{mod}^e\text{-}T(B)$ as announced at the beginning of this section. In the following, we shall show it is possible to make \tilde{F} a stable functor: $\underline{\text{mod}}\text{-}T(A) \rightarrow \underline{\text{mod}}\text{-}T(B)$.

Let $X_i = (L_i \oplus V_i, \begin{pmatrix} 0 & 0 \\ \lambda_i & \phi_i \end{pmatrix})$ ($i=1, 2$) be $T(A)$ -modules in $\text{mod}^e\text{-}T(A)$ and $\theta = \begin{pmatrix} g & 0 \\ h & f \end{pmatrix}: X_1 \rightarrow X_2$ a $T(A)$ -morphism. Since $0 \rightarrow eA \xrightarrow{\alpha} P \xrightarrow{\beta} \tau^{-1}eA \rightarrow 0$ is an AR -sequence, we can define A -morphism g' , g'' and h' by the following commutative diagrams:

$$\begin{array}{ccccccc} 0 & \rightarrow & L_1 & \xrightarrow{\alpha_{L_1}} & P(L_1) & \xrightarrow{\beta_{L_1}} & T(L_1) \rightarrow 0 \\ & & g \downarrow & & g' \downarrow & & g'' \downarrow \\ 0 & \rightarrow & L_2 & \xrightarrow{\alpha_{L_2}} & P(L_2) & \xrightarrow{\beta_{L_2}} & T(L_2) \rightarrow 0 \text{ and} \\ & & & & & & \\ 0 & \rightarrow & L_1 & \xrightarrow{\alpha_{L_1}} & P(L_1) & \xrightarrow{\beta_{L_1}} & T(L_1) \rightarrow 0 \\ & & h \downarrow & & \swarrow h' & & \\ & & V_2 & & & & \end{array}$$

Here, g' , and g'' are uniquely determined by g since $\text{Hom}_A(\tau^{-1}eA, A) = 0$ but h' is not by h .

For a given $T(A)$ -morphism $\theta = \begin{pmatrix} g & 0 \\ h & f \end{pmatrix}$, fixing the representative h' of h , we shall define the $T(B)$ -morphism:

$$\tilde{F}(\theta) = \begin{pmatrix} F(g') & 0 & 0 \\ F(h') & F(f) & 0 \\ 0 & 0 & F(g') \otimes DB \end{pmatrix}: \tilde{F}(X_1) \rightarrow \tilde{F}(X_2)$$

LEMMA 2.4. $\tilde{F}(\theta)$ is a $T(B)$ -morphism.

PROOF. We have to verify the following three properties:

- (i) $\phi_2^* \cdot F(f) \otimes DB = F(f) \cdot \phi_1^*$,
- (ii) $F(\beta_{L_2}) \otimes DB \cdot F(g') \otimes DB = F(g'') \otimes DB \cdot F(\beta_{L_1}) \otimes DB$ and
- (iii) $\lambda_2^* \cdot F(g') \otimes DB + \phi_2^* \cdot F(h') \otimes DB = F(f) \cdot \lambda_1^*$.

(i) and (ii) are easily verified. We shall show (iii) only. Since $\theta_{P(L_2)} \cdot F(g') \otimes DB = F(g' \otimes DA) \cdot \theta_{P(L_1)}$ and $\theta_{V_1} \cdot F(h') = F(h' \otimes DA) \cdot \theta_{P(L_1)}$, we have following equalities:

$$\begin{aligned} F(\hat{\lambda}) \cdot \theta_{P(L_2)} \cdot F(g') \otimes DB &= F(\hat{\lambda}_2) \cdot F(g' \otimes DA) \cdot \theta_{P(L_1)} \text{ and} \\ F(\phi_2) \cdot \theta_{V_2} \cdot F(h') &= F(\phi_2) \cdot F(h' \otimes DA) \cdot \theta_{P(L_1)}, \end{aligned}$$

thus it is enough to show $F(\hat{\lambda}_2) \cdot F(g' \otimes DA) + F(\phi_2) \cdot F(h' \otimes DA) = F(f) \cdot F(\hat{\lambda}_1)$. From the equalities $f \cdot \lambda_1 = \lambda_2 \cdot g \otimes DA + \phi_2 \cdot h \otimes DA$, $\lambda_2 = \hat{\lambda}_2 \cdot \alpha_{L_2}$ and $h \otimes DA = h' \otimes DA \cdot \alpha_{L_1} \otimes DA$,

we have $f \cdot \hat{\lambda}_1 \cdot \alpha_{L_1} \otimes DA = (\hat{\lambda}_2 \cdot g' \otimes DA + \phi_2 \cdot h' \otimes DA) \cdot \alpha_{L_1} \otimes DA$ and $f \cdot \hat{\lambda}_1 = \hat{\lambda}_2 \cdot g' \otimes DA + \phi_2 \cdot h' \otimes DA$ since $\alpha_{L_1} \otimes DA$ is an epimorphism. Hence we obtain the desired equality.

LEMMA 2.5. $\tilde{F}(\theta)$ is uniquely determined in the stable category $\underline{\text{mod}}\text{-}T(B)$ for a morphism θ .

PROOF. Let $h' \cdot \alpha_{L_1} = h = h'' \cdot \alpha_{L_1}$ and $h' - h'' = z \cdot \beta_{L_1}$:

$$\begin{array}{ccccccc} 0 & \rightarrow & L_1 & \xrightarrow{\alpha_{L_1}} & P(L_1) & \xrightarrow{\beta_{L_1}} & T(L_1) & \longrightarrow & 0 \\ & & & & \downarrow h' - h'' & & \nearrow z & & \\ & & & & V_2 & & & & \end{array}$$

Let $\tilde{F}(\theta)'$ be the morphism constructed from θ and h'' . Then $\tilde{F}(\theta) - \tilde{F}(\theta)'$ factors through projective: $F(X_1) \xrightarrow{\chi} Q \xrightarrow{\zeta} F(X_2)$, where χ, ζ and Q are defined as follows:

$$Q = \left(F(T(L)) \oplus F(T(L_1)) \otimes_B DB, \begin{pmatrix} 0 & 0 \\ 1_{F(T(L_1)) \otimes_B DB} & 0 \end{pmatrix} \right) \text{ and}$$

$$\chi = \begin{pmatrix} F(\beta_{L_1}) & 0 & 0 \\ 0 & 0 & 1_{F(T(L_1)) \otimes_B DB} \end{pmatrix}, \quad \zeta = \begin{pmatrix} 0 & 0 \\ F(z) & 0 \\ 0 & 0 \end{pmatrix}.$$

We have shown the main property of the correspondence \tilde{F} .

It is easy to see that \tilde{F} maps any projective module in $\text{mod}^e\text{-}T(A)$ to projective $T(B)$ -module and it induces a stable functor: $\underline{\text{mod}}\text{-}T(A) \rightarrow \underline{\text{mod}}\text{-}T(B)$. In fact, if θ factors through $eT(A)$ then $\tilde{F}(\theta)$ factors through $F(P) \otimes_B T(B)$.

By the symmetry of the partial Coxeter functor, we can define a stable functor $\tilde{F}': T(B)\text{-}\underline{\text{mod}} \rightarrow T(A)\text{-}\underline{\text{mod}}$. Using this functor \tilde{F}' , we define the stable functor $\tilde{G}: \underline{\text{mod}}\text{-}T(B) \rightarrow \underline{\text{mod}}\text{-}T(A)$ as the composite $D \cdot \tilde{F}' \cdot D$. In the next section, we shall show the equivalence $\tilde{G} \cdot \tilde{F} \simeq 1_{\underline{\text{mod}}\text{-}T(A)}$.

3. Proof of Theorem

In order to compute $\tilde{G} \cdot \tilde{F}$, it is convenient to give $\tilde{G}(Y)$ concretely for a $T(B)$ -module $Y_{T(B)}$ in $\text{mod}^e\text{-}T(B)$.

Let $Y_{T(B)} = \left(W \oplus K, \begin{pmatrix} \phi & 0 \\ \mu & 0 \end{pmatrix} \right)$ be a $T(B)$ -module with $W \in \mathcal{Y}$ and $K \in \mathcal{X}$. If $Y_{T(B)}$ is in $\text{mod}^e\text{-}T(B)$, then there is a morphism $\hat{\mu}$ such that $\mu = \gamma_K \cdot \hat{\mu}$. Using this morphism $\hat{\mu}$, $\tilde{G}(Y)$ is defined as follows:

$$\tilde{G}(Y) = \left(\text{Hom}_A(DA, W(K) \otimes_B T) \oplus I(K) \otimes_B T \oplus W \otimes_B T, \begin{pmatrix} 0 & 0 & 0 \\ \delta_K \otimes T \cdot \varepsilon_{W(L)}^A & 0 & \mu^* \\ 0 & 0 & \phi^* \end{pmatrix} \right).$$

$\mu^* = \hat{\mu} \otimes T$ and $\phi^* = \phi \otimes T$.

It is easy to see that $K = F(T(L))$, $W = F(P(L)) \oplus F(V)$ and $\phi = \begin{pmatrix} 0 & 0 \\ \lambda^* & \phi^* \end{pmatrix}$, $\mu = (F(\beta_L) \otimes DB, 0)$ if we put $Y = \tilde{F}(X)$. From here, the following Lemma is proved directly.

LEMMA 3.1. For a given $T(A)$ -module $X = (L \oplus V, \begin{pmatrix} 0 & 0 \\ \lambda & \phi \end{pmatrix})$ with $V \in \mathcal{T}$ and $L \in \mathcal{F}$. If $X_{T(A)}$ has no direct summand isomorphic to $eT(A)$, then $\tilde{G} \cdot \tilde{F}(X)_{T(A)}$ is of the form $\left(L \oplus P(L) \oplus P(L) \otimes_A DA \oplus V, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_L \otimes DA & 1_{P(L)} \otimes DA & 0 & 0 \\ 0 & \lambda & 0 & 0 \end{pmatrix} \right)$.

Now we shall show $\tilde{G} \cdot \tilde{F}(X)_{T(A)} \cong P(L) \otimes_A T(A) \oplus X$, by making use of the form in the above Lemma.

Let us define two $T(A)$ -morphisms χ and ζ .

$$\begin{aligned} \chi : P(L) \otimes_A T(A) &= \left(P(L) \oplus P(L) \otimes_A DA, \begin{pmatrix} 0 & 0 \\ 1_{P(L) \otimes DA} & 0 \end{pmatrix} \right) \\ &\xrightarrow{\begin{pmatrix} 0 & 0 \\ 1_{P(L)} & 0 \\ 0 & 1_{P(L) \otimes DA} \\ 0 & \lambda \end{pmatrix}} \tilde{G} \cdot \tilde{F}(X) = L \oplus P(L) \oplus P(L) \otimes_A DA \oplus V, \\ \zeta : \tilde{G} \cdot \tilde{F}(X) &= L \oplus P(L) \oplus P(L) \otimes_A DA \oplus V \xrightarrow{\begin{pmatrix} -1_L & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 1_V \end{pmatrix}} X = L \oplus V. \end{aligned}$$

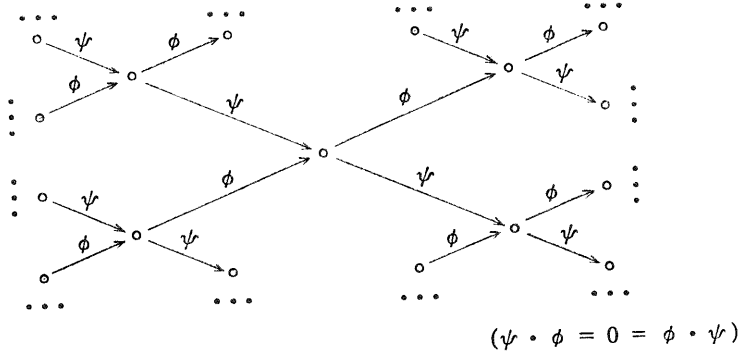
Then it is easy to verify that χ is a $T(A)$ -monomorphism and ζ is a $T(A)$ -epimorphism and $\zeta \cdot \chi = 0$. Further, the sum of lengths of $P(L) \otimes_A T(A)$ and X is just that of $\tilde{G} \cdot \tilde{F}(X)$. Hence we know that the sequence $0 \rightarrow P(L) \otimes_A T(A) \xrightarrow{\chi} \tilde{G} \cdot \tilde{F}(X) \xrightarrow{\zeta} X \rightarrow 0$ is exact. Since $P(L) \otimes_A T(A)_{T(A)}$ is injective, the above sequence splits. Therefore, $\tilde{G} \cdot \tilde{F}(X)$ is isomorphic to the direct sum of X and $P(L) \otimes_A T(A)$. It is easy to prove that the isomorphism $\tilde{G} \cdot \tilde{F}(X) \cong X$ in the category $\underline{\text{mod}}\text{-}T(A)$ has naturality on X . This completes the proof of Theorem. $\tilde{F} \cdot \tilde{G} \cong 1_{\underline{\text{mod}}\text{-}T(B)}$ is given by the symmetry of partial Coxeter functor.

4. An Application

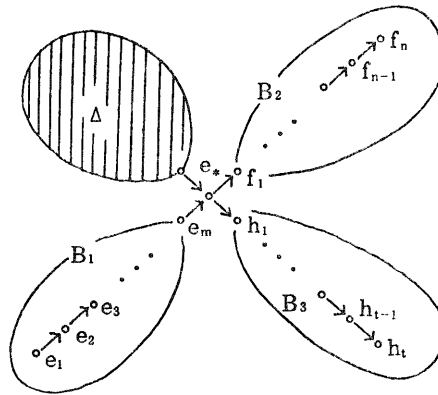
As an application of our Theorem, we shall prove that any DJK-algebra of a Brauer quiver of multiplicity 1 is stably equivalent to a serial self-injective algebra as announced before.

It is easy to show that any DJK-algebra of a Brauer quiver of multiplicity 1 is a self-injective trivial extension of some algebra A . By the characterization of generalized tilted algebras of Dynkin class A_n which is given by Assem and Happel [1], we know that the algebra A has to be a generalized tilted algebra of Dynkin

class A_n . Hence, A is the algebra of a connected finite full subquiver of the following infinite tree :



Clearly, there is a joint whose branches have no joints except at most one branch. Here, a vertex is said to be a joint if it has at least three neighbours. Consider such a joint e_* as follows :



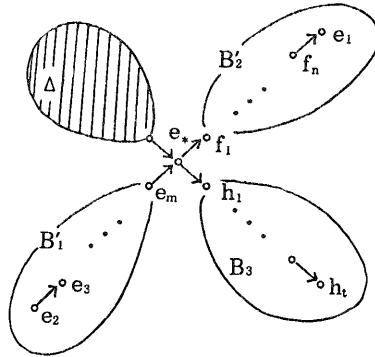
The starting point e_1 of a branch B_1 corresponds to a simple projective module e_1A . Thus, A is of the form :

$$\begin{pmatrix} e_1Ae_1 & 0 \\ (1-e_1)Ae_1 & (1-e_1)A(1-e_1) \end{pmatrix} = e_1Ae_1 \times (1-e_1)A(1-e_1) \times (1-e_1)Ae_1.$$

Let us put

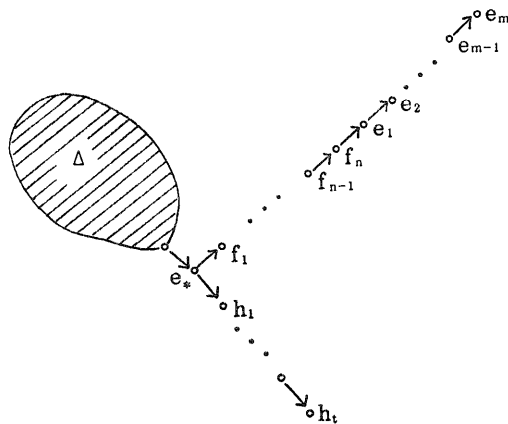
$$A' = e_1Ae_1 \times (1-e_1)A(1-e_1) \times D((1-e_1)Ae_1) = \begin{pmatrix} e_1Ae_1 & D((1-e_1)Ae_1) \\ 0 & (1-e_1)A(1-e_1) \end{pmatrix}.$$

Then $T(A)$ and $T(A')$ are isomorphic to each other as algebras. The quiver of the algebra A' is as follows :

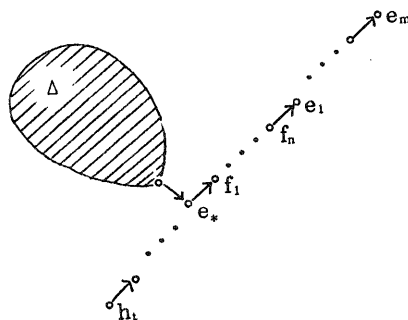


The length of the branch B'_1 (resp. B'_2) is shorter (resp. longer) than that of B_1 (resp. B_2) just 1.

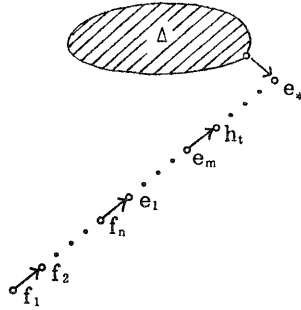
We can continue this process and have an algebra A'' of the following quiver:



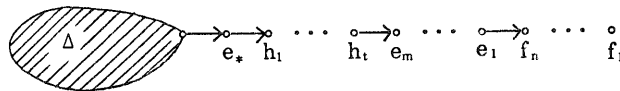
Then we apply the reflection processes (these correspond to partial Coxeter functors defined at the vertices h_t, h_{t-1}, \dots) to the algebra A'' and finally, we have an algebra B of the following quiver:



Further, we apply the inverse processes of $A \mapsto A'$ to the algebra B , then we have an algebra B' of the following quiver :



At last, we apply the reflection processes to B' , then it becomes an algebra $A_{(1)}$ for which the number of joints in the quiver is smaller than that of the original algebra A :



Hence, iterating the processes $A \mapsto A_{(1)} \mapsto A_{(2)} \mapsto A_{(3)} \dots$, we have a hereditary algebra of the quiver :



By our Theorem, it is easy to see that $T(A)$ is stably equivalent to the trivial extension algebra of the final hereditary algebra. On the other hand, the trivial extension algebra of the above hereditary algebra is clearly serial. Thus every DJK-algebra of a Brauer quiver of multiplicity 1 is stably equivalent to a serial self-injective algebra.

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