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HOMOGENEOUS TOTALLY REAL SUBMANIFOLDS OF S^6

By

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It is well-known that the 6-dimensional sphere S^6 admits an almost complex structure. Among all submanifolds of almost complex manifold, there are two typical classes of submanifolds: namely the class of holomorphic submanifolds and the class of totally real submanifolds. By using a recent result of Harvey and Lawson [5], a cone over a 2-dimensional holomolphic (or a 3-dimensional totally real) submanifold of S^6 is a stable minimal submanifold of R^7.

On the existence of such submanifolds, a satisfying result on holomorphic submanifolds was obtained by R. Bryant [1], i.e., he proved that every compact Riemann surface can be realized as a holomorphic curve of S^6.

On the contrary, we do not have such a satisfying result on the existence of 3-dimensional totally real submanifolds of S^6. In this paper we classify 3-dimensional compact totally real submanifolds of S^6, which are obtained as orbits of closed subgroups of G_2.

1. Cayley algebra and the exceptional simple Lie group G_2

In this section we give a brief review on Cayley algebra and the exceptional simple Lie group G_2.

Let $\boldsymbol{H}$ be the skew field of all quaternions. Then the Cayley algebra $\boldsymbol{Ca}$ over $\boldsymbol{R}$ is $\boldsymbol{Ca}=\boldsymbol{H}+\boldsymbol{H}$ with the following multiplication:

$$(q, r)\cdot(s, t)=(qs-\bar{t}r,\ tq+r\bar{s}),\quad q,\ r,\ s,\ t\in\boldsymbol{H}$$

where "—" means the conjugation in $\boldsymbol{H}$. We define a conjugation in $\boldsymbol{Ca}$ by $\overline{(q, r)}=(\bar{q}, -r)$, q, $r\in\boldsymbol{H}$, and an inner product $\langle\ ,\ \rangle$ by

$$\langle x, y\rangle=(x\cdot\bar{y}+y\cdot\bar{x})/2,\quad x,\ y\in\boldsymbol{Ca}.$$

Let 1, i, j, k be the standard basis of $\boldsymbol{H}$. Then $e_0=(1, 0)$, $e_1=(i, 0)$, $e_2=(j, 0)$, $e_3=(k, 0)$, $e_4=(0, 1)$, $e_5=(0, i)$, $e_6=(0, j)$, $e_7=(0, k)$ form an orthonormal basis of $\boldsymbol{Ca}$. We put

$$\boldsymbol{Ca}_0=\{x\in\boldsymbol{Ca}\mid x+\bar{x}=0\}=\textstyle\sum_{j=1}^{7}\boldsymbol{R}e_j.$$

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Then we have the following multiplication table

(1.1) $e_i \cdot e_j=$

i/j	1	2	3	4	5	6	7
1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The Cayley algebra $\boldsymbol{Ca}$ is neither commutative nor associative. But we have the following

LEMMA 1.1. (1) *If* $x, y \in \boldsymbol{Ca}_0$, *then* $x \cdot y = -y \cdot x$.
(2) *For any* $x, y, z \in \boldsymbol{Ca}$,

$$\bar{x} \cdot (x \cdot y) = (\bar{x} \cdot x) \cdot y\,,$$
$$\langle x \cdot y,\ x \cdot z\rangle = \langle x, x\rangle\langle y, z\rangle\,.$$

(3) *Let* x, y, z $\boldsymbol{Ca}$ *be mutually orthogonal unit vectors. Then*

$$x \cdot (y \cdot z) = y \cdot (z \cdot x) = z \cdot (x \cdot y)\,.$$

For the proof, we refer to [4].

It is well-known that the group of all automorphisms of $\boldsymbol{Ca}$ is a compact connected simple Lie group of type $\mathfrak{G}_2$ ([4]). So we denote it by G_2. Then G_2 leaves the vector e_0 and the subspace $\boldsymbol{Ca}_0$ invariant. Furthermore G_2 leaves the inner product $\langle\,,\rangle$ invariant. So we may regard G_2 as a subgroup of $SO(7)=SO(\boldsymbol{Ca}_0)$.

LEMMA 1.2. *Let* v_1, v_2, v_3 *be mutually orthogonal unit vectors in* $\boldsymbol{Ca}_0$ *with* $\langle v_1 \cdot v_2, v_3\rangle = 0$. *Then there exists a (unique) automorphism* g *of* $\boldsymbol{Ca}$ *such that* $g(e_i)=v_i$, $i=1, 2, 3$.

For the proof of Lemma 1.2, we refer to [5].

Let G_{ij}, $1 \leqq i \neq j \leqq 7$ be the skew symmetric transformation on $\boldsymbol{Ca}_0$ defined by

$$G_{ij}(e_k) = \begin{cases} e_i, & \text{if } k=j\,, \\ -e_j, & \text{if } k=i\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then the Lie algebra $\mathfrak{G}_2$ of G_2 is spanned by the following vectors in the Lie algebra $\mathfrak{so}(7)$ of $SO(7)$.

$$aG_{23}+bG_{45}+cG_{76}\,,$$

$$aG_{31}+bG_{46}+cG_{57}\,,$$

$$aG_{12}+bG_{47}+cG_{65}\,,$$

$$aG_{51}+bG_{73}+cG_{62}\,,$$

$$aG_{14}+bG_{72}+cG_{36}\,,$$

$$aG_{17}+bG_{24}+cG_{53}\,,$$

$$aG_{61}+bG_{34}+cG_{25}\,,\quad a+b+c=0,\ a,\ b,\ c\in \boldsymbol{R}\,.$$

§ 2. Stability of cones over totally real submanifolds

In this section we briefly summalize the results of Harvey and Lawson [5] and study the cones over 3-dimensional totally real submanifolds of S^6.

Let M be an n-dimensional Riemannian manifold and let $G_p(M)$ be the bundle of p-planes of M. Then we can regard $G_p(M)$ as a subset of the p-th exterior power $\Lambda^p(M)$ of the tangent bundle of M in a natural manner. Then any exterior p-form on M can be considered as a function on $G_p(M)$. The comass of an exterior p-form ϕ is defined by

$$||\phi||^*=\sup_{\xi\in G_p(M)}\phi(\xi)\,.$$

Assuming $||\phi||^*=1$, we put

$$G(\phi)=\{\xi\in G_p(M)|\phi(\xi)=1\}\,.$$

A p-dimensional, oriented C^1-submanifold S of M is called a *ϕ-manifold* if the oriented p-plane $T_x(S)$ is contained in $G(\phi)$ for all $x\in S$.

THEOREM 2.1 (Harvey and Lawson, [5]). *Let ϕ be a closed* p*-form with $||\phi||^*=1$ and S be a compact ϕ-manifold. Then $Vol(S)\leqq Vol(S')$ for any compact submanifold S' of M which is homologous to S.*

In [5], Harvey and Lawson considered 2 calibrations on $\boldsymbol{Ca}_0$. Let ϕ be a trilinear function on $\boldsymbol{Ca}_0$ defined by

$$\phi(x,\ y,\ z)=\langle x,\ y\cdot z\rangle,\ x,\ y,\ z\in \boldsymbol{Ca}_0\,.$$

Then by Lemma 1.1 it is easily seen that ϕ is a closed 3-form on $\boldsymbol{Ca}_0$. Furthermore ϕ has comass one ([5]). We fix an orientation on $\boldsymbol{Ca}_0$ such that $e_1, e_2, \cdots, e_7$ is an oriented basis and let * be the Hodge star operator. Then $*\phi$ is a closed 4-form and also has comass one ([5]). A ϕ-manifold is called an *associative submanifold* and a $*\phi$-manifold is called a *coassociative submanifold.*

Let S^6 be the unit sphere in $\boldsymbol{Ca}_0$ centered at the origin. Then S^6 has an almost complex structure J defined by

$$J_p(X)=p\cdot X, \quad X\in T_p(S^6).$$

From the definition, J is preserved by G_2. A submanfold M of S^6 is called a *holomorphic submanifold* (resp. *totally real submanifold*) if $J(T_p(M))=T_p(M)$ (resp. $J(T_p(M))$ is contained in the normal space $N_p(M)$) for any $p\in M$. We denote by CM the cone over M.

THEOREM 2.2. *Let M be a 2-dimensional submanifold of S^6. Then M is a holomorphic submanifold of S^6 if and only if $CM-\{0\}$ is an associative submanifold of $\boldsymbol{Ca}_0$.*

The proof of the above Theorem is easy so that we omit it.

THEOREM 2.3. *Let M be a 3-dimensional submanifold of S^6. Then M is a totally real submanifold of S^6 if and only if $CM-\{0\}$ is a coassociative submanifold of $\boldsymbol{Ca}_0$.*

It is well-known that a submanifold M of S^N is a minimal submanifold if and only if $CM-\{0\}$ is a minimal submanifold of $\boldsymbol{R}^{N+1}$. Thus we have the following

COROLLARY 2.4 (Ejiri, [3]). *Any 3-dimensional totally real submaifold of S_1^6 is a minimal submanifold.*

Proof of Theorem 2.3. "if" part. Let p be a point of M, and put $p=u_4$. Let u_5, u_6, u_7 be an orthonormal basis of $T_p(M)$ and u_1, u_2, u_3 be an orthonormal basis of the normal space $N_p(M)$ of M at p in S^6 such that $u_1, u_2, \cdots, u_7$ is oriented. Since $*\phi(u_4\wedge u_5\wedge u_6\wedge u_7)=\phi(u_1\wedge u_2\wedge u_3)=\pm 1$, we get $u_1\cdot u_2=\pm u_3$, $u_2\cdot u_3=\pm u_1$ and $u_3\cdot u_1=\pm u_2$. By Lemma 1.1, we get

$$\begin{aligned}\langle J(u_1),\ u_1\rangle&=\langle u_4\cdot u_1,\ u_1\rangle=-\langle u_1\cdot u_4,\ u_1\rangle\\&=-\langle u_4,\ \bar{u}_1,\ u\rangle=0,\\ \langle J(u_1),\ u_2\rangle&=\langle u_4\cdot u_1,\ u_2\rangle=-\langle u_1\cdot u_4,\ u_2\rangle\\&=-\langle u_4,\ \bar{u}_1\cdot u_2\rangle=0,\\ \langle J(u_1),\ u_3\rangle&=\langle u_4\cdot u_1,\ u_3\rangle=-\langle u_1\cdot u_4,\ u_3\rangle\\&=-\langle u_4,\ \bar{u}_1\cdot u_3\rangle=0,\end{aligned}$$

i.e., $J(u_1)$ is contained in $T_p(M)$. Similarly we see that $J(u_2)$ and $J(u_3)$ are contained in $T_p(M)$. Since J is non-singular and dim $N_p(M)=$dim $T_p(M)$, we get $J(T_p(M))=N_p(M)$.

"only if" part. Let u_5, u_6, u_7 be an orthonormal basis of $T_p(M)$. By a simple calculation, $(\nabla_{u_5}J)(u_6)$ is equal to the tangential part of $u_5\cdot u_6$ to S^6, where ∇ is the covariant derivative of S^6. Since M is a totally real submanifold, $u_5\cdot u_6$ is normal to p. So $(\nabla_{u_5}J)(u_6)=u_5\cdot u_6$. In [3], Ejiri proved that $(\nabla_{u_5}J)(u_6)$ is normal to M. By Lemma 1.1 (ii), $u_5\cdot u_6$ is normal to $p\cdot u_5$ and $p\cdot u_6$. Since $N_p(M)$ is of dimension 3 and $u_5\cdot u_6$, $p\cdot u_7$ are unit vectors, $u_5\cdot u_6=\pm p\cdot u_7$. Similarly $u_6\cdot u_7=\pm p\cdot u_5$ and $u_7\cdot u_5=\pm p\cdot u_6$. Then it is easily seen that $p\cdot u_5, p\cdot u_6, p\cdot u_7$ form an orthonormal basis of $T_p(M)$. Thus by Lemma 1.1 we obtain

$$*\phi(p\wedge u_5\wedge u_6\wedge u_7)=\pm\phi(p\cdot u_5\wedge p\cdot u_6\wedge p\cdot u_7)$$
$$=\pm\langle p\cdot u_5, (p\cdot u_6)\cdot(p\cdot u_7)\rangle=\pm 1.$$

Therefore $CM-\{0\}$ is a coassociative submanifold. Q.E.D.

LEMMA 2.5. *Let M, M' be two totally real submanifolds of constant curvature 1/16 in S_1^6. Then there exists $g\in G_2$ such that $g(M)=M'$.*

Proof. In [3], Ejiri proved that the normal connection $\nabla^\perp$ of a totally real submanifold M of S_1^6 is

$$\nabla^\perp{}_X JY=X\cdot Y+J(\nabla_X Y),\ X,\ Y\in TM. \tag{2.1}$$

Furthermore he proved that if M is a space of constant curvature then there exists a local orthonormal frame field e_1, e_2, e_3 such that

$$\alpha(e_1, e_1)=15^{1/2}\ Je_1/2,$$
$$\alpha(e_2, e_2)=(-15^{1/2}\ Je_1+10^{1/2}\ Je_2)/4,$$
$$\alpha(e_3, e_3)=(-5^{1/5}\ Je_1-10^{1/2}\ Je_2)/4,$$
$$\alpha(e_1, e_2)=-5^{1/2}\ Je_2,$$
$$\alpha(e_2, e_3)=-10^{1/2}\ Je_3/4,$$
$$\alpha(e_1, e_3)=-5^{1/2}\ Je_3/4,$$
$$\nabla_{e_i}e_i=0,\ i=1,\ 2,\ 3,$$
$$\nabla_{e_1}e_2=-\nabla_{e_2}e_1=-e_3/4,$$
$$\nabla_{e_2}e_3=-\nabla_{e_3}e_1=-e_1/4,$$
$$\nabla_{e_3}e_1=-\nabla_{e_1}e_3=-e_2/4,$$
$$J(e_1, e_2)=e_3,\ J(e_2, e_3)=e_1,\ J(e_3, e_1)=e_2.$$

where α is the second fundamental form of M.

Take a point $p\in M$ and a point $p'\in M'$. Let e_1, e_2, e_3 be an orthonormal frame of M at p and e_1', e_2', e_3', be an orthonormal frame of M' at p' with the above properties. Then by a well-known rigidity theorem there exists a rigid motion

$\sigma \in O(7)$ such that $\sigma(p)=p'$, $\sigma(e_i)=e_i'$, $i=1, 2, 3$. Since σ preserves the second fundamental form α, $\sigma(Je_i)=Je_i'$, $i=1, 2, 3$. Put $v_1=e_2 \cdot e_3$, $v_2=-J(e_3)$, $v_3=e_3 \cdot e_1$, $v_4=p$, $v_5=e_1$, $v_6=e_3$ and $v_7=e_2$. Define $v_1', \cdots, v_7'$ in a similar manner. Then by Lemma 1.1, it is easily seen that $v_1, \cdots, v_7$ and $v_1', \cdots, v_7'$ satisfy the same table of multiplication (1.1), i.e., σ is contained in G_2. Q.E.D.

§ 3. 3-dimensional closed subgroups of G_2

Let $\mathfrak{G}$ be a compact simple Lie algebra and $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{G}$. Let $\mathfrak{l}$ be a complex simple 3-dimensional subalgebra of $\mathfrak{G}^C$. Then there exists a basis H, X_+, X_- of $\mathfrak{l}$ such that

$$[H, X_+]=2X_+,\ [H, X_-]=-2X_-,\ [X_+, X_-]=H. \tag{3.1}$$

We may assume that H is contained in $\mathfrak{t}^C$, in fact in $(-1)^{1/2}\mathfrak{t}^C$. Hence $\alpha(H)$ is a real number for every root α of $\mathfrak{G}^C$ with respect to $\mathfrak{t}^C$. Furthermore $\alpha(H)=0$, 1 or 2 if α is a simple root [2, p. 166]. The weighted Dynkin diagram with weight $\alpha(H)$ added to each vertex α of the Dynkin diagram of $\mathfrak{G}^C$ is called the characteristic diagram of $\mathfrak{l}$. Let $\mathfrak{l}$ and $\mathfrak{l}'$ be 3-dimensional simple Lie subalgebras of $\mathfrak{G}^C$. Then $\mathfrak{l}$ and $\mathfrak{l}'$ are mutually conjugate if and only if $\mathfrak{l}$ and $\mathfrak{l}'$ have the same characteristic diagrams.

Mal'cev [7] classified the 3-dimensional complex simple subalgebras of $\mathfrak{G}_2^C$. From his classification, $\mathfrak{G}_2^C$ has 4 types of 3-dimensional simple subalgebras as follows.

I $\overset{1}{\circ}\Rrightarrow\overset{0}{\circ}$ II $\overset{0}{\circ}\Rrightarrow\overset{1}{\circ}$

III $\overset{2}{\circ}\Rrightarrow\overset{0}{\circ}$ IV $\overset{2}{\circ}\Rrightarrow\overset{2}{\circ}$

Let $\mathfrak{l}$ be a 3-dimensional simple subalgebra of $\mathfrak{G}_2$. Then the complexification $\mathfrak{l}^C$ of $\mathfrak{l}$ in $\mathfrak{G}_2^C$ corresponds to one of the above 4 characteristic diagrams. As a special case of a Theorem of Siebenthal ([8], p. 252), we have the folowing.

LEMMA 3.1. *Let $\mathfrak{l}$ and $\mathfrak{l}'$ be 3-dimensional simple subalgebras of $\mathfrak{G}_2$. If $\mathfrak{l}^C$ and $\mathfrak{l}'^C$ correspond to the characteristic diagram* $\overset{2}{\circ}\Rrightarrow\overset{2}{\circ}$, *then $\mathfrak{l}$ and $\mathfrak{l}'$ are conjugate in $\mathfrak{G}_2$.*

Similarly we can prove

LEMMA 3.2. *Let $\mathfrak{l}$ and $\mathfrak{l}'$ be 3-dimensional simple subalgebras of $\mathfrak{G}_2$. If $\mathfrak{l}^C$ and*

$\mathfrak{l}'^{C}$ *correspond to the characteristic diagram* $\overset{1}{\circ}\Longrightarrow\overset{0}{\circ}$ *or* $\overset{0}{\circ}\Longrightarrow\overset{1}{\circ}$, *then* $\mathfrak{l}$ *and* $\mathfrak{l}'$ *are conjugate in* $\mathfrak{G}_2$.

Now we give here an example of a basis X_1, X_2, X_3 of $\mathfrak{l}$ with $[X_1, X_2]=2X_3$, $[X_2, X_3]=2X_1$, $[X_3, X_1]=2X_2$. If $\mathfrak{l}$ corresponds to the characteristic diagram I, then

$$X_1=-G_{45}+G_{76},$$
$$X_2=-G_{46}+G_{57}, \tag{3.6}$$
$$X_3=-G_{47}+G_{65}.$$

If $\mathfrak{l}$ corresponds to the characteristic diagram II, then

$$X_1=-2G_{23}+G_{45}+G_{76},$$
$$X_2=-2G_{31}+G_{46}+G_{57}, \tag{3.7}$$
$$X_3=-2G_{12}+G_{47}+G_{65}.$$

If $\mathfrak{l}$ corresponds to the characteristic diagram IV, then

$$X_1=4G_{32}+2G_{54}-6G_{76},$$
$$X_2=6^{1/2}(G_{37}+G_{26}-2G_{15})+10^{1/2}(G_{42}-G_{35}), \tag{3.8}$$
$$X_3=6^{1/2}(G_{63}+G_{27}-2G_{41})+10^{1/2}(G_{25}-G_{34}).$$

LEMMA 3.3. *Let* $\mathfrak{l}$ *be a 3-dimensional simple subalgebra of* $\mathfrak{G}_2$. *If* $\mathfrak{l}^C$ *corresponds to the characteristic diagram* $\overset{2}{\circ}\Longrightarrow\overset{0}{\circ}$, *then* $\mathfrak{l}$ *is spanned by the following basis* X_1, X_2, X_3 *for some* θ:

$$X_1=-2G_{21}-2G_{65},$$
$$X_2=-2\cos\theta(G_{32}+G_{76})-2\sin\theta(G_{72}+G_{63}), \tag{3.9}$$
$$X_3=-2\cos\theta(G_{31}+G_{75})-2\sin\theta(G_{53}+G_{71}).$$

Proof. A simple computation shows that $\mathfrak{l}^C$ is conjugate to the Lie subalgebra spanned by

$$H=\ 2(-1)^{1/2}(G_{21}+G_{65}),$$
$$X_+=-2(G_{32}+G_{76})+2(-1)^{1/2}(G_{31}+G_{75}), \tag{3.10}$$
$$X_-=\ 2(G_{32}+G_{76})+2(-1)^{1/2}(G_{31}+G_{75}).$$

Hence it is easily seen that $\sum_{i=1}^{3}\boldsymbol{R}e_i$, $\boldsymbol{R}e_4$, $\sum_{i=5}^{7}\boldsymbol{R}e_i$ are invariant irreducible components of $\overset{C}{\boldsymbol{Ca}}_0$ under the action of the subalgebra spanned by H, X_+ and X_- defined by (3.10). Therefore $\boldsymbol{Ca}_0$ has 2 invariant irreducible components V_1, V_2

of dimension 3 and an invariant irreducible component V_0 of dimension 1 under the action of $\mathfrak{k}$. Let L be the Lie subgroup of G_2 generated by $\mathfrak{k}$. Remark that L is isomorphic to $SO(3)$ and the actions of L on V_1 and V_2 are equivalent to the standard action of $SO(3)$ on $\boldsymbol{R}^3$. Let v_4 be a unit vector in V_0. Take a one parameter subgroup K in L. Then there are determined (up to sign) unit vectors v_1 in V_1 and v_5 in V_2. Since $v_1 \cdot v_4$ is also a K-fixed vector and is normal to v_1 and v_4, $v_1 \cdot v_4$ is equal to v_5 or $-v_5$. By a change of sign (if necessary) we have $v_1 \cdot v_4 = v_5$. Let v_2 be a unit vector in V_1 which is orthogonal to v_1 and K' be the isotropy subgroup at v_2. Then by a similar argument, we can choose a unit vector v_6 in V_2 such that $v_2 \cdot v_4 = v_6$. Put $v_3 = v_1 \cdot v_2$ and $v_7 = v_3 \cdot v_4$. Then by Lemma 1.2, there exists an automorphism g of $\boldsymbol{Ca}$ such that $g(e_i) = v_i$ for $i=1, 2, 4$. Since g is an automorphism of $\boldsymbol{Ca}$, we have $g(e_3) = v_3$, $g(e_5) = v_5$, $g(e_6) = v_6$ and $g(e_7) = v_7$. Hence $v_1, v_2, \cdots, v_7$ satisfy the same multiplication table (1.1) as $e_1, e_2, \cdots, e_7$. Let v_3', be a unit vector in V_1 which is orthogonal to v_1 and v_2. Then v_3' is of the form

$$v_3' = (\cos\theta) v_3 + (\sin\theta) v_7 .$$

Take a suitable basis X_1, X_2, X_3 of $\mathfrak{k}$. Then the restrictions of X_1, X_2 and X_3 to V_1 are represented by the following matrices with respect to the basis v_1, v_2 and v_3':

$$X_1 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Put $v_7' = v_4 \cdot v_3 = -(\sin\theta) v_3 + (\cos\theta) v_7$. Then v_5, v_6 and v_7', form an orthonormal basis of V_2. Since X_1 is contained in $\mathfrak{G}_2$,

$$\begin{aligned} X_1(v_5) &= X_1(v_1 \cdot v_4) = X_1(v_1) \cdot v_4 + v_1 \cdot X_1(v_4) \\ &= v_1 \cdot X_1(v_4) = -2v_6 . \end{aligned}$$

By similar calculations, we get the representations of X_1, X_2 and X_3 restricted to V_2 with respect to v_5, v_6 and v_7'. They are of the same form as X_1, X_2 and X_3 as above. Express X_1, X_2 and X_3 with respect to $v_1, \cdots, v_7$. Then we see that $\mathfrak{k}$ is conjugate to the subalgbra spanned by the following basis

$$\begin{aligned} X_1 &= -2G_{21} - 2G_{65} , \\ X_2 &= -2\cos\theta (G_{32} + G_{76}) - 2\sin\theta (G_{72} + G_{63}) , \\ X_3 &= -2\cos\theta (G_{31} + G_{75}) - 2\sin\theta (G_{53} + G_{71}) . \end{aligned}$$

Q.E.D.

§ 4. Homogeneous totally real submanifolds of S^6

In this section we classify 3-dimensional compact homogeneous totally real submanifolds of S^6, which are obtained as orbits of closed subgroups of G_2.

First we study one by one the 4 types of subgroups which are generated by subalgebras listed in § 3. In some cases it is convenient for us to find all orbits which are 3-dimensional minimal submanifolds of S_1^6, since a 3-dimensional totally real submanifold of S_1^6 is a minimal submanifold by Corollary 2.4.

CASE I. $\boldsymbol{R}e_1$, $\boldsymbol{R}e_2$, $\boldsymbol{R}e_3$ and $\sum_{j=4}^{7} \boldsymbol{R}e_j$ are irreducible invariant subspaces so that each orbit is a small sphere or a great sphere. Therefore the orbit we are looking for is a trivial one.

CASE II. This case was studied by Harvey and Lawson [5].

THEOREM 4.1. *Let L be the subgroup of G_2 generated by the subalgebra spanned by X_1, X_2 and X_3 defined by* (3.7). *Then there exists exactly one orbit which is a 3-dimensional totally real submanifold of S^6. It is the orbit through* $(5^{1/2}/3)e_1+(2/3)e_5$, *which we denote by M_1.*

CASE III. For this case we have the following

THEOREM 4.2. *Let L_θ be the subgroup of G_2 generated by the subalgebra spanned by X_1, X_2 and X_3 defined by* (3.5). *Then there exists exactly one orbit under L_θ which is a 3-dimensional totally real submanifold of S^6. It is the orbit through* $(2^{1/2}/2)(e_2+e_5)$, *which we denote by M_2.*

PROOF. In this case, L_θ is isomorphic to $SO(3)$ and the action of L_θ on $\boldsymbol{Ca}_0$ is equivalent to the direct sum of the adjoint action of $SO(3)$ on $\mathfrak{so}(3, \boldsymbol{C})$ and the trivial action of $SO(3)$ on $\boldsymbol{R}$. Therefore by calculating the volume of each orbit ([6]), we can easily see that the only orbit through $p=(2^{1/2}/2)(e_2+e_5)$ is a 3-dimensional minimal submanifold of S^6 under the action of L_θ on $\boldsymbol{Ca}_0$. The tangent space of the orbit at p is spanned by

$$X_1(p)= 2^{1/2}(e_1-e_6),$$

$$X_2(p)=-2^{1/2}(\cos\theta)e_3-2^{1/2}(\sin\theta)e_7,$$

$$X_3(p)= 2^{1/2}(\sin\theta)e_3-2^{1/2}(\cos\theta)e_7.$$

Consulting the multiplication table (1.1), we get

$$J(X_1(p))=p\cdot X_1(p)=2e_4,$$

$$J(X_2(p))=p\cdot X_2(p)=-\cos\theta(e_1+e_6)+\sin\theta(-e_2+e_5),$$

$$J(X_3(p))=p\cdot X_3(p)=\cos\theta(-e_2+e_5)+\sin\theta(e_1+e_6).$$

Therefore the orbit is a totally real submanifold. Q.E.D.

CASE IV. For this case we have the following

THEOREM 4.3. *Let L be the subgroup of G_2 generated by the Lie subalgebra spanned by X_1, X_2 and X_3 defined by* (3.8). *Then, under the action of L on $\boldsymbol{Ca}_0$, there exist exactly 2-types of orbits in S^6 which are 3-dimensional totally real submanifold of S^6 up to the action of G_2. They are*

(1) *the orbit through e_2, which we denote by M_3.*

(2) *the orbit through e_6, which we denote by M_4.*

It is easily seen that M_3 is of constant curvature 1/16. The proof of this Theorem will be given in § 6.

Let M be a compact 3-dimensional totally real submanifold of S^6, which is obtained as an orbit of a closed subgroup L of G_2. It is well-known that the dimension of L is smaller than or equal to 6 ([10]). If dim $L=6$, then M is a space of constant curvature and, by a Theorem of Ejiri, the curvature of M is 1/16 ([3]). And by Theorem 2.5 it (if exists) is congruent to M_3 of Theorem 4.3. It is known that if dim $L\leqq 5$, then dim $L\leqq 4$ ([10]). If dim $L=4$, then the Lie algebra $\mathfrak{l}$ of L must be isomorphic to $\mathfrak{u}(2)$, since L is compact. By a direct calculation we see that it is isomorphic to the Lie subalgebra of $\mathfrak{G}_2$ which is spanned by

$$X_1=-2G_{23}+G_{45}+G_{76},$$

$$X_2=-2G_{31}+G_{46}+G_{57},$$

$$X_3=-2G_{12}+G_{47}+G_{65},$$

$$J=a(G_{45}-G_{76})+b(G_{46}-G_{57})+b(G_{47}-G_{65}),\quad a,\ b,\ c\in\boldsymbol{R}.$$

Let G_s be the Lie subgroup of L whose Lie algebra is $\boldsymbol{R}X_1+\boldsymbol{R}X_2+\boldsymbol{R}X_3$. Then it is easily seen that $L(p)=G_s(p)$ for any $p\in S_1^6$. Thus we have the following

THEOREM 4.4. *Let M be a 3-dimensional totally real submanifold of S_1^6 which is obtained as an orbit of a closed subgroup of G_2. Then M is congruent to one of the M_1, M_2, M_3 or M_4, unless it is a great sphere.*

§ 5. Orbits in a sphere

In this section we prepare some Lemmata to prove Theorem 4.3.

Let G be a Lie subgroup of $SO(N+1)$. Then G acts on the unit sphere S_1^N in $\boldsymbol{R}^{N+1}$ centered at the origin in a natural manner. Take a point p in S_1^N and let

M be the orbit of the action of G through p.

Let $\mathfrak{G}$ be the Lie algebra of G. We denote by A^* the vector field on S_1^N induced by $A\in\mathfrak{G}$. Then, by regarding A as a skew symmetric transfomation on $\boldsymbol{R}^{N+1}$, we have

$$A^*_{|p}=A(p), \quad A\in\mathfrak{G}, \quad p\in S_1^N.$$

Therefore the tangent space of M at p is

$$T_p(M)=\{A(p)|A\in\mathfrak{G}\}.$$

Let $N_p(M)$ be the normal space of M in S_1^N at p. Regard the tangent space $T_p(M)$ and the normal space $N_p(M)$ as subspaces of $\boldsymbol{R}^{N+1}$. Then $\boldsymbol{R}^{N+1}$ is decomposed into the direct sum

$$\boldsymbol{R}^{N+1}=\boldsymbol{R}p+T_p(M)+N_p(M). \tag{5.1}$$

For a vector X in $\boldsymbol{R}^{N+1}$, we denote by X^T (resp. X^N) the $T_p(M)-$(resp. $N_p(M)-$) component of X with respect to the decomposition (5.1).

LEMMA 5.1. *Let G be a Lie subgroup of $SO(N+1)$. Let α be the second fundamental form of the orbit $M=G(p)$. Then*

$$\alpha(A^*, B^*)_{|p}=(A(B(p)))^N, \tag{5.2}$$

$$\nabla_{B^*}A^*_{|p}=(A(B(p)))^T, \quad A, B\in\mathfrak{G}. \tag{5.3}$$

where ∇ is the Riemannian connection of M.

Proof. Let D be the Riemannian connection of $\boldsymbol{R}^{N+1}$. Then

$$\begin{aligned} D_{B^*}A^*_{|p} &= d/dt_{|t=0}A^*_{|\exp(tB)(p)} \\ &= d/dt_{|t=0}\, A(\exp tB)(p) \\ &= A(B(p)). \end{aligned}$$

Since $\alpha(A^*, B^*)_{|p}=(D_{B^*}A^*_{|p})^N$ and $\nabla_{B^*}A^*_{|p}=(D_{B^*}A^*_{|p})^T$, we get (5.2) and (5.3).

Q.E.D.

LEMMA 5.2. *Let G be a Lie subgroup of $SO(N+1)$ and fix an orbit $M=G(p)$. Let S be the complete connected totally geodesic submanifold of S_1^N such that $T_p(S)=N_p(M)$. Then each G-orbit in S_1^N contains at least one point of S.*

PROOF. Take an arbitrary orbit $M'=G(p')$. Then there exists a point p_1 in M and a point p_2 in M' such that the distance between M and M' is attained by p_1 and p_2. Let τ be the shortest geodesic joining p_1 and p_2. Take an element $\sigma\in G$ such that $\sigma(p_1)=p$. Since σ is an isometry of S_1^N, $\sigma(\tau)$ is also a geodesic and is normal to M at p. Therefore $\sigma(\tau)$ is contained in S and $\sigma(p_2)$ is contained in $S\cap M$. Q.E.D.

Now we consider the case that G is isomorphic to $SU(2)$ or $SO(3)$. Let B be the Killing form of $\mathfrak{su}(2)$. Then the basis X_1, X_2 and X_3 with $[X_1, X_2]=2X_3$, $[X_2, X_3]=2X_1$ and $[X_3, X_1]=2X_2$ is orthonormal with respect to $-B/8$. Let g_0 be the Riamannian metric on G which is the bi-invariant extension of $-B/8$.

LEMMA 5.3 (Sugahara, [9]). *Let g be an inner product on $\mathfrak{su}(2)$. Then there exists an element σ in G such that*
(i) $X'_i=Ad(\sigma)(X_i)$, $i=1, 2, 3$, *are mutually orthogonal with respect to g.*
(ii) $g=\lambda_1\omega_1{}^2+\lambda_2\omega_2{}^2+\lambda_3\omega_3{}^2$, *where λ_i are positive constants and* $\omega_i(\cdot)=g_0(X_i, \cdot)$, $i=1, 2, 3$.

REMARK 5.4. (i) *Put* $\sigma=\exp(\pi X_i/4)$. *Then* $Ad(\sigma)(X_1)=X_1$, $Ad(\sigma)(X_2)=X_3$ *and* $Ad(\sigma)(X_3)=-X_2$ *so that λ_2 and λ_3 of Lemma* 5.3 *can be permuted. Similarly λ_1 and λ_2* (resp. *λ_1 and λ_3*) *are permuted by* $Ad(\exp(\pi X_3/4))$ (resp. $Ad(\exp(\pi X_2/4))$).
(ii) *(G, g) is a space of constant curvature k if and only if* $\lambda_1=\lambda_2=\lambda_3=1/k$, i.e., $g=(1/k)g_0$.

LEMMA 5.5 (Sugahara, [9]). *Let X_1', X_2' and X_3' be as in Lemma* 5.3. *Then the one parameter subgroups* $\tau_{X_i'}(t)=\exp tX_i'$, $i=1, 2, 3$, *are geodesics of* (G, g).

Let (V, ρ) be an orthogonal representation of G and $\langle , \rangle$ be a G-invariant inner product on V. Let $M=G(p)$ be an orbit in the unit sphere S_1 through p.

LEMMA 5.6. *If* $\dim M=3$, *then there exists an element σ in G such that*

$$\langle\rho(X_i)(\sigma(p)),\ \rho(X_j)(\sigma(p))\rangle=0 \text{ for } i\neq j.$$

Proof. Define a map $f: G\longrightarrow S_1$ by

$$f(\sigma)=\rho(\sigma)(p), \quad \sigma\in G. \tag{5.4}$$

Then $f_*(X_i)=\rho(X_i)(p)$. Let g be the metric on G induced by f. Then g is a left invariant metric. Consider the inner product $g_{|e}$ on the tangent space $T_e(G)$ at the unit element e. Then by Lemma 5.3 there exists an element σ in G such that $Ad(\sigma^{-1})(X_i)$, $i=1, 2, 3$, are mutually orthogonal with respect $g_{|e}$. Let R_σ and L_σ be the right and left translations by σ respectively. Then we have

$$f_*(dR_\sigma(X))=d/dt_{|t=0}f(\exp(tX)\sigma(p))=\rho(X)(\sigma(p)),\ X\in\mathfrak{su}(2),\ p\in S_1. \tag{5.5}$$

Since $Ad(\sigma^{-1})(X_i)$ and $Ad(\sigma^{-1})(X_j)$ are orthogonal if $i\neq j$, it follows from (5.5) that

$$0=g(Ad(\sigma^{-1})(X_i),\quad Ad(\sigma^{-1})(X_j))$$

$$=g(dL_{\sigma^{-1}}(dR_\sigma(X_i)),\quad dL_{\sigma^{-1}}(dR_\sigma(X_j)))$$

$$=g(dR_\sigma(X_i)),\quad dR_\sigma(X_j))$$

$$=\langle f_*(dR_\sigma(X_i)),\quad f_*(dR_\sigma(X_j))\rangle$$

$$=\langle \rho(X_i)(\sigma(p)),\quad \rho(X_j)(\sigma(p))\rangle. \qquad \text{Q.E.D.}$$

Hereafter we may assume

$$\langle \rho(X_i)(p),\ \rho(X_j)(p)\rangle=0,\ i\neq j, \tag{5.6}$$

if the orbit $M=G(p)$ is of dimension 3.

LEMMA 5.7. *Let $M=G(p)$ be a 3-dimensional orbit. Then $f: G\longrightarrow S_1$ defined by* (5.4) *is a minimal immersion if and only if*

$$\textstyle\sum_{i=1}^3 X_i(X_i(p))/\lambda_i=-3p,$$

where $\lambda_i=\langle X_i(p),\ X_i(p)\rangle$, $i=1, 2, 3$.

PROOF. Since (5.6) holds at the initial point p, $X_i'=X_i/\lambda_i^{1/2}$ is an orthonormal frame of $T_e(G)$. By the G-equivalence of the immersion f, we have only to verify $\sum_{i=1}^3 \alpha(X_i',\ X_i')_{|e}=0$. Since $\tau_{X_i'}(t)=\exp tX_i'$ are geodesics of $(G,\ g)$, by (5.2) we get

$$\nabla_{X_i*}X_i*_{|e}=(X_i(X_i(p))^T=0. \tag{5.8}$$

By (5.3) f is a minimal immersion if and only if $\sum_{i=1}^3 \alpha(X_i',\ X_i')=0$. Therefore $\sum_{i=1}^3 X_i'(X_i'(p))$ is proportional to p if and only if f is a minimal immersion.

Now we assume that $\sum_{i=1}^3 X_i'(X_i'(p))=\text{cp}$ for some constant c. Then

$$c=\langle \textstyle\sum_{i=1}^3 X_i(X_i(p))/\lambda_i,\ p\rangle$$

$$=-\textstyle\sum_{i=1}^3 \langle X_i(p),\ X_i(p)\rangle/\lambda_i$$

$$=-3. \qquad \text{Q.E.D.}$$

§6. Proof of Theorem 4.3

Let L be the Lie subgroup of G_2 generated by the Lie subalgebra defined by (3.8) and let $p=\sum_{j=1}^7 x_je_j$ be a point on S_1^6. Then the tangent space of $L(p)$ is spanned by

$$X_1(p)=-4x_3e_2+4x_2e_3-2x_5e_4+2x_4e_5-6x_7e_6+6x_6e_7,$$

$$X_2(p)=-2\cdot 6^{1/2}x_5e_1+(6^{1/2}x_6-10^{1/2}x_4)e_2+(6^{1/2}x_7-10^{1/2}x_5)e_3$$
$$+10^{1/2}x_2e_4+(2\cdot 6^{1/2}x_1+10^{1/2}x_3)e_5-6^{1/2}x_2e_6-6^{1/2}x_3e_7,$$

$$X_3(p)=2\cdot 6^{1/2}x_4e_1+(6^{1/2}x_7+10^{1/2}x_5)e_2-(6^{1/2}x_6+10^{1/2}x_4)e_3$$
$$+(10^{1/2}x_3-2\cdot 6^{1/2}x_1)e_4-10^{1/2}x_2e_5+6^{1/2}x_3e_6-6^{1/2}x_2e_7.$$

We may assume that (5.6) holds at p, i.e.,

$$(6.1)_1 \qquad 5(x_3x_6-x_2x_7)+15^{1/2}(x_2x_5-x_3x_4)-2x_1x_4=0,$$

$$(6.1)_2 \qquad 5(x_2x_6+x_3x_7)+15^{1/2}(x_2x_4+x_3x_5)-2x_1x_5=0,$$

$$(6.1)_3 \qquad 15^{1/2}(2x_1x_2+x_4x_7-x_5x_6)+6x_4x_5=0.$$

Then by Lemma 5.5, the orbit $L(p)$ is a minimal submanifold of S_1^6 if and only if

$$(6.2)_1 \qquad -24(1/\lambda_2+1/\lambda_3)x_1-4\cdot 15^{1/2}(1/\lambda_2-1/\lambda_3)x_3 \qquad =-3x_1,$$

$$(6.2)_2 \qquad -16(1/\lambda_1+1/\lambda_2+1/\lambda_3)x_2 \qquad =-3x_2,$$

$$(6.2)_3 \qquad -16(1/\lambda_1+1/\lambda_2+1/\lambda_3)x_3-4\cdot 15^{1/2}(1/\lambda_2-1/\lambda_3)x_1=-3x_3,$$

$$(6.2)_4 \qquad -(4/\lambda_1+10/\lambda_2+34/\lambda_3)x_4+2\cdot 15^{1/5}(1/\lambda_2-1/\lambda_3)x_7=-3x_4,$$

$$(6.2)_5 \qquad -(4/\lambda_1+34/\lambda_2+10/\lambda_3)x_5+2\cdot 15^{1/2}(1/\lambda_2-1/\lambda_3)x_7=-3x_5,$$

$$(6.2)_6 \qquad -(36/\lambda_1+6/\lambda_2+6/\lambda_3)x_6+2\cdot 15^{1/2}(1/\lambda_2-1/\lambda_3)x_4 \ =-3x_6,$$

$$(6.2)_7 \qquad -(36/\lambda_1+6/\lambda_2+6/\lambda_3)x_7+2\cdot 15^{1/2}(1/\lambda_2-1/\lambda_3)x_5 \ =-3x_7,$$

where

$$(6.3)_1 \qquad \lambda_1=16(x_2^2+x_3^2)+4(x_4^2+x_5^2)+36(x_6^2+x_7^2),$$

$$(6.3)_2 \qquad \lambda_2=24x_1^2+16(x_2^2+x_3^2)+10x_4^2+34x_5^2+6(x_6^2+x_7^2)$$
$$+4\cdot 15^{1/2}(2x_1x_3-x_4x_6-x_5x_7),$$

$$(6.3)_3 \qquad \lambda_3=24x_1^2+16(x_2^2+x_3^2)+34x_4^2+10x_5^2+6(x_6^2+x_7^2)$$
$$-4\cdot 15^{1/2}(2x_1x_3-x_4x_6-x_5x_7).$$

LEMMA 6.1. *If* $x_1, x_2, \cdots, x_7$, $\sum_{i=1}^7 x_i^2=1$, *satisfy* (6.1) *and* (6.2) *then* $(\lambda_1, \lambda_2, \lambda_3)$ *is* (16, 16, 16), (36, 6, 6) *or* $(20+4\cdot 15^{1/2}, 8, 20-4\cdot 15^{1/2})$ *up to permutation.*

PROOF. By adding $(6.3)_1$, $(6.3)_2$, and $(6.3)_3$, we get

$$(6.4) \qquad \lambda_1+\lambda_2+\lambda_3=48(x_1^2+\cdots+x_7^2)=48.$$

If $x_2\neq 0$, then we get $1/\lambda_1+1/\lambda_2+1/\lambda_3=3/16$ from $(6.2)_2$. Thus we have

$$16=(\lambda_1+\lambda_2+\lambda_3)/16\geqq(\lambda_1\lambda_2\lambda_3)^{1/2}$$
$$\geqq 3/(1/\lambda_1+1/\lambda_2+1/\lambda_3)=16.$$

The equalities hold if and only if $\lambda_1=\lambda_2=\lambda_3=16$. Hereafter we assume $x_2=0$.

CASE 1. $\lambda_i=\lambda_j$ for some i, j, $1\leqq i\neq j\leqq 3$.

By Remark 5.2, we may assume $\lambda_2=\lambda_3$ without loss of generality.

If $x_1\neq 0$, then we get $\lambda_1=\lambda_2=\lambda_3=16$ by $(6.2)_1$, and (6.4). If $x_3\neq 0$, we get $\lambda_1=\lambda_2=\lambda_3=16$ by an argument similar to the case of $x_2\neq 0$. If $x_4\neq 0$ or $x_5\neq 0$, we get $(\lambda_1, \lambda_2, \lambda_3)=(16, 16, 16)$ or $(4, 22, 22)$. If $(\lambda_1, \lambda_2, \lambda_3)=(4, 22, 22)$, then $x_1=x_2=x_3=x_6=x_7=0$ by $(6.2)_1$, $(6.2)_2$, $(6.2)_3$, $(6.2)_6$ and $(6.2)_7$ so that $x_4\cdot x_5=0$ by $(6.1)_3$. But from $(6.3)_2$ and $(6.3)_3$, we get $x_4^2=x_5^2$. This is a contradiction. Thus we have $(\lambda_1, \lambda_2, \lambda_3)\neq(4, 22, 22)$. If $x_6\neq 0$ or $x_7\neq 0$, then we get $(\lambda_1, \lambda_2, \lambda_3)=(16, 16, 16)$ or $(36, 6, 6)$.

CASE 2. λ_1, λ_2 and λ_3 are mutually different.

In this case we may assume $\lambda_1>\lambda_2>\lambda_3$ by Remark 5.2.

If $x_1=0$ (resp. $x_3=0$), then $x_3=0$ by $(6.2)_1$ (resp. $x_1=0$ by $(6.2)_3$).

If $x_4=0$ (resp. $x_6=0$), then $x_6=0$ by $(6.2)_4$ (resp. $x_4=0$ by $(6.2)_6$).

If $x_5=0$ (resp. $x_7=0$), then $x_7=0$ by $(6.2)_5$ (resp. $x_5=0$ by $(6.2)_7$).

By $(6.2)_6$ and $(6.2)_7$, we get

$$x_4x_7-x_5x_6=0.$$

By $(6.2)_4$ and $(6.2)_5$, we get

$$2\cdot 15^{1/2}(1/\lambda_2-1/\lambda_3)(x_5x_6-x_4x_7)-24(1/\lambda_3-1/\lambda_2)x_4x_5=0.$$

Thus we have $x_4x_5=0$. Finally we have the following five subcases.

Subcase 2.1. $x_2=x_5=x_7=0$, $x_1x_3x_4x_6\neq 0$.

Subcase 2.2. $x_2=x_4=x_5=x_6=x_7=0$, $x_1x_3\neq 0$.

Subcase 2.3. $x_1=x_2=x_3=x_4=x_6=0$, $x_5x_7\neq 0$.

Subcase 2.4. $x_2=x_4=x_6=0$, $x_1x_3x_5x_7\neq 0$.

Subcase 2.5. $x_1=x_2=x_3=x_5=x_7=0$, $x_4x_6\neq 0$.

SUBCASE 2.1. Put $\mu_i=1/\lambda_i$, $i=1, 2, 3$. Since $(6.2)_1$ and $(6.2)_3$ (resp. $(6.2)_4$ and $(6.2)_6$) have a non-trivial solution (x_1, x_3) (resp. (x_4, x_6)), we get

$$(6.5)_1 \qquad 0=(1/3)\det\begin{bmatrix}3-24(\mu_2+\mu_3) & 4\cdot 15^{1/2}(\mu_2-\mu_3)\\ 4\cdot 15^{1/2}(\mu_2-\mu_3) & 3-16(\mu_1+\mu_2+\mu_3)\end{bmatrix}$$

$$=3-8(2\mu_1+5\mu_2+5\mu_3)+128(\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3)-80(\mu_2-\mu_3)^2,$$

$$(6.5)_2 \qquad 0=(1/3)\det\begin{bmatrix}3-(4\mu_1+10\mu_2+34\mu_3) & 2\cdot 15^{1/2}(\mu_2-\mu_3)\\ 2\cdot 15^{1/2}(\mu_2-\mu_3) & 3-(36\mu_1+6\mu_2+6\mu_3)\end{bmatrix}$$

$$=3-8(5\mu_1+2\mu_2+5\mu_3)+4(2\mu_1+17\mu_2+5\mu_3)(6\mu_1+\mu_2+\mu_3)-20(\mu_2-\mu_3)^2.$$

By subtracting $(6.5)_2$ from $(6.5)_1$, we get

$$0=(\mu_1-\mu_2)(1-2\mu_1-2\mu_2-12\mu_3).$$

Since $\mu_1>\mu_2$, we get

(6.6) $$\mu_1+\mu_2=1/2-6\mu_3.$$

By adding $(6.5)_1$ and $(6.5)_2$, we get

(6.7) $$0=3-28(\mu_1+\mu_2)-40\mu_3+24(\mu_1+\mu_2)^2+48\mu_3{}^2+80\mu_1\mu_2+272(\mu_1+\mu_2)\mu_3.$$

By substituting (6.6) into (6.7), we get

$$0=-(1-12\mu_3)^2+16\mu_1\mu_2.$$

From (6.4) and (6.6), we obtain

$$\mu_1\mu_2=\mu_3(1-12\mu_3)/2(48\mu_3-1).$$

Therefore we get

$$0=-(1-12\mu_3)^2+8\mu_3(1-12\mu_3)/(48\mu_3-1).$$

As solutions of the above equation, we get $\lambda_3=1/\mu_3=12$, 16, 36. If $\mu_3=1/12$, we get $\mu_1=\mu_2=0$ from (6.6). Thus $\lambda_3\neq 12$. For $\lambda_3=16$, 36 we get $(\lambda_1, \lambda_2, \lambda_3)=(16, 16, 16)$, $(36, 6, 6)$ by (6.6) and (6.8) respectively. Therefore Subcase 2.1 cannot occur.

SUBCASE 2.2. By $(6.2)_1$, $(6.3)_2$ and $(6.3)_3$, we get

$$16x_1{}^4-18x_1{}^2+5=0.$$

As solutions of the above equation we get $x_1{}^2=5/8$, $1/2$. If $x_1{}^2=5/8$, then $(\lambda_1, \lambda_2, \lambda_3)=(6, 36, 6)$, $(6, 6, 36)$. Thus $x_1{}^2\neq 5/8$. If $x_1{}^2=1/2$, $(\lambda_1, \lambda_2, \lambda_3)=(8, 20+4\cdot 15^{1/2}, 20-4\cdot 15^{1/2})$.

SUBCASE 2.3. Let $p=x_5e_5+x_7e_7$ be a solution of (6.1) and (6.2). Then from Remark 5.2, $\exp(\pi X_2/4)(p)$ is also a solution and λ's for $\exp(\pi X_2/4)(p)$ coincide (up to permutation) with λ's for p.

It is easily seen that $V_1=\boldsymbol{R}e_1+\boldsymbol{R}e_3+\boldsymbol{R}e_5+\boldsymbol{R}e_7$ is invariant under $\exp(\pi X_3/4)$. We can see that the restriction $\exp(\pi X_3/4)|V_1$ is $\begin{bmatrix} * & & * & \\ * & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}$ with respect to e_1, e_3, e_5, e_7. Thus Subcase 2.3 is reduced to the Subcase 2.2.

SUBCASE 2.4. Let $(x_1, 0, x_3, 0, x_5, 0, x_7)$, $x_1x_3x_5x_7\neq 0$, be a solution of (6.1) and (6.2) with $(\lambda_1, \lambda_2, \lambda_3)=(a, b, c)$. Then $(-x_1, 0, x_3, -x_5, 0, x_7, 0)$ is also a solution of (6.1) and (6.2) with $(\lambda_1, \lambda_2, \lambda_3)=(a, c, b)$. Thus Subcase 2.4 is reduced to Subcase 2.1.

SUBCASE 2.5. By an argument similar to Subcase 2.3, Subcase 2.5 is reduced to Subcase 2.2. Q.E.D.

Now we prove the existence and uniqueness (up to the action of G_2) of orbits in S_1^6 which are minimal submanifolds of S_1^6 and $(\lambda_1, \lambda_2, \lambda_3)$ of Lemma 6.1 is (16, 16, 16), (36, 6, 6) or $(4, 20-4\cdot 15^{1/2}, 20+4\cdot 15^{1/2})$. First we prove the following

LEMMA 6.2. *There exists an orbit which is a totally real submanifold of* S_1^6 *and* $(\lambda_1, \lambda_2, \lambda_3)$ *of Lemma* 6.1 *is* (16, 16, 16). *Such an orbit is unique up to the action of* G_2.

PROOF. Put $(x_1, x_2, \cdots, x_7)=(0, 1, 0, \cdots, 0)$. Then we can easily verify that $(x_1, \cdots, x_7)$ is a solution of (6.1) and (6.2) with $\lambda_1=\lambda_2=\lambda_3=16$.

Apply the Lemma 5.2 to the orbit $M_4=L(e_6)$. Then each orbit contains at least one point of $S=\{x_1e_1+x_4e_4+x_5e_5+x_6e_6|x_1^2+x_4^2+x_5^2+x_6^2=1\}$.

Assume that an orbit M through $p=x_1e_1+x_4e_4+x_5e_5+x_6e_6$ is a minimal submanifold of S_1^6 with $\lambda_1=\lambda_2=\lambda_3=16$. Then, since the induced metric on L is bi-invariant, (5.6) must hold at any point on the orbit. Therefore we get

$$x_1x_4=x_1x_5=-15^{1/2}x_5x_6+6x_4x_5=0.$$

Under the above conditions, we solve the equation $\lambda_1=\lambda_2=\lambda_3=16$. Then we have $(x_1, x_4, x_5, x_6)=\pm(0, 10^{1/2}/4, 0, \pm 6^{1/2}/2)$, $\pm(5^{1/2}/3, 0, 0, \pm 2/3)$ or $\pm(0, 10^{1/2}/4, \pm 6^{1/2}/4, \pm 30^{1/2}/8)$. It is easily verified that an orbit through each of the above points is a totally real submanifold of S_1^6 and of constant curvature 1/16. Thus by Lemma 2.5, they are congruent under the action of G_2. Q.E.D.

LEMMA 6.3. *There exists a unique orbit which is a totally real submanifold of* S_1^6 *and* $(\lambda_1, \lambda_2, \lambda_3)$ *of Lemma* 6.1 *is* (36, 6, 6) *up to permutation.*

PROOF. By Remark 5.2, we may assume that $\lambda_1=36$, $\lambda_2=\lambda_3=6$. Since $\lambda_1=36$, $(6.3)_1$ and (6.4) yield

$$\begin{aligned}0&=36(x_1^2+\cdots+x_7^2)-16(x_2^2+x_3^2)-4(x_4^2+x_5^2)-36(x_6^2+x_7^2)\\&=36x_1^2+20(x_2^2+x_3^2)+32(x_4^2+x_5^2)\end{aligned}$$

so that $x_i=0$, $i=1, \cdots, 5$. It is easily verified that $(x_1, \cdots, x_5, x_6, x_7)=(0, \cdots, 0, \cos\theta, \sin\theta)$ is a solution of (6.1) and (6.2) with $\lambda_1=36$, $\lambda_2=\lambda_3=6$. By a simple computation, we get $\exp(\theta X_3/6)(e_6)=(\cos\theta)e_6+(\sin\theta)e_7$. Hence these points are contained in exactly one orbit. Furthermore we can easily see that this orbit is a totally real submanifold of S_1^6. Q.E.D.

LEMMA 6.5. *There exists a minimal submanifold of* S_1^6 *such that* $(\lambda_1, \lambda_2, \lambda_3)$ *of Lemma* 6.1 *is equal to* $(8, 20+4\cdot 15^{1/2}, 20-4\cdot 15^{1/2})$. *But it is not a totally real submanifold.*

PROOF. It is easily verified that an orbit through each of the points $\pm(2^{1/2}/2, 0, \pm 2^{1/2}/2, 0, \cdots, 0)$ is a minimal submanifold of S_1^6. In the way of proving Lemma 6.1, we proved that any orbit in S_1^6 which is a minimal submanifold such that $(\lambda_1, \lambda_2, \lambda_3)$ is equal to $(8, 20+4\cdot 15^{1/2}, 20-4\cdot 15^{1/2})$ is congruent to one of the orbits through $\pm(2^{1/2}/2, 0, \pm 2^{1/2}/2, 0, \cdots, 0)$ under the action of G_2. But by direct calculations, they are not totally real submanifolds of S_1^6. Q.E.D.

Added in proof. Recently Dr. Tasaki proved the following; Let $\mathfrak{l}$ and $\mathfrak{l}'$ be semisimple Lie subalgebras of a compact semisimple Lie algebra $\mathfrak{g}$. If $\mathfrak{l}^C$ and $\mathfrak{l}'^C$ are conjugate in $\mathfrak{g}^C$, then $\mathfrak{l}$ and $\mathfrak{l}'$ are conkugate in $\mathfrak{g}$. Thus subalgebras in Lemma 3.9, are conjugate in $\mathfrak{g}_2$.

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