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ON RAMANUJAN SUMS ON ARITHMETICAL SEMIGROUPS

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1. Introduction.

Let $f: N\rightarrow C$ be an arithmetic function and let $f^{*}=\mu*f$ denote the Dirichlet convolution and the Möbius function μ, so that

(1.1) $$f^{*}(n)=\sum_{d|n}\mu(d)f\left(\frac{n}{d}\right), \quad n\geqq 1 .$$

Let

(1.2) $$c_{q}(n)=\sum_{\substack{h=1\\(h,q)=1}}^{q}\exp\left(2\Pi i\frac{hn}{q}\right)$$

be the Ramanujan's trigonometric sum. A Ramanujan series is a series of the form

(1.3) $$\sum_{q=1}^{\infty}a_{q}c_{q}(n)$$

where $c_{q}(n)$ is Ramanujan's sum and

(1.4) $$a_{q}=\sum_{m=1}^{\infty}\frac{f^{*}(mq)}{mq}.$$

Important result concerning Ramanujan's expansions of certain arithmetical functions has been given by Delange [2]. He proved the following result:

THEOREM A. *If* $\sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n}|f^{*}(n)|<\infty$, *where* $\omega(n)$ *is the number of distinct prime divisors of* n, *then* $\sum_{q=1}^{\infty}|a_{q}c_{q}(n)|<\infty$ *for every* n *and* $\sum_{q=1}^{\infty}a_{q}c_{q}(n)=f(n)$.

In his proof, Delange used the inequality

(1.5) $$\sum_{d|k}|c_{d}(n)|\leqq 2^{\omega(k)}n ,$$

see [2; Lemma, p. 263] and conjectured [2, p. 264] that his Lemma is best possible.

In [3] we proved the following identity:

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$$(*) \qquad \sum_{d\mid k} |c_k(n)| = 2^{\omega(k/(k,n))}(k, n)$$

for all positive integers k and n.

D. Redmond [7] generalized (*) to a larger class of functions and K.R. Johnson [4] evaluated the left hand side of (*) for second variable. Further generalizations connected with (*) have been given by K.R. Johnson [5], J. Chidambaraswamy and D.V. Krishnaiah [1] and D. Redmond [8].

In this paper by using (*) we give a theorem inverse to the Theorem A. Moreover we obtain an evaluation of Ramanujan's sum defined on an arithmetical semigroup.

2. Inverse Theorem to the Theorem A.

We prove the following

THEOREM 1. *If* $\sum_{k=1}^{\infty} U_k < \infty$, *where*

$$U_k = \sum_{mq=k} \frac{|f^*(mq)|}{mq} |c_q(n)|, \tag{2.1}$$

then

$$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n} |f^*(n)| < \infty .$$

PROOF. Let us suppose $\sum_{k=1}^{\infty} U_k < \infty$, where U_k is given by (2.1) above. From (2.1) we get

$$\sum_{k=1}^{\infty} \frac{|f^*(k)|}{k} \sum_{q\mid k} |c_q(n)| < \infty . \tag{2.2}$$

By (2.2) and (*) we obtain

$$\sum_{k=1}^{\infty} \frac{|f^*(k)|}{k} 2^{\omega(k/(k,n))}(k, n) < \infty . \tag{2.3}$$

Now, by well-known properties of the function $\omega(n)$ it follows that

$$2^{\omega(k/(k,n))} \geqq \frac{2^{\omega(k)}}{2^{\omega((k,n))}} \tag{2.4}$$

and if $D=(k, n)$ then

$$D \geqq 2^{\omega(D)} . \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$2^{\omega(k/(k,n))}(k,\ n) \geqq 2^{\omega(k)} \frac{(k,\ n)}{2^{\omega((k,n))}} \geqq 2^{\omega(k)}\ . \tag{2.6}$$

By (2.3) and (2.6) our theorem follows.

3. Ramanujan's sum on an arithmetical semigroup.

Let G denote a commutative semigroup with identity element 1, relative to multiplication operation. Suppose that G has a finite or countably infinite subset P such that every element $a \neq 1$ in G has a unique with to up to order of the factor indicated of the factorization of the form

$$a = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \tag{3.1}$$

where $p_i \in P$, $p_i \neq p_j$ for $i \neq j$ and α_i are positive rational integers. It there exists a real-valued norm mapping $\|\cdot\|$ on G such that

$$\begin{cases} \text{(i)} \quad \|1\| = 1\ , \quad \|p\| > 1 & \text{if } p \in P \\ \text{(ii)} \quad \|ab\| = \|a\|\|b\| & \text{for all } a,\ b \in G\ , \\ \text{(iii)} \quad N_G(x) = \text{card}\,\{a \in G : \|a\| \leqq x\} < \infty & \text{for } x < 0 \end{cases} \tag{3.2}$$

then the semigroup G will be called the arithmetical semigroup.

We have the following;

LEMMA 1. *Let $c_r(a)$ denote the Ramanujan sum defined on an arithmetical semigroup G as follows*

$$c_r(a) = \sum_{d \mid (r,s)} \mu\left(\frac{r}{d}\right) \|d\|^{\delta} \tag{3.3}$$

where $r,\ a \in G$; $\delta > 0$, μ denotes the Möbius function on G and $(r,\ a)$ is the g.c.d. of $r,\ a$ in G. Then we have

$$c_{p^m}(a) = \begin{cases} \|p\|^{\delta m} - \|p\|^{\delta(m-1)} & \textit{if } p^m \mid a \\ -\|p\|^{\delta(m-1)} & \textit{if } p^{m-1} \mid a \textit{ and } p^m \nmid a \\ 0 & \textit{if } p^{m-1} \nmid a \end{cases} \tag{3.4}$$

for any prime element $p \in P$ and positive integer m. Moreover $c_1(a) = 1$ and for any fixed $a \in G$ the function $c_r(a)$ is a multiplicative function with respect to variable r.

The proof of this Lemma follows from the results given by J. Knopfmacher see [6, pp. 185-186].

Now, we can prove the following:

THEOREM 2. *Let G be a given arithmetical semigroup and $c_r(a)$ denote the Ramanujan sum on G. Then for any $r, a\in G$ we have*

$$\sum_{d\mid r}|c_d(a)|=2^{\omega(r/(r,a))}\|(r, a)\|^{\delta} \tag{3.5}$$

where $\delta>0$ and $\omega(D)=k$ if $D=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$.

PROOF. Let $a\in G$ be fixed. Then the function $f(r)=(r, a)$ is a multiplicative function of r. Hence by (ii) of (3.2) it follows that

$$\|(rs, a)\|^{\delta}=\|(r, a)\|^{\delta}\|(s, a)\|^{\delta} \quad \text{for } r, s\in G \text{ such that } (r, s)=1\,. \tag{3.6}$$

Let $g=p^{\alpha_1}\cdots p^{\alpha_k}$, $p_i\in P$, $p_i\neq p_j$ for $i\neq j$, then we have $\omega(g)=k$. Consider the function $F(g)=2^{w(g)}$.

It is easy to see that $F(g)$ is a multiplicative function. Hence

$$F\left(\frac{rs}{(rs, a)}\right)=F\left(\frac{r}{(r, a)}\right)F\left(\frac{s}{(s, a)}\right) \qquad \text{for } (r, s)=1\,;\ r, s\in G\,. \tag{3.7}$$

From (3.6) and (3.7) it follows that the function

$$P(r, a)=2^{\omega(r/(r,a))}\|(r, a)\|^{\delta}$$

is a multiplicative functions of $r\in G$ for any fixed $a\in G$.

By Lemma 1 it follows that the left hand side of (3.5) is also multiplicative of $r\in G$ for any fixed $a\in G$. Thus it suffices to verify (3.5) for $r=p^m$, where $p\in P$ and m is a positive integer. Denote by $L(r, a)$ the left hand side of (3.5) and suppose that $p^1\| a$.

If $0\leqq 1<m$ then we have

$$L(p^m, a)=\sum_{j=0}^{m}|c_{p^j}(a)|=|c_1(a)|+\sum_{j=1}^{1}|c_{p^j}(a)|+|c_{p^{l+1}}(a)|. \tag{3.8}$$

By Lemma 1 and (3.8) it follows that

$$L(p^m, a)=1+\sum_{j=1}^{1}(\|p\|^{\delta j}-\|p\|^{\delta(j-1)})+\|p\|^{\delta 1}=2\|p\|^{\delta 1}\,.$$

If $1\geqq m$ then by Lemma 1 we obtain

$$L(p^m, a)=\sum_{j=0}^{m}|c_{p^j}(a)|=1+\sum_{j=1}^{m}(\|p\|^{\delta j}-\|p\|^{\delta(j-1)})=\|p\|^{\delta m}\,.$$

Comparing the functions $L(p^m, a)$ and $P(p^m, a)$ we get $L(p^m, a)=P(p^m, a)$ and the proof of Theorem 2 is complete.

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