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DUAL LIE ALGEBRAS OF HEISENBERG POISSON LIE GROUPS

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Abstract. In this note, we shall classify all the dual Lie algebra structures induced by multiplicative Poisson tensors on an arbitrary dimensional Heisenberg Lie group.

1. Introduction

According to Drinfel'd [1], a Poisson structure of a Lie group L is called *multiplicative* if the multiplication of L is a Poisson map with respect to the Poisson structure. A Lie group with a multiplicative is called a Poisson (Lie) group.

In terms of Poisson tenson $\pi: L \rightarrow \wedge^2 TL$, multiplicativity is equivalent to

$$\pi(ab) = Tl_a \pi(b) + Tr_b \pi(a) \qquad a, b \in L$$

where l_a and r_a are the left and right translation of $a \in L$ respectively.

Let $\pi_r: L \rightarrow \wedge^2 \mathfrak{l}$ be a Posisson tensor after identifying TL with $L \times \mathfrak{l}$ by right translations where $\mathfrak{l}$ is the Lie algebra of L. Then multiplicativity is also equivalent to π_r being a 1-cocycle of Adjoint representation ([6]).

A multiplicative Poisson structure defines a Lie algebra structure for the dual space $\mathfrak{l}^*$ of Lie algebra $\mathfrak{l}$ by

$$\langle [\sigma, \tau], \xi \rangle = \langle \sigma \wedge \tau, d_e \pi_r(\xi) \rangle$$

where $\sigma, \tau \in \mathfrak{l}^*$, $\xi \in \mathfrak{l}$ (cf. [1]).

In cases of semi-simple Lie groups, all multiplicative Poisson structures are classified by terms of cohomology of its Lie algebra. In cases of abelian Lie groups, the situation is rather simple. The simplest case of non-abelian Lie group is $(ax+b)$-group and the multiplicative Poisson structures on $(ax+b)$-group is characterized in [3]. A Heisenberg group is neither semi-simple nor abelian, but is *almost* abelian. In this note, we shall classify all the dual Lie algebras structures induced by multiplicative Poisson tensors on an arbitrary

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dimensional Heisenberg Lie group. We prepare notation and state some results on Heisenberg Lie algebras/groups in section 2. In section 3, we shall classify the dual Lie algebra structures induced by arbitrary dimensional Heisenberg Poisson Lie groups of the first type. In section 4, we shall classify multiplicative Poisson tensors of the second type on arbitrary dimensional Heisenberg Lie groups and its dual Lie algebra structures.

2. Notation and review of Heisenberg Poisson Lie groups

We recall the structure of Heisenberg Lie algebra along to [4].

A Lie algebra $\mathfrak{h}$ is Heisenberg if $\mathcal{Z}(\mathfrak{h})(:=$the center of $\mathfrak{h})$ is 1-dimensional and the derived algebra $\mathfrak{h}'$ of $\mathfrak{h}$ is $\mathcal{Z}(\mathfrak{h})$.

For a Heisenberg Lie algebra $\mathfrak{h}$, $\mathfrak{a}=\mathfrak{h}/\mathcal{Z}(\mathfrak{h})$ is abelian. Fix a non-zero vector $v_0\in\mathcal{Z}(\mathfrak{h})$. Then we have a linear symplectic structure Ω_0 on $\mathfrak{a}$ by the relation

$$[h_1, h_2]=\Omega_0(proj(h_1), proj(h_2))v_0 \qquad \text{for } h_1, h_2\in\mathfrak{h}$$

where $proj: \mathfrak{h}\rightarrow\mathfrak{a}$ is the canonical projection.

Take a $\sigma_0\in\mathfrak{h}^*$ satisfying $\langle\sigma_0, v_0\rangle=1$. Then $\mathfrak{h}=Ker(\sigma_0)\oplus\boldsymbol{R}v_0$ and $Ker(\sigma_0)\cong\mathfrak{a}$. Hereafter we use this decomposition $\mathfrak{h}=\mathfrak{a}\oplus\boldsymbol{R}v_0$ and often use symplectic terminology as

1. $\xi^\flat(\eta)=\langle\xi^\flat, \eta\rangle:=\Omega_0(\xi, \eta)$
2. $\Omega^0=$the "inverse" of Ω_0, which is defined as $\Omega^0(p, \xi^\flat)=\langle p, \xi\rangle$
3. $\Omega^0(\xi^\flat, \eta^\flat)=\Omega_0(\xi, \eta)$
4. $\sigma^\#$ is defined by $\langle\sigma, \xi\rangle=\Omega_0(\sigma^\#, \xi)$. $(\sigma^\#)^\flat=\sigma$ and $(\xi^\flat)^\#=\xi$
5. For the dual map N of $M\in End(\mathfrak{a})$ defined by $\langle Np, \xi\rangle=\langle p, M\xi\rangle$, we have

$$\Omega^0(N\xi^\flat, \eta^\flat)=\langle\xi^\flat, M\eta\rangle=\Omega_0(\xi, M\eta)$$

6. Let $S^\perp=\{\xi\in\mathfrak{a}\,|\,\Omega_0(\xi, S)=0\}$ for a subset $S\subset\mathfrak{a}$.

If $\mathfrak{h}$ is a Heisenberg Lie algebra, the corresponding connected and simply connected Lie group H is called the Heisenberg group. We can parametrize H with $\mathfrak{h}$ by using the exponential map as

$$h_1h_2\cong h_1+h_2+[h_1, h_2]/2\,.$$

I. Szymczak and S. Yakrzewski [4] list up all the multiplicative Poisson tensors on a Heisenberg group as follows.

PROPOSITION 1 (cf. [4]). *Each Multiplicative Poisson structure π on a Heisenberg group H is one of the following two forms:*

(i) $\pi_r(h)=v_0\wedge M(\xi)$, *where* $h=\xi+\lambda v_0\in\mathfrak{a}\oplus \boldsymbol{R}v_0$ *and* $M\in End(\mathfrak{a})$

(ii) $\pi_r(h)=v_0\wedge(\lambda A+M(\xi))+1/2(\xi\wedge A+\Omega_0(\xi, A)\Omega^0)$,

where $h=\xi+\lambda v_0\in\mathfrak{a}\oplus\boldsymbol{R}v_0$, A *is a non-zero vector of* $\mathfrak{a}=Ker(\sigma_0)$, *and* $M\in End(\mathfrak{a})$ *must satisfy*

$$M\xi=\Omega_0(A, B)\xi+\Omega_0(\xi, B-lA)A+\Omega_0(A, \xi)B$$

for some vector $B\in\mathfrak{a}=Ker(\sigma_0)$ *and some constant* l.

Among multiplicative Poisson tensors, there is a subclass including those induced by r-matrices. In case of Heisenberg groups, we have the following.

PROPOSITION 2. *Multiplicative Poisson structure* π *of the form*

$$\pi(h)=Tl_h\boldsymbol{r}-Tr_h\boldsymbol{r} \qquad h\in L$$

is of the first type with $MJ_0=J_0{}^tM$, *where* J_0 *is a matrix representation of the linear symplectic form* Ω_0 *corresponding to the Heisenberg Lie algebra.*

In particular, multiplicative Poisson structure that comes from r-matrix is characterized by $MJ_0=J_0{}^tM$ *and* $M^2=0$.

PROOF. Let $\{v_1, \cdots, v_{2n}\}$ be a basis of $\mathfrak{a}$ consisting of left invariant vector fields. They are related as

$$[v_i, v_j]=\Omega_0(v_i, v_j)v_0$$
$$[v_i, v_0]=0.$$

Let $\boldsymbol{r}=(1/2)\sum_{i,j} p_{ij}v_i\wedge v_j+\sum_k q_kv_k\wedge v_0$ be an arbitrary constant 2-vector, where $P=(p_{ij})_{ij}$ is a skew-symmetric matrix. For $\pi(h)=Tl_h\boldsymbol{r}-Tr_h\boldsymbol{r}$, we have $\pi_r(h)=Ad_h(\boldsymbol{r})-\boldsymbol{r}$ and $d_e\pi_r(\xi+\lambda v_0)=ad_{(\xi+\lambda v_0)}(\boldsymbol{r})$. Thus,

$$d_e\pi_r(v_0)=0$$
$$d_e\pi_r(v_i)=v_0\wedge\sum_{j,k}\Omega_0(v_i, v_j)p_{jk}v_k\,.$$

We see that this π is a multiplicative Poisson tensor by Proposition 1 or calculating the Schouten bracket $[\pi, \pi]$ directly. π is of the first type with $M=(\sum_j\Omega_0(v_i, v_j)p_{jk})_{i,k}$, that is, $M=J_0P$. Skew-symmetry of P implies $MJ_0=J_0{}^tM$.

Since

$$[\boldsymbol{r}, \boldsymbol{r}]=-v_0\wedge\sum_{i,j,k,l}p_{ij}\Omega_0(v_j, v_k)p_{kl}v_i\wedge v_l$$

we have $[\boldsymbol{r}, \boldsymbol{r}]=0$ if and only if $\sum_{j,k}p_{ij}\Omega_0(v_j, v_k)p_{kl}=0$ for each i, l. That is equivalent to $M^2=0$. ∎

When the dimension of a Lie group is 2 or 3, there is a notion of non-degeneracy for multiplicative Poisson tensors due to Drinfel'd [2]. For 3-dimensional Heisenberg group, we have the following.

PROPOSITION 3. *On 3-dimensional Heisenberg group, multiplicative Poisson structures of the second type are non degenerate in the sense of Drinfel'd* [2].

PROOF. Use the same notation in the proof of Proposition 2 with $n=1$. We consider the following diagram.

$$\Lambda^2\mathfrak{h} \xrightarrow{[\cdot,\,\cdot]} \mathfrak{h} \xrightarrow{d_e\pi_r} \Lambda^2\mathfrak{h}$$

Through this map, $v_0\wedge v_1$ and $v_0\wedge v_2$ go to zero, and $v_1\wedge v_2$ goes to $d_e\pi_r(v_0)$. We see that

$$d_e\pi_r(v_0)=0 \qquad \text{(if the first type)}$$

$$d_e\pi_r(v_0)=v_0\wedge A\neq 0 \qquad \text{(if the second type)}$$

and the conclusion follows. ∎

3. Structure of dual Lie algebras induced by multiplicative Poisson tensors of the first type

It is natural to ask if given two multiplicative Poisson tensors are equivalent or not. On a general Lie group L, equivalence of two multiplicative Poisson tensors π and π' means that there is a Lie group isomorphism Φ that is a Poisson map between π and π'. Then we have $\pi'(\Phi(a))=\Phi_*(\pi(a))$ and $\pi'_r(a)=\Phi_*\pi_r(\Phi^{-1}(a))$. We thereby, have

$$d_e\pi'_r(x)=\Phi_* d_e\pi_r((\Phi_*)^{-1}(x))$$

where $x\in\mathfrak{l}$. We call π' the induced Poisson tensor by π and Φ.

In case of Heisenberg groups, Lie group automorphisms are equal to Lie algebra automorphisms. We use the same symbol Φ instead of the differential Φ_* of Φ. A Lie algebra isomorphism Φ of the Heisenberg Lie algebra is defined by

$$\Phi(\xi+\lambda v_0)=\varphi\xi+(\mu(\xi)+\lambda c)v_0$$

where $c\neq 0$ and φ satisfies $\Omega_0(\varphi\xi, \varphi\eta)=c\Omega_0(\xi, \eta)$.

PROPOSITION 4. *A multiplicative Poisson tensor π' of the first type induced by π and Φ above is also of the first type with the corresponding matrix $M'=c\varphi M\varphi^{-1}$.*

A multiplicative Poisson tensor π' of the second type induced by π and Φ above is also of the second type and the relations are

$$A'=\varphi(A)$$

$$\begin{aligned}M'(\xi)&=c\varphi M\varphi^{-1}(\xi)-\frac{1}{2}\mu\circ\varphi^{-1}(\xi)\varphi(A)-\frac{1}{2}\mu(A)\xi-\frac{1}{2c}\Omega_0(\xi,\ \varphi(A))\varphi(\mu^{\#})\\&=\Omega_0\Big(\varphi A,\ \varphi B+\frac{1}{2c}\varphi(\mu^{\#})\Big)\xi+\Omega_0\Big(\xi,\ \varphi B+\frac{1}{2c}\varphi(\mu^{\#})-l\varphi A\Big)\varphi A\\&\quad+\Omega_0(\varphi A,\ \xi)\Big(\varphi B+\frac{1}{2c}\varphi(\mu^{\#})\Big).\end{aligned}$$

PROOF. The case of the first type is easy so that we omit proof of it.

We discuss the second type. Denote $d_e\pi_r$ and $d_e\pi'_r$ by γ and γ', respectively. We have

$$\begin{aligned}\gamma'(\xi+\lambda v_0)&=\Phi\Big(\gamma\Big(\varphi^{-1}(\xi)+\frac{1}{c}(-\mu\circ\varphi^{-1}(\xi)+\lambda)v_0\Big)\Big)\\&=\Phi\Big\{v_0\wedge\Big(\frac{1}{c}(-\mu\circ\varphi^{-1}(\xi)+\lambda)A+M\varphi^{-1}(\xi)\Big)\\&\quad+\frac{1}{2}\varphi^{-1}(\xi)\wedge A+\frac{1}{2}\Omega_0(\varphi^{-1}(\xi),\ A)\Omega^0\Big\}\\&=\Phi(v_0)\wedge\Big(\frac{1}{c}(-\mu\circ\varphi^{-1}(\xi)+\lambda)\Phi(A)+\Phi(M\varphi^{-1}(\xi))\Big)\\&\quad+\frac{1}{2}\Phi(\varphi^{-1}(\xi)\wedge\Phi(A)+\frac{1}{2}\Omega_0(\varphi^{-1}(\xi),\ A)\Phi(\Omega^0)\\&=cv_0\wedge\Big(\frac{1}{c}(-\mu\circ\varphi^{-1}(\xi)+\lambda)(\varphi A+\mu(A)v_0)+\varphi M\varphi^{-1}(\xi)+\mu(M\varphi^{-1}(\xi))v_0\Big)\\&\quad+\frac{1}{2}\Big(\varphi(\varphi^{-1}(\xi))+\mu\circ\varphi^{-1}(\xi)v_0\Big)\wedge(\varphi(A)+\mu(A)v_0)+\frac{1}{2}\Omega_0(\varphi^{-1}(\xi),\ A)\Phi(\Omega^0)\\&=v_0\wedge((-\mu\circ\varphi^{-1}(\xi)+\lambda)\varphi A+c\varphi M\varphi^{-1}(\xi))\\&\quad+\frac{1}{2}(\xi+\mu\circ\varphi^{-1}(\xi)v_0)\wedge(\varphi(A)+\mu(A)v_0)+\frac{1}{2}\Omega_0(\varphi^{-1}(\xi),\ A)\Phi(\Omega^0)\\&=v_0\wedge((-\mu\circ\varphi^{-1}(\xi)+\lambda)\varphi A+c\varphi M\varphi^{-1}(\xi))+\frac{1}{2}\xi\wedge\varphi(A)\\&\quad+\frac{1}{2}\mu\circ\varphi^{-1}(\xi)v_0\wedge\varphi(A)-\frac{1}{2}\mu(A)v_0\wedge\xi+\frac{1}{2}\Omega_0(\varphi^{-1}(\xi),\ A)\Phi(\Omega^0)\end{aligned}$$

$$=v_0\wedge\Big((-\mu\circ\varphi^{-1}(\xi)+\lambda)\varphi A+c\varphi M\varphi^{-1}(\xi)+\frac{1}{2}\mu\circ\varphi^{-1}(\xi)\varphi(A)-\frac{1}{2}\mu(A)\xi\Big)$$

$$+\frac{1}{2}\xi\wedge\varphi(A)+\frac{1}{2}\Omega_0(\varphi^{-1}(\xi),\ A)\Phi(\Omega^0)$$

$$=v_0\wedge\Big(\lambda\varphi A+c\varphi M\varphi^{-1}(\xi)-\frac{1}{2}\mu\circ\varphi^{-1}(\xi)\varphi(A)-\frac{1}{2}\mu(A)\xi\Big)$$

$$+\frac{1}{2}\xi\wedge\varphi(A)+\frac{1}{2c}\Omega_0(\xi,\ \varphi(A))\Phi(\Omega^0)$$

$$=v_0\wedge\Big(\lambda\varphi A+c\varphi M\varphi^{-1}(\xi)-\frac{1}{2}\mu\circ\varphi^{-1}(\xi)\varphi(A)-\frac{1}{2}\mu(A)\xi\Big)$$

$$+\frac{1}{2}\xi\wedge\varphi(A)+\frac{1}{2c}\Omega_0(\xi,\ \varphi(A))(c\Omega^0-v_0\wedge\varphi(\mu^{\#}))$$

$$=v_0\wedge(\lambda\varphi A+c\varphi M\varphi^{-1}(\xi)-\frac{1}{2}\mu\circ\varphi^{-1}(\xi)\varphi(A)$$

$$-\frac{1}{2}\mu(A)\xi-\frac{1}{2c}\Omega_0(\xi,\ \varphi(A))\varphi(\mu^{\#}))+\frac{1}{2}\xi\wedge\varphi(A)+\frac{1}{2}\Omega_0(\xi,\ \varphi(A))\Omega^0$$

using the next lemma. Thus, we complete the proof. ■

With the same notation in Proposition 4, we have

LEMMA 1.

$$\Phi^*(\xi^\flat)=c(\varphi^{-1}\xi)^\flat \qquad \Phi^*(\sigma_0)=c\sigma_0+\mu$$

$$\Phi(\Omega^0)=c\Omega^0-v_0\wedge\varphi(\mu^{\#})$$

PROOF.

$$\langle\Phi^*(\xi^\flat),\ \eta\rangle=\langle\xi^\flat,\ \Phi(\eta)\rangle=\langle\xi^\flat,\ \varphi(\eta)+\mu(\eta)v_0\rangle$$

$$=\Omega_0(\xi,\ \varphi(\eta))=c\Omega_0(\varphi^{-1}\xi,\ \eta)=c\langle(\varphi^{-1}\xi)^\flat,\ \eta\rangle$$

$$\langle\Phi^*(\xi^\flat),\ v_0\rangle=\langle\xi^\flat,\ \Phi(v_0)\rangle=\langle\xi^\flat,\ cv_0\rangle=0$$

$$\langle\Phi^*(\sigma_0),\ \eta\rangle=\langle\sigma_0,\ \Phi(\eta)\rangle=\langle\sigma_0,\ \varphi(\eta)+\mu(\eta)v_0\rangle=\mu(\eta)$$

$$\langle\Phi^*(\sigma),\ v_0\rangle=\langle\sigma,\ \Phi(v_0)\rangle=\langle\sigma_0,\ cv_0\rangle=c$$

$$\Phi(\Omega^0)(\xi^\flat,\ \eta^\flat)=\Omega^0(\Phi^*(\xi^\flat),\ \Phi^*(\eta^\flat))=\Omega^0(c(\varphi^{-1}\xi)^\flat,\ c(\varphi^{-1}\eta)^\flat)$$

$$=c^2\Omega_0(\varphi^{-1}\xi,\ \varphi^{-1}\eta)=c\Omega_0(\xi,\ \eta)$$

$$\Phi(\Omega^0)(\sigma_0,\ \eta^\flat)=\Omega^0(\Phi^*\sigma_0,\ \Phi^*(\eta^\flat))=\Omega^0(c\sigma_0+\mu,\ c(\varphi^{-1}\eta)^\flat)$$

$$=c\Omega^0(\mu,\ (\varphi^{-1}\eta)^\flat)=c\langle\mu,\ \varphi^{-1}\eta\rangle$$

$$
\begin{aligned}
&= c\Omega_0(\mu^\#, \varphi^{-1}\eta) = \Omega_0(\varphi(\mu^\#), \eta) \\
&= -\Omega_0(\eta, \varphi(\mu^\#)) = -\langle \eta^\flat, \varphi(\mu^\#)\rangle \\
&= -\langle v_0 \wedge \varphi(\mu^\#), \sigma_0 \wedge \eta^\flat \rangle
\end{aligned}
\qquad \blacksquare
$$

We consider the dual Lie algebra induced by multiplicative Poisson tensor

$$\pi_r(h) = v_0 \wedge M(\xi) \qquad (h = \xi + \lambda v_0 \in \mathfrak{a} \oplus \mathfrak{Z}(\mathfrak{h}))$$

of the first type on a Heisenberg group.

Let $\{\sigma_1, \sigma_2, \cdots, \sigma_{2n}, \sigma_0\}$ be the dual basis of $\{v_1, v_2, \cdots, v_{2n}, v_0\}$. From the definition of dual Lie algebra, we have

$$[\sigma_j, \sigma_k] = 0 \quad \text{and} \quad [\sigma_0, \sigma_j] = N\sigma_j \quad (j, k = 1, \cdots, 2n),$$

where N is the dual of M. We denote by $[\cdot, \cdot]_N$ this Lie bracket in order to show that it is defined by N.

If $M=0$, then the dual Lie algebra is abelian. We consider the case of $M \neq 0$. Then, the induced Lie algebra is a semidirect product of the abelian Lie algebra $\boldsymbol{R}^{2n}$ and $\boldsymbol{R}$. If two dual Lie algebra structures induced by M and M' are isomorphic, then the derived Lie algebras are also isomorphic and it turns out that the rank of M and M' should be equal.

Two Lie algebra structures $[\cdot, \cdot]_N$ and $[\cdot, \cdot]_{N'}$ are isomorphic if and only if there is a linear isomorphism Ψ of $\boldsymbol{R}^{2n+1}$ satisfying

$$\Psi([x, y]_N) = [\Psi(x), \Psi(y)]_{N'} \qquad (x, y \in \boldsymbol{R}^{2n+1}).$$

Identifying $\boldsymbol{R}^{2n+1} \cong \boldsymbol{R}^{2n} \times \boldsymbol{R}$, we can split the linear map Ψ on $\boldsymbol{R}^{2n+1}$ as

$$
\begin{aligned}
&\Psi(\sigma) = \psi(\sigma) + \mu(\sigma)\sigma_0 \\
&\Psi(\sigma_0) = \tau_0 + \lambda\sigma_0
\end{aligned}
$$

for $\forall \sigma \in \boldsymbol{R}^{2n}$, where μ is a linear functional on $\boldsymbol{R}^{2n}$, τ_0 is a constant vector in $\boldsymbol{R}^{2n}$ and λ is some constant number.

The relations that they must satisfy are the following.

$$
\begin{aligned}
&\mu(\sigma)N'\psi(\tau) - \mu(\tau)N'\psi(\sigma) = 0 \\
&\mu(N(\sigma)) = 0 \\
&\psi(N(\sigma)) = \lambda N'(\psi(\sigma)) - \mu(\sigma)N'(\tau_0)
\end{aligned}
$$

Suppose that a linear isomorphism Ψ provided with $\mu \neq 0$. Then we may assume that $\mu(\sigma_1) = 1$ and $Ker(\mu) = \langle\!\langle \sigma_2, \cdots, \sigma_{2n} \rangle\!\rangle$. The relations above yield

$$N'\psi(\sigma_j) = 0 \qquad (j \geqq 2)$$

$$Im(N)\subset Ker(\mu)$$
$$\phi(N(\sigma_1))=\lambda N'(\phi(\sigma_1))-N'(\tau_0)$$
$$\phi(N(\sigma_j))=\lambda N'(\phi(\sigma_j))=0 \qquad (j\geqq 2).$$

Since Ψ is a linear isomorphism, $\Psi_{|R^{2n}}$ is a monomorphism. Thus, $\mu(N(\sigma_j))=0$ and $\phi(N(\sigma_j))=0$ imply $N(\sigma_j)=0$ for $j\geqq 2$ and the rank of N and N' must be at most 1. In other words, if we deal with M and M' with the rank greater than 1, μ must be 0. Then, it turns out ϕ is a linear isomorphism of $\boldsymbol{R}^{2n}$, $\lambda\neq 0$, and $\phi(N(\sigma))=\lambda N'(\phi(\sigma))$. Conversely, if N and N' satisfy the relations above, then we can define a Lie algebra isomorphism between $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_{N'}$.

If the rank of N is equal to 1, then we have to study 2 cases depending on $N^2=0$ or not. If $N^2=0$, then the derived algebra is a subalgebra of the center, i.e., $[\mathfrak{h}^*,\mathfrak{h}^*]_N\subset\mathfrak{Z}(\mathfrak{h}^*,[\cdot,\cdot]_N)$. If $N^2\neq 0$, then the derived algebra is not a subalgebra of the center, i.e., $[\mathfrak{h}^*,\mathfrak{h}^*]_N\not\subset\mathfrak{Z}(\mathfrak{h}^*,[\cdot,\cdot]_N)$. Thus, the Lie algebras defined by N with $N^2=0$ and N with $N^2\neq 0$ are not isomorphic.

Let us consider N and N' with $N^2=N'^2=0$. Then we find a linear bijection ϕ on $\boldsymbol{R}^{2n}$ satisfying $\phi N=N'\phi$ and can define a Lie algebra isomorphism Ψ between $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_{N'}$.

Consider N and N' with $N^2\neq 0$ and $N'^2\neq 0$. Then we find a non-zero constant λ and a linear bijection ϕ on $\boldsymbol{R}^{2n}$ satisfying $\phi N=\lambda N'\phi$ and can define a Lie algebra isomorphism Ψ between $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_{N'}$ by ϕ and λ.

Thus, we have the following result.

THEOREM 1. *On a Heisenberg group, the dual Lie algebras induced by multiplicative Poisson structures of the first type are solvable and characterized as follows.*

1) *If $M=0$, then the induced dual Lie algebra is abelian.*

2) *If $M\neq 0$, then the dual Lie algebra is semidirect product of the abelian Lie algebra $\boldsymbol{R}^{2n}$ and $\boldsymbol{R}$ and the Lie algebra isomorphism classes are characterized by the relation*

$$PM=\lambda M'P$$

for some non-singular matrix P and some non-zero constant λ.

In the case of 3-dimensional, we can state our theorem in another way by using Tasaki-Umehara invariant (cf. [5]).

Let $\{\sigma_1,\sigma_2,\sigma_0\}$ be the dual basis of $\{v_1,v_2,v_0\}$. From the definition of dual Lie algebra, we have

$$[\sigma_1, \sigma_2]=0 \qquad [\sigma_0, \sigma_1]=N\sigma_1 \qquad [\sigma_0, \sigma_2]=N\sigma_2$$

where N is the dual of M. We consider the case of $M\neq 0$. By direct calculation of Killing form and cofactor matrix of structure constants, we see that Tasaki-Umehara invariant is

$$\chi=\frac{trace(N)^2}{det(N)}=\frac{trace(M)^2}{det(M)}$$

if $det(M)\neq 0$, because $trace(N)=trace(M)$, $det(N)=det(M)$.

When $det(N)=det(M)=0$, Tasaki-Umehara invariant of the dual Lie algebra is ∞ if $trace(M)\neq 0$.

When $det(N)=det(M)=0$ and $trace(N)=trace(M)=0$, the dual Lie algebra is a Heisenberg Lie algebra.

Thus, we have the following result which is a corollary of Theorem 1.

CORLOLARY 1. *On 3-dimensional Heisenberg group, the dual Lie algebras induced by multiplicative Poisson structures of the first type are solvable and are characterized as follows.*

1) *If $M=0$, then the induced dual Lie algebra is abelian.*

2) *If $M\neq 0$ and $trace(M)=det(M)=0$, then the induced dual Lie algebra is a Heisenberg Lie algebra.*

3) *If $det(M)$ or $trace(M)\neq 0$, then the induced dual Lie algebra is characterized by Tasaki-Umehara invariant $trace(M)^2/det(M)$.*

4. Structures of multiplicative Poisson tensors of the second type and their dual Lie algebras

In this section, we restrict ourselves to the multiplicative Poisson tensors of the second type. We first study of endomorphisms M which appear in multiplicative Poisson tensors of the second type. Since multiplicative Poisson tensor M of the second type in Proposition 1 is defined by A, B and l, we can write M as $M_{(A,B,l)}$.

PROPOSITION 5. *For $M=M_{(A,B,l)}\in End(\mathfrak{a})$,*

1) *M satisfies $\Omega_0(M\xi, \eta)-\Omega_0(\xi, M\eta)=2l\Omega_0(A, \xi)\Omega_0(A, \eta)$ for $\forall\xi, \eta\in\mathfrak{a}$.*

2) *If A and B are linearly dependent, then we have two cases.*

   a) *If $l=0$, then $M=0$.*

   b) *If $l\neq 0$, then $Ker(M)=A^{\perp}$, $Im(M)=\boldsymbol{R}A$, and rank $M=1$.*

3) *If A and B are linearly independent, then we have two cases.*

   a) *If $\Omega_0(A, B)=0$, then $Ker(M)=(\boldsymbol{R}A+\boldsymbol{R}B)^{\perp}$, $Im(M)=\boldsymbol{R}A+\boldsymbol{R}B$, and* *rank* $M=2$.

b) *If* $\Omega_0(A, B)\neq 0$, *then* M *is non-singular.*

PROPOSITION 6. *For the function* $(A, B, l)\mapsto M_{(A,B,l)}$, $M_{(A,B,l)}=M_{(A',B',l')}$ *if and only if* $A\wedge B=A'\wedge B'$ *and* $ll'\neq 0$ *and* $\begin{cases} A'=\alpha A \\ l'=l/\alpha^2 \end{cases}$, *or* $A\wedge B=A'\wedge B'$ *and* $l=l'=0$.

PROOF. Comparing the rank of the both sides of $M_{(A,B,l)}=M_{(A',B',l')}$, we have two conditions below.

$$\Omega_0(A, B)=\Omega_0(A', B') \tag{C-1}$$

$$\Omega_0(\xi, B-lA)A+\Omega_0(A, \xi)B=\Omega_0(\xi, B'-l'A')A'+\Omega_0(A', \xi)B' \tag{C-2}$$

When $A\wedge B\neq 0$, then the *rank* $M_{(A,B,l)}\geqq 2$. Using Proposition 5, we see $A'\wedge B'\neq 0$. Condition (C-2) implies that

$$A'=\alpha A+\beta B$$

$$B'=\alpha' A+\beta' B\,.$$

Plugging them to (C-1), we have the following.

If $l=0$, then $l'=0$ and $A\wedge B=A'\wedge B'$.

If $l\neq 0$, then $l'\neq 0$ and $A'=\alpha A$, $l'=l/\alpha^2$, and $A\wedge B=A'\wedge B'$.

Similarly, when $A\wedge B=0$, then the *rank* $M_{(A,B,l)}\leqq 1$. Using Proposition 5, we see $A'\wedge B'=0$. Condition (C-2) implies that

$$l=0 \text{ implies } l'=0, \quad \text{or} \quad l\neq 0 \text{ implies } \begin{cases} A'=\alpha A \\ l'=l/\alpha^2\,. \end{cases}$$

■

THEOREM 2. *The space of equivalence classes of all multiplicative Poisson tensors of the second type is* 1*-dimensional real space* $\boldsymbol{R}$. *Their representatives are* $(A_0, 0, l)$ *where* A_0 *is any fixed non-zero vector.*

PROOF. Each multiplicative Poisson tensor of the second type is determined by $(A, M_{(A,B,l)})$ and $M_{(A,B,l)}$ is determined as explained in Proposition 5 and is parametrized in the space

$$\{(A, A\wedge B, l) \mid A, B\in\mathfrak{a}\}\,.$$

Lie algebra automorphism action on the parameter space is as

$$\Phi\cdot(A, A\wedge B, l)=\left(\varphi A,\ \varphi A\wedge\varphi\left(B+\frac{1}{2c}\mu^{\#}\right),\ l\right)$$

where Φ is defined by

$$\Phi(\xi+\lambda v_0)=\varphi\xi+(\mu(\xi)+\lambda c)v_0,\quad c\neq 0,\quad \varphi \text{ satisfying } \Omega_0(\varphi\xi,\varphi\eta)=c\Omega_0(\xi,\eta).$$

We can take μ as $B+(1/2c)\mu^*=0$ and we can take φ so that $\varphi A=A_0$ (constant vector). ■

THEOREM 3. *On a Heisenberg group, the dual Lie algebras induced by multiplicative Poisson tensors of the second type are all solvable and are classified into two classes by $l\neq 0$ or $l=0$.*

PROOF. Theorem 2 guarantee that we can put $A=A_0$ (constant) and $B=0$ for each multiplicative Poisson tensor of the second type. Then $M\xi=\Omega_0(\xi,-lA_0)A_0$. According to the definition of the dual Lie algebra induced by multiplicative Poisson tensor, we have

$$[\sigma_0,\tau]_l=\langle\tau,A_0\rangle(\sigma_0+lA_0^\flat)$$

$$[\sigma,\tau]_l=\frac{1}{2}(\langle\tau,A_0\rangle\sigma-\langle\sigma,A_0\rangle\tau)-\frac{1}{2}\langle\sigma\wedge\tau,\Omega^0\rangle A_0^\flat.$$

When $l\neq 0$, using the transformation f such that $f(\sigma_0)=(1/l)\sigma_0$ and $f_{|anih(v_0)}=identity$, we calculate the induced Lie algebra structure $[[\cdot,\cdot\cdot]]=f^{-1}[f(\cdot),f(\cdot\cdot)]_l$. Then we have

$$[[\sigma_0,\tau]]=\langle\tau,A_0\rangle(\sigma_0+A_0^\flat)$$

$$[[\sigma,\tau]]=\frac{1}{2}(\langle\tau,A_0\rangle\sigma-\langle\sigma,A_0\rangle\tau)-\frac{1}{2}\langle\sigma\wedge\tau,\Omega^0\rangle A_0^\flat.$$

This is independent of l and is equal to $[\cdot,\cdot\cdot]_1$. Thus, f is a Lie algebra isomorphism between $[\cdot,\cdot\cdot]_l$ and $[\cdot,\cdot\cdot]_1$.

Next, we show that $[\cdot,\cdot\cdot]_1$ and $[\cdot,\cdot\cdot]_0$ are never isomorphic. We can take a symplectic basis $\{\sigma_1,\sigma_2,\cdots,\sigma_{2n-1},\sigma_{2n}\}$ with respect to Ω^0 so that

$$[\sigma_0,\sigma_1]_l=\sigma_0+l\sigma_2$$

$$[\sigma_0,\sigma_k]_l=0\qquad(k\geqq 2)$$

$$[\sigma_1,\sigma_k]_l=-\frac{\sigma_k}{2}-\frac{\Omega^0(\sigma_1,\sigma_k)}{2}\sigma_2\qquad(k\geqq 2)$$

$$[\sigma_j,\sigma_k]_l=-\frac{\Omega^0(\sigma_j,\sigma_k)}{2}\sigma_2\qquad(j,\ k\geqq 2)$$

where $A_0^\flat=\sigma_2$ and $\Omega^0(\sigma_1,\sigma_2)=\Omega^0(\sigma_3,\sigma_4)=\cdots=\Omega^0(\sigma_{2n-1},\sigma_{2n})=1$. For each vector v, we consider the kernel of ad_v that is the stabilizer at v of adjoint action, and its dimension. We call here this number by the nullity of v, namely $nullity(v)=\dim Ker(ad_v)$. For example, $nullity(\sigma_0)=2n$, $nullity(\sigma_1)=1$, $nullity(\sigma_2)=2n$, $nullity(\sigma_k)=2n-1 (k\geqq 3)$. The nullity is invariant under Lie algebra isomorphisms, namely if $f:\mathfrak{g}\to\mathfrak{l}$ is a Lie algebra isomorphism, then $nullity_{\mathfrak{g}}(v)=nullity_{\mathfrak{l}}(f(v))$ for each $v\in\mathfrak{g}$. On the Lie algebra $(\mathfrak{h}^*, [\cdot, \cdot\cdot]_l)$ we can calculate the nullity of an arbitrary vector $\sigma=\sum_{j=0}^{2n} a_j\sigma_j$ as

$$nullity(\sigma)=1 \quad \text{if } a_1\neq 0$$

$$nullity(\sigma)=2n-1 \quad \text{if } a_1=0 \text{ and } a_j\neq 0 \text{ for some } j(\geqq 3)$$

$$nullity(\sigma)=2n \quad \text{if } a_1=0,\ a_j=0(j\geqq 3), \text{ but } a_0 \text{ or } a_2\neq 0.$$

(It may be remarkable that the statements above do not include the parameter l.)

Suppose that there is a Lie algebra isomorphism Ψ between $(\mathfrak{h}^*, [\cdot, \cdot\cdot]_1)$ and $(\mathfrak{h}^*, [\cdot, \cdot\cdot]_0)$. Then, considering the nullity of each vector yields that the matrix representation of Ψ has the form

$$\Psi(\sigma_0)=b_{00}\sigma_0+b_{02}\sigma_2$$

$$\Psi(\sigma_1)=\sum_{j=0}^{2n} b_{1j}\sigma_j \qquad (b_{11}\neq 0)$$

$$\Psi(\sigma_2)=b_{20}\sigma_0+b_{22}\sigma_2$$

$$\Psi(\sigma_j)=\sum_{j=0}^{2n} b_{jk}\sigma_j \qquad (b_{j1}=0).$$

Since Ψ is a linear isomorphism, we have

$$b_{11}(b_{00}b_{22}-b_{02}b_{20})\,Det[(b_{jk})_{j,k\geqq 3}]\neq 0\,.$$

$\Psi([\sigma_0, \sigma_1]_1)=[\Psi\sigma_0, \Psi\sigma_1]_0$ implies

$$a_{00}(1-a_{11})+a_{20}=0 \quad \text{and} \quad a_{02}(1-a_{11})+a_{22}=0\,.$$

$\Psi([\sigma_1, \sigma_2]_1)=[\Psi\sigma_1, \Psi\sigma_2]_0$ implies

$$a_{20}(1-a_{11})=0 \quad \text{and} \quad a_{22}(1-a_{11})=0\,.$$

These 5 equations can not coincide, thereby we have no Lie algebra isomorphism between $(\mathfrak{h}^*, [\cdot, \cdot\cdot]_1)$ and $(\mathfrak{h}^*, [\cdot, \cdot\cdot]_0)$.

Finally, we show that these Lie algebras are solvable. It comes from direct computation of their derived algebras. In fact, $\mathcal{D}^2\mathfrak{g}=\boldsymbol{R}z_2$ and $\mathcal{D}^3\mathfrak{g}=0$. In particular, since $-ad_{z_1}z_j=\begin{cases} z_0+lz_2 & j=0 \\ z_2 & j=2 \end{cases}$, we have

$$(-ad_{z_1})^k z_0=z_0+klz_2 \qquad (k>1).$$

Thus, they are not nilpotent. ∎

REMARK. In the 3-dimensional case, we can compute Tasaki-Umehara invariant. If $l=0$, then the matrix of structure constants is skew-symmetric.

If $l \neq 0$, then $\chi = 4$ and is independent of l. Thus, we have two equivalence classes induced by multiplicative Poisson structures of the second type.

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