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AVERAGE ORDER OF THE DIVISOR FUNCTIONS WITH NEGATIVE POWER WEIGHT

Dedicated to Professor Katsumi Shiratani on his 60th birthday

By

Makoto ISHIBASHI

1. Introduction.

In this paper we are primarily concerned with the study of the sums of the sum-of-divisors function $\sigma_a(n)$ with negative power weight $n^{-t}(t>0)$, i.e. the sums of the form

$$\sum_{n\leqq x} n^{-t}\sigma_a(n)$$

and we also study the averages of associated error terms. Throughout the paper, we shall refer to [6] as I and whose results we cite e.g. as I-Theorem 1. First we consider the case $0\leqq a-t\in \boldsymbol{Z}$, where $\boldsymbol{Z}$ denotes the set of all rational integers, and prove Theorem 1 which generalizes and in some cases corrects MacLeod's Theorem 8[8]. This case is easier to handle although the needed calculations are rather long. And the special case $a=t$ of this is the starting point of the investigation of the case $a<t$. In this case our approach, which depends on MacLeod's back-track method (Lemma 1 below), is not so effective for a large, and we have to restrict ourselves to the narrower range $0\leqq a\leqq 3$ which, however, covers and interpolates all the formulas obtained by MacLeod. In the case of general t we appeal to induction, and in order to guess the forms of the formulas, we have to calculate out all the cases $t=a+1$, $t=a+2$, $t=a+3$, the last being the initial value of t for induction. Here we take the instructive standpoint and calculated out all these three cases successively and then give the form for $t\geqq a+3$, since each independent formula seems to have its own interest. Except for integral values of a, our interpolating formulas involve various negative powers of x with extremely complicated and clumsy coefficients, but in some cases they are absorbed in the error terms by just multiplying the log-factor. The main reasons why we restrict ourselves to $0\leqq a\leqq 3$ are the complication of these coefficients as well as inapplicability of Lemma 8. However, we state the formulas for $a>3$ as well, only for $t=$

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$a+1$ (Theorem 2), though, for want of a more effective method to compute the sums in question. Of course, in principle, we can continue to calculate further cases $t=a+2$, $a+3$, $\cdots$ starting from the case $t=a+1$, using the back-track method. But the effort made does not seem to deserve it since the main difficulty lies in the explicit determination of the coefficients. Our Theorem 2 covers and at some points corrects Theorem 10, and the first half of Theorem 12, of MacLeod [8], and Corollary to it covers Corollary to his Theorem 10 and gives a new result as a counterpart of the possible Corollary (which is non-existent) to the first half of Theorem 12. In particular, it follows from our Theorem 2 and [5] that

$$\sum_{n\leq x}\frac{d(n)}{n}=\frac{1}{2}\log^2 x+2\gamma\log x+\gamma^2+2\gamma_1+O(x^{-15/22}(\log x)^{69/22})\,.$$

which improves a result of Amitsur [1]. We note that our formulas in the Corollary are proved while MacLeod's are conjectured, since by Lemma 8 an extended form of MacLeod's conjecture is proved. Lemma 8 seems interesting in its own right, indicating a great deal of cancellation of $G_{a,k}(x)$ for x large. Our subsequent Theorem 3-(1) reduces to Theorem 4-(1). However, for the formulas (2) and (3), the case $t=a+2$ is still exceptional, and by only looking at the case $t=a+3$ we can guess the general form of main terms. As in [6], we note that the error estimates claimed in MacLeod's Theorem 12 are not yet proved since they depend on Segal's yet unproven estimates [17], and the error estimates claimed in our Theorems and Corollaries are the best that are known to date.

As in [6], main results of the paper are asymptotic formulas for our sums in terms of $G_{a,k}$-functions. For completeness' sake we collect some information on $G_{a,k}$. They are defined as

$$G_{a,k}(x)=\sum_{n\leq\sqrt{x}}n^a\bar{B}_k\left(\frac{x}{n}\right).$$

for $a\in\boldsymbol{R}$, $k\in\boldsymbol{N}$, where $B_k(y)$ is the k-th Bernoulli polynomial, $\bar{B}_k(y)=B_k(y-[y])$, with $[y]$ the integral part of y, $\boldsymbol{R}$ and $\boldsymbol{N}$ denoting the set of all real numbers and the set of all natural numbers respectively. As regards the order estimates of $G_{a,k}(x)$, there is a famous conjecture due to Chowla and Walum saying (in a more precise form)

$$\alpha_k(a)=\begin{cases} a/2+1/4 & (a\geqq -1/2),\\ 0 & (a\leqq -1/2),\end{cases}\tag{S}$$

where $\alpha_k(a)$ denotes the least α for which $G_{a,k}(x)=O\ (x^{\alpha+\varepsilon})$ for each $\varepsilon>0$

(For detailed introduction to these functions we refer the reader to Pétermann [12, 13]). The looser form ($S\leqq$) of the conjecture (S), where we replace the equality sign in (S) by $\leqq$, has been known for $k\geqq 2$, $a\geqq 1/2$, and non-trivial eatimates have been known for $k\geqq 2$, $0\leqq a<1/2$, $k=1$, $-1<a<0$. Very recently, Pétermann has succeeded in sharpening the last estimates by applying the theory of one dimensional exponent pairs. We will quote these eatimates from [13], referring the details to it. E is the set of all exponent pairs (κ, λ) $(0\leqq\kappa\leqq 1/2\leqq\lambda\leqq 1)$ and S is Rankin's set $\subset E$ obtained by applying Theorem 4.1 [13] a finite (but arbitrary) number of times starting with the trivial exponent pair (0, 1). S may be seen as the convex hull $\operatorname{conv} T$ of T obtainable from (0, 1) by applying to it all the finite compositions built with operators A and BA. Also, if M denotes the set $\{h(\varepsilon)|0<\varepsilon\leqq 5/56\}$ of exponent pairs $h(\varepsilon)=(9/56+\varepsilon, 1/2+9/56+\varepsilon)$ found by Huxley and Watt, we construct the set $S_1=S_1(M)$ by apllying Theorem 4.1 to M repeatedly. If $R=\partial S$, ∂S_1 denotes the borders of S, S_1 respectively, the infimum $I(R, \theta)$ on R of some function θ gives the optimal result.

Now define the function Γ: $[0, 1/2]\to[1/2, 1]$ by

$$\partial S_1 := \{(\kappa, \Gamma(\kappa)), \kappa\in[0, 1/2]\}.$$

Then Γ is continuous, decreasing and convex. Also, with

$$\theta_2(\kappa, \lambda)=\frac{\kappa+\lambda}{2(\kappa+1)},$$

define $\alpha := I(S_1, \theta_2)=0.32894\cdots$ and define β to be the larger one of α and $(\lambda-\kappa)/(\kappa+1)$ for (κ, λ) for which $\theta(\kappa, \lambda)=\alpha$ (see [12] for the numerical values of α and β).

O-THEOREM (Huxley [5], Pétermann [12, 13], Walfisz [18])

$$\alpha_1(a)\leqq\begin{cases}\phi(a), & -1<a<-\beta, \\ a/2+\alpha, & a\geqq-\beta,\ a\neq 0,\end{cases} \tag{1.1}$$

$$G_{0,1}(x)=O(x^{7/22}(\log x)^{69/22}), \qquad G_{-1,1}(x)=O((\log x)^{2/3}), \tag{1.2}$$

and for $k\geqq 2$,

$$\alpha_k(a)\leqq\begin{cases}\psi(a), & -1\leqq a<1/2, \\ a/2+1/4, & a\geqq 1/2,\end{cases} \tag{1.3}$$

where $\phi(a)$ (resp. $\psi(a)$) is the value of $\kappa/(\kappa+1)$ (resp. κ) at the unique argument κ satisfying $a=(\kappa-\Gamma(\kappa))/(\kappa+1)$ (resp. $a=2\kappa-\Gamma(\kappa)$).

Ω-THEOREM (Hafner [3, 4], Pétermann [10, 11, 14, 15])

$$(1.4)\qquad G_{a,1}(x)=\begin{cases}\Omega_{+}((x\log x)^{1/4}(\log\log x)^{(3+2\log 2)/4}\exp(-A\sqrt{\log\log\log x})) \\ \Omega_{-}(x^{1/4}\exp(c(\log\log x)^{1/4}(\log\log\log x)^{-3/4}), & a=0, \\ \Omega_{\pm}\left(\exp\left(c_a\dfrac{(\log x)^{a+1}}{\log\log x}\right)\right), & -1<a<0, \\ \Omega_{\pm}(\log\log x), & a=-1, \\ \Omega_{\pm}(1), & a<-1,\end{cases}$$

$$(1.5)\qquad G_{a,2}(x)=\begin{cases}\Omega_{\pm}(x^{a/2+1/4}g_a(x)), & 0<a<1/2, \\ \Omega_{+}(x^{a/2+1/4}f_a(x)), & 0\leqq a\leqq 1/2, \\ \Omega_{\pm}(x^{a/2+1/4}f_a(x)), & -1/2\leqq a<0, \\ \Omega_{+}\left(\exp\left(c_a\dfrac{(\log x)^{a+1}}{\log\log x}\right)\right), & -1<a<-1/2, \\ \Omega_{\pm}(\log\log x), & a=-1, \\ \Omega_{\pm}(1), & a<-1,\end{cases}$$

for some positive constant A, c, c_a, where

$$f_a(x)=\begin{cases}(\log x)^{1/4-a/2}, & -1/2\leqq a<1/2, \\ \log\log x, & a=1/2,\end{cases}$$

and

$$g_a(x)=\exp\left(c\frac{(\log\log x)^{1/4-a/2}}{(\log\log\log x)^{3/4+a/2}}\right).$$

The interested reader should ask the author for a detailed version of the paper at the address indicated at the end.

2. Statement of results.

THEOREM 1. *Let $t>0$ and $0\leqq a-t\in\boldsymbol{Z}$. Then*

(1)

$$\sum_{n\leqq x}n^{-t}\sigma_a(n)=\frac{\zeta(a+1)}{a-t+1}x^{a-t+1}+\sum_{r=1}^{a-t+1}\frac{(-1)^r}{r}\binom{a-t}{r-1}x^{a-t+1-r}G_{r-a-1,r}(x)$$
$$+\begin{cases}\zeta(1-a)\log x+\gamma\zeta(1-a)+\zeta(1-a), & t=1, \\ \zeta(t)\zeta(t-a)+\dfrac{\zeta(t-a)}{1-t}x^{1-t} & t\neq 1,\end{cases}$$
$$-x^{-t}G_{a,1}(x)+O(x^{a/2-t}),$$

where γ denotes Euler's constant.

(2) *Define $E^{a}_{-t}(x)$ by*

$$E^{a}_{-t}(x)=\sum_{n\leq x}n^{-1}\sigma_a(n)-g^{a}_{-t}(x),$$

where

$$g^{a}_{-t}(x)=\frac{\zeta(a+1)}{a-t+1}x^{a-t+1}+\begin{cases}0, & t>2 \text{ or } 0<t\leqq 2,\ t\neq 1 \text{ or } t=1,\ a>2,\\ \zeta(1-a)\log x, & t=1,\ a\leqq 2.\end{cases}$$

and let $G^{a}_{-t}(x)=\sum_{n\leq x}E^{a}_{-t}(n)$.

Then

(i) $$G^{a}_{-a}(x)=-\frac{1}{2x^a}G_{a+1,2}(x)+\frac{1}{2}G_{1-a,2}(x)+O(x^{(1-a)/2})$$

$$+\begin{cases}\dfrac{\zeta(1-a)}{(1-a)(2-a)}x^{2-a}+\dfrac{\zeta(a+1)-\zeta(a)}{2}x+\Big(\dfrac{1}{2}-\bar{B}_1(x)\Big)\dfrac{\zeta(1-a)}{1-a}x^{1-a}, & 0<a=t<1,\\ \dfrac{1}{2}(\zeta(2)-\gamma-\log 2\pi)x-\dfrac{1}{4}\log x-\Big(\dfrac{1}{2}-\bar{B}_1(x)\Big)G_{-1,1}(x), & a=t=1,\\ \dfrac{\zeta(a+1)-\zeta(a)}{2}x+\dfrac{\zeta(1-a)}{(1-a)(2-a)}x^{2-a}+\dfrac{\zeta(a-1)+\zeta(a+1)}{12}-\dfrac{1}{2}\Big(\dfrac{1}{2}-\bar{B}_1(x)\Big)\zeta(a) & \\ -\dfrac{\zeta(a+1)}{2}\bar{B}_2(x)-\Big(\dfrac{1}{2}-\bar{B}_1(x)\Big)G_{-a,1}(x), & 1<a=t<2,\\ \dfrac{\zeta(3)-\zeta(2)}{2}x+\dfrac{1}{12}\log x+\dfrac{\gamma}{12}+A_1+\dfrac{1}{24}+\dfrac{\zeta(3)}{12}-\dfrac{\zeta(3)}{2}\bar{B}_2(x) & \\ -\dfrac{1}{2}\Big(\dfrac{1}{2}-\bar{B}_1(x)\Big)\zeta(2)-\Big(\dfrac{1}{2}-\bar{B}_1(x)\Big)G_{-2,1}(x), & a=t=2,\end{cases}$$

where $A_1=-\zeta'(-1)$ (*see* I-*Theorem* 1).

(ii) $$G^{a}_{-t}(x)=\frac{\zeta(a+1)}{2(a-t+1)}x^{a-t+1}+\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}G_{m-a,m+1}(x)$$

$$+\frac{\zeta(a+1)}{a-t+1}\sum_{m=1}^{a-t+1}\frac{(-1)^m}{m+1}\binom{a-t+1}{m}\bar{B}_{m+1}(x)x^{a-t+1-m}$$

$$+\Big(\frac{1}{2}-\bar{B}_1(x)\Big)\frac{1}{a-t+1}\sum_{m=1}^{a-t+1}(-1)^m\binom{a-t+1}{m}x^{a-t+1-m}G_{m-a-1,m}(x)$$

$$-\frac{1}{2x^t}G_{a+1,2}(x)+\zeta(t-a)\zeta(t)x+\frac{(-1)^{a-t+1}}{a-t+2}G_{1-t,a-t+2}(x)$$

$$+\frac{B_{a-t+2}}{a-t+1}\zeta(t-1)+\frac{B_{a-t+2}}{(a-t+1)(a-t+2)}\zeta(a+1)+\Big(\frac{1}{2}-\bar{B}_1(x)\Big)\zeta(t-a)\zeta(t)$$

$$+\begin{cases}F_2x^{2-t}, & 2<t<3,\\ 0, & t\geqq 0,\end{cases}+O(x^{(a+1)/2-t}),$$

where

$$F_1=-\frac{1}{a(a-t+1)(a-t+2)}-\frac{(-1)^{a-t}}{a-t+2}\Big(\sum_{u=0}^{a-t+1}\binom{a-t+2}{u}\frac{B_u}{u-a}$$
$$-\frac{\zeta(1-a)}{\binom{2-t}{a-t+2}}\Big)+\frac{\zeta(t-a)}{1-t}.$$

(iii) $$G^{a}_{-a+1}(x)=\frac{\zeta(a+1)}{4}x^2+\frac{\zeta(1-a)}{(a-2)(a-3)}x^{3-a}-\Big(\frac{\zeta(a-1)}{12}+\frac{\zeta(a+1)}{2}\bar{B}_2(x)\Big)x$$
$$+\Big(\frac{1}{2}-\bar{B}_1(x)\Big)\frac{\zeta(1-a)}{2-a}x^{2-a}-\frac{x}{2}G_{1-a,2}(x)-\frac{x^{1-a}}{2}G_{1+a,2}(x)$$
$$-\Big(\frac{1}{2}-\bar{B}_1(x)\Big)xG_{-a,1}(x)+O(x^{(3-a)/2}),\qquad 0<t<1,\ a-t=1,$$

(iv) $G^{a}_{-t}(x)$
$$=\frac{\zeta(a+1)}{2(a-t+1)}x^{a-t+1}+F_4x^{2-t}-\frac{\zeta(a+1)}{a-t+1}\sum_{m=1}^{a-t-1}\frac{(-)^{m+1}}{m+1}\binom{a-t+1}{m}\bar{B}_{m+1}(x)x^{a-t+1-m}$$
$$+\sum_{m=1}^{a-t}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}G_{m-a,m+1}(x)-\frac{1}{2x^t}G_{1+a,2}(x)$$
$$+\Big(\frac{1}{2}-\bar{B}_1(x)\Big)\frac{x^{1-t}}{a-t+1}\sum_{m=1}^{a-t-1}(-1)^m\binom{a-t+1}{m}x^{a-m}G_{m-a-1,m}(x)+O(x^{(a+1)/2-t})$$
$$0<t<1,\ a-t\geqq 2,$$

where

$$F_2=(-1)^{a-t+1}\sum_{u=0}^{a-t}\binom{a-t+1}{u}\frac{B_u}{u-a}\Big(\frac{1}{a-t+2-u}-\frac{a-t}{a-t+1}+\frac{(a-t-1)(a-t+1-u)}{2(a-t+1)}\Big)$$
$$+(-1)^{a-t}\zeta(1-a)\binom{1-t}{a-t+1}^{-1}\Big(\frac{1}{2-t}-\frac{a-t}{a-t+1}-\frac{(a-t-1)(1-t)}{a-t+1}\Big),$$

(v) $$G^{2}_{-1}(x)=\frac{\zeta(3)}{4}x^2-\Big(\frac{1}{24}+\frac{\gamma}{12}+A_1+\frac{\zeta(3)}{2}\bar{B}_2(x)\Big)x-\frac{x}{2}G_{-1,2}(x)$$
$$-\frac{1}{2x}G_{3,2}(x)-\Big(\frac{1}{2}-\bar{B}_1(x)\Big)xG_{-2,1}(x)+O(x^{1/2}),\qquad t=1,\ a=2,$$

(vi) $$G^{a}_{-t}(x)=\frac{\zeta(a+1)}{2(a-t+1)}x^{a-t+1}+\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a-t}{m+1}x^{a-t+1-m}G_{m-a,m+1}(x)$$
$$+\frac{\zeta(a+1)}{a-t+1}\sum_{m=1}^{a-t}\frac{(-1)^m}{m+1}\binom{a-t+1}{m}\bar{B}_{m+1}(x)x^{a-t+1-m}$$
$$+\Big(\frac{1}{2}-\bar{B}_1(x)\Big)\frac{1}{a-t+1}\sum_{m=1}^{a-t}(-1)^m\binom{a-t+1}{m}x^{a-t+1-m}G_{m-a-1,m}(x)-\frac{1}{2x^t}G_{a+1,2}(x)$$

$$+\begin{cases} 0, & t=1,\ a-t\geqq 2, \\ F_1x^{2-t}+\zeta(t-a)\zeta(t)x, & 1<t<2,\ a-t\geqq 1, \\ -\dfrac{B_{a-1}}{a-1}\zeta(2)x+\dfrac{(-1)^{a+1}}{a}G_{-1,a}(x), & t=2,\ a-t\geqq 1, \end{cases} +O(x^{(a+1)/2-t}).$$

(3) *We have*

(i) $$\int_1^x E_{-a}^a(u)du=-\frac{1}{2x^a}G_{a+1,2}(x)-\frac{1}{2}G_{1-a,2}(x)+O(x^{(a+1)/2-t})$$

$$+\begin{cases} \dfrac{\zeta(1-a)}{(1-a)(2-a)}x^{2-a}-\dfrac{\zeta(a)}{2}x, & 0<a=t<1, \\ -\dfrac{1}{2}(\log 2\pi+\gamma)x, & a=t=1, \\ -\dfrac{\zeta(a)}{2}x+\dfrac{\zeta(1-a)}{(1-a)(2-a)}x^{2-a}+\dfrac{\zeta(a-1)}{12}+\dfrac{\zeta(a+1)}{2}, & 1<a=t<2, \\ -\dfrac{\zeta(2)}{2}x+\dfrac{1}{12}\log x+\dfrac{\gamma}{12}+A_1+\dfrac{1}{24}+\dfrac{\zeta(3)}{2}, & a=t=2, \end{cases}$$

(ii) $$\int_1^x E_{-t}^a(u)du$$

$$=\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}G_{m-a,m+1}(x)-\frac{1}{2x^t}G_{a+1,2}(x)+\zeta(t-a)\zeta(t)x$$

$$+\frac{B_{a-t+2}}{a-t+2}\zeta(t-1)+\frac{\zeta(a+1)}{(a-t+1)(a-t+2)}+\frac{(-1)^{a-t+1}}{a-t+2}G_{1-t,a-t+2}(x)$$

$$+\begin{cases} 0, & t\geqq 3, \\ F_1x^{2-t}, & 2<t<3, \end{cases} +O(x^{(a+1)/2-t}),$$

(iii) $$\int_1^x E_{-a+1}^a(u)du=-\frac{x}{2}G_{1-a,2}(x)-\frac{x^{1-a}}{2}G_{1+a,2}(x)+\frac{1}{3}G_{2-a,3}(x)$$

$$+\frac{\zeta(1-a)}{(a-2)(a-3)}x^{3-a}-\frac{\zeta(a-1)}{12}x+O(x^{(a+1)/2-t}),$$

$$0<t<1,\ a-t=1,$$

(iv) $$\int^x E_{-t}^a(u)du=\sum_{m=1}^{a-1}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}G_{m-a,m+1}(x)-\frac{1}{2x^t}G_{a+1,2}(x)$$

$$+F_2x^{2-t}+O(x^{(a+1)/2-t}),\qquad 0<t<1,\ a-t\geqq 2,$$

(v) $$\int_1^x E_{-1}^2(u)du$$

$$=-\frac{x}{2}G_{-1,2}(x)-\left(\frac{1}{24}+\frac{\gamma}{12}+A_1\right)x-\frac{1}{2x}G_{3,2}(x)+O(\sqrt{x}),\qquad t=1,\ a=2,$$

(vi) $$\int_1^x E_{-t}^{a}(u)du=\sum_{m=1}^{a-t}\frac{(-1)^m}{m+1}\binom{a-t}{m-1}x^{a-t+1-m}G_{m-a,m+1}(x)-\frac{1}{2x^t}G_{a+1,2}(x)$$
$$+\begin{cases}0, & t=1,\ a-t\geqq 2,\\ F_3x^{2-t}+\zeta(t-a)\zeta(t)x, & 1<t<2,\ a-t\geqq 1,\\ \zeta(2-a)\zeta(2)x+(-1)^{a+1}a^{-1}G_{-1,a}(x), & t=2,\ a-t\geqq 1,\end{cases}\quad +O(x^{(a+1)/2-t})$$

where

$$F_3=-\frac{1}{a(a-t+1)(a-t+2)}-\frac{\zeta(t-a)}{1-t}-\frac{\zeta(t-1-a)}{2-t}$$
$$-\frac{(-1)^{a-t}}{a-t+2}E_{1-t,a-t+2}+(-1)^{a-t}\frac{a-t}{a-t+1}E_{-t,a-t+1}.$$

Theorem 1, (1.1)-(1.5) imply the following Corollary.

COROLLARY 1. *For every $\varepsilon>0$, we have*

(i) $$\sum_{n\leq x}\frac{\sigma_a(n)}{n^a}=\begin{cases}x\log x+(2\gamma-1)x,\ ([6]) & a=0,\\ \zeta(a+1)x-\frac{\zeta(a)}{2}+\frac{\zeta(1-a)}{1-a}x^{1-a}, & 0<a<1,\\ \zeta(2)x-\frac{1}{2}\log x-\frac{1}{2}(\log 2\pi+\gamma), & a=1,\end{cases}$$
$$+\begin{cases}O(x^{-a/2+\alpha+\varepsilon}), & 0\leqq a\leqq\beta,\\ O(x^{\phi(-a)+\varepsilon}), & \beta<a<1,\\ O((\log x)^{2/3}), & a=1.\end{cases}$$

In particular

$$\sum_{n\leq x}\frac{\sigma(n)}{n}-\frac{\pi^2}{6}x+\frac{1}{2}\log x=\Omega_{\pm}(\log\log x),\qquad a=1.$$

(ii) $$\sum_{n\leq x}E_{-1}^{1}(n)=\frac{1}{2}(\zeta(2)-\gamma-\log 2\pi)x+\begin{cases}O(x^{\phi(0)+\varepsilon}),\\ \Omega_{+}(x^{1/4}(\log x)^{1/4}),\end{cases}$$

$$\int_1^x E_{-1}^{1}(u)du=-\frac{1}{2}(\log 2\pi+\gamma)x+\begin{cases}O(x^{\phi(0)+\varepsilon}),\\ \Omega_{+}(x^{1/4}(\log x)^{1/4}),\end{cases}$$

(iii) $$\sum_{n\leq x}E_{-2}^{2}(n)=\frac{\zeta(3)-\zeta(2)}{2}x+\begin{cases}O(x^{\varepsilon})\\ \frac{1}{12}\log x+\Omega_{+}(\log\log x)\end{cases}$$

$$\int_1^x E_{-2}^{2}(u)du=-\frac{\zeta(2)}{2}x+\begin{cases}O(x^{\varepsilon}),\\ \frac{1}{12}\log x+\Omega_{+}(\log\log x),\end{cases}$$

(iv) $$\sum_{n\leq x} E_{-1}^2(n)=\frac{\zeta(3)}{4}x^2+\begin{cases} O(x^{1+\varepsilon}), \\ \Omega_+(x \operatorname{loglog} x). \end{cases}$$

$$\int_1^x E_{-1}^2(u)du=\begin{cases} O(x^{1+\varepsilon}), \\ \Omega_+(x \operatorname{loglog} x). \end{cases}$$

Similar results hold for other values of a.

REMARK 1. These formulas improve and correct MacLeod's corresponding results in [8].

In what follows we shall use the notation $\bar{\delta}_{i,j}=1-\delta_{i,j}$, where $\delta_{i,j}=1$ for $i=j$ and 0 otherwise.

THEOREM 2.

(1) $$\sum_{n\leq x}\frac{\sigma_a(n)}{n^{a+1}}=\begin{cases} \frac{1}{2}\log^2 x+2\gamma\log x+\gamma^2+2\gamma_1, & a=0, \\ \zeta(a+1)\log x+\zeta'(a+1)+\gamma\zeta(a+1), & a>0, \end{cases}$$

$$-x^{-1}G_{-a,1}(x)-x^{-a-1}G_{a,1}(x)$$

$$+\begin{cases} 0, & a=0, \\ -\frac{\zeta(1-a)}{a}x^{-a}, & 0<a<2, \\ -\frac{1}{2x^2}G_{-1,2}(x) & a=2, \\ \left(\frac{\zeta(a+1)}{6}+\frac{\zeta(a)}{4}\right)x^{-2}-\frac{1}{2x^2}G_{1-a,2}(x) & 2<a<3, \\ \sum_{1<r<a/2+2} c_r(x)x^{-1-r}+\sum_{n>x}\frac{G_{1-a,2}(n)}{n(n+1)^2}+2\sum_{n>x}\frac{G_{-a,1}(n)}{n(n+1)^2} & \\ -\frac{1}{2x^2}G_{1-a,2}(x)-\sum_{n>x}\frac{G_{-a,1}(n)}{n^2(n+1)}+\frac{1}{2}\sum_{n>x}\frac{G_{1-a,2}(n)}{n^2(n+1)^2} & \\ +\sum_{n>x}\frac{G_{-a,1}(n)}{n^2(n+1)^2}, & a>3, \end{cases}$$

$$+O(x^{-a/2-1}),$$

where $c_1(x)=\frac{\zeta(a+1)}{6}+\frac{\zeta(a)}{4}$,

$$c_2(x)=(-2\bar{B}_3(x)+3\bar{B}_2(x)-1)\frac{\zeta(a+1)}{6}+(3\bar{B}_1(x)-1)\frac{\zeta(a)}{6}$$

$$+(\bar{B}_2(x)-\bar{B}_1(x)+3^{-1})G_{-a,1}(x),$$

$$c_r(x)=\frac{1}{r+1}\Big[\sum_{u=0}^{r}(-1)^{r-u}\binom{r+1}{u}\Big\{\frac{\zeta(a+1)(u-1)(2r-u)}{2(r+2-u)}-\frac{\zeta(a)}{2}$$

$$+(r-1)G_{-a,1}(x)\Big\}\bar{B}_u(x)-2^{-1}\zeta(a+1)r(r-1)\bar{B}_{r+1}(x)\Big],\quad r\geqq 3,$$

and γ_1 *is the constant term in the Euler-Maclaurin expansion of* $\sum_{n\leqq x} n^{-1}\log n$ *(see* I-*Lemma* 8).

(2) *On defining* $E^a_{-a-1}(x)$ *by*

$$E^a_{-a-1}(x)=\sum_{n\leqq x}\frac{\sigma_a(n)}{n^{a+1}}-\begin{cases} 2^{-1}\log^2 x+2\gamma\log x+\gamma^2+2\gamma_1, & a=0,\\ \zeta(a+1)\log x+\zeta'(a+1)+\gamma\zeta(a+1), & a>0,\end{cases}$$

we have

$$\sum_{n\leqq x}E^a_{-a-1}(n)=$$

$$\begin{cases} \dfrac{1}{4}\log^2 x+\gamma\log x+W^0_{-1}+2x\displaystyle\sum_{n>x}\frac{G_{1,2}(n)}{n^3}-x^{-1}G_{1,2}(x), & a=0,\\[2ex] \dfrac{\zeta(2)+1}{2}\log x+W^1_{-2}+\Big(\bar{B}_1(x)-\dfrac{1}{2}\Big)x^{-1}G_{-1,1}(x)-\dfrac{1}{2x}(x^{-1}G_{2,2}(x)+G_{0,2}(x)), & a=1,\\[2ex] \dfrac{\zeta(a+1)}{2}\log x+W^a_{-a-1}+\begin{cases} -\dfrac{\zeta(1-a)}{a(1-a)}x^{1-a}+\dfrac{\zeta(1-a)}{a}\Big(\bar{B}_1(x)-\dfrac{1}{2}\Big)x^{-a}, & 0<a<1,\\[2ex] -\dfrac{\zeta(1-a)}{a(1-a)}\bar{\bar{\delta}}_{a,2}\bar{\bar{\delta}}_{a,3}x^{1-a}+f_1(x)x^{-1}, & 1<a\leqq 3,\\[2ex] \displaystyle\sum_{1\leqq r\leqq(a+1)/2}f_r(x)x^{-r}+2x\sum_{n>x}\frac{G_{-a,1}(n)}{n(n+1)^2}\\ \displaystyle -x\sum_{n>x}\frac{G_{-a,1}(n)}{n^2(n+1)}+x\sum_{n>x}\frac{G_{-a,1}(n)}{n^2(n+1)^2}-\Big(\bar{B}_1(x)-\frac{1}{2}\Big)\\ \displaystyle \cdot\Big\{\sum_{n>x}\frac{G_{1-a,2}(n)}{n(n+1)^2}+2\sum_{n>x}\frac{G_{-a,1}(n)}{n(n+1)^2}-\sum_{n>x}\frac{G_{-a,1}(n)}{n^2(n+1)}\\ \displaystyle +\frac{1}{2}\sum_{n>x}\frac{G_{1-a,2}(n)}{n^2(n+1)^2}+\sum_{n>x}\frac{G_{-a,1}(n)}{n^2(n+1)^2}-\frac{1}{2x^2}G_{1-a,2}(x)\\ \displaystyle +x\sum_{n>x}\frac{G_{1-a,2}(n)}{n(n+1)^2}+\frac{1}{2}x\sum_{n>x}\frac{G_{1-a,2}(n)}{n^2(n+1)^2}, & a>3,\end{cases}\\[2ex] -\dfrac{1}{2x}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x)) \end{cases}$$

$$+O(x^{-(a+1)/2}),$$

where,

$$f_1(x)=\frac{\zeta(a+1)}{6}+\frac{\zeta(a)}{4}-\frac{\zeta(a+1)}{2}\bar{B}_2(x)+\Big(\bar{B}_1(x)-\frac{1}{2}\Big)G_{-a,1}(x),$$

$$f_2(x)=(-8\bar{B}_3(x)+6\bar{B}_2(x)-2\bar{B}_1(x)-1)\frac{\zeta(a+1)}{12}+(6\bar{B}_1(x)-1)\frac{\zeta(a)}{24}$$
$$+(\bar{B}_2(x)-\bar{B}_1(x)+3^{-1})G_{-a,1}(x),$$
$$f_r(x)=c_r(x)-(\bar{B}_1(x)-1/2)c_{r-1}(x), \qquad r\geqq 3,$$
$$W^0_{-1}=\gamma^2+2\gamma_1-\frac{\zeta''(0)}{2}+\frac{3}{4}-\gamma\log 2\pi, \qquad a=0,$$
$$W^1_{-2}=(\log 2\pi-\zeta(2)\log 2\pi+2\zeta'(2)+2\gamma\zeta(2)+\gamma+1)/2, \qquad a=1,$$
$$W^a_{-a-1}=\frac{\zeta(a)}{2}+\zeta'(a+1)+\gamma\zeta(a+1)-\frac{\zeta(a+1)}{2}\log 2\pi, \qquad a\neq 0,\ 1,$$

(3) $$\int_1^x E^a_{-a-1}(u)du=Y^a_{-a-1}-\frac{1}{2x}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x))$$
$$+\begin{cases} 2x\sum_{n>x}\dfrac{G_{1,2}(n)}{n^3}, & a=0,\\ \dfrac{1}{2}\log x, & a=1,\\ -\dfrac{\zeta(1-a)}{a(1-a)}\bar{\delta}_{a,2}\bar{\delta}_{a,3}x^{1-a} + \begin{cases} 0, & 0<a<1,\\ \left(\dfrac{\zeta(a+1)}{6}+\dfrac{\zeta(a)}{4}\right)x^{-1}, & 1<a\leqq 3,\\ \sum_{1\leq r\leq (a+1)/2}c_r(x)x^{-r}+2x\sum_{n>x}\dfrac{G_{-a,1}(n)}{n(n+1)^2} & \\ \quad -x\sum_{n>x}\dfrac{G_{-a,1}(n)}{n^2(n+1)}+x\sum_{n>x}\dfrac{G_{1+a,2}(n)}{n^{a+1}(n+1)^2} & \\ \quad +x\sum_{n>x}\dfrac{G_{1-a,2}(n)}{n(n+1)^2}+\dfrac{1}{2}x\sum_{n>x}\dfrac{G_{1-a,2}(n)}{n^2(n+1)^2}, & a>3,\end{cases}\end{cases}$$
$$+O(x^{-(a+1)/2}(\log x)^{\delta_{a,1}}),$$

where,

$$Y^a_{-a-1}=\begin{cases}\gamma^2+2\gamma_1-2\gamma+3/4, & a=0,\\ (\log 2\pi+\gamma+1)/2+\zeta'(2)+\gamma\zeta(2)-\zeta(2), & a=1,\\ \zeta'(a+1)+\gamma\zeta(a+1)-\zeta(a+1)+\zeta(a)/2, & a\neq 0,\ 1,\end{cases}$$

By Lemma 8 and (1.1), (1.2), (1.4), we have

COROLLARY 2. *For any* $\varepsilon>0$,

(i) $\sum_{n\le x}\frac{d(n)}{n}=\frac{1}{2}\log^2 x+2\gamma\log x+\gamma^2+2\gamma_1+O(x^{-15/22+\varepsilon})$,

(ii) $\sum_{n\le x}E_{-1}^{0}(n)=\frac{1}{4}\log^2 x+\gamma\log x+W_{-1}^{0}+\begin{cases} O(x^{-1/4}), \\ \Omega_{\pm}(x^{-1/4}),\end{cases}$

$$\int_1^x E_{-1}^{0}(u)du=Y_{-1}^{0}+\begin{cases} O(x^{-1/4}), \\ \Omega_{\pm}(x^{-1/4}),\end{cases}$$

(iii) $\sum_{n\le x}E_{-2}^{1}(n)=\frac{\zeta(2)+1}{2}\log x+W_{-2}^{1}+\begin{cases} O(x^{\phi(0)-1+\varepsilon}), \\ \Omega_{+}(x^{-3/4}(\log x)^{1/4}),\end{cases}$

$$\int_1^x E_{-2}^{1}(u)du=\frac{1}{2}\log x+Y_{-2}^{1}+\begin{cases} O(x^{\phi(0)-1+\varepsilon}), \\ \Omega_{+}(x^{-3/4}(\log x)^{1/4}),\end{cases}$$

(iv) $\sum_{n\le x}E_{-3}^{2}(n)=\frac{\zeta(3)}{2}\log x+W_{-3}^{2}+f_1(x)x^{-1}+\begin{cases} O(x^{-1+\varepsilon}), \\ \Omega_{+}(x^{-1}\log\log x),\end{cases}$

$$\int_1^x E_{-3}^{2}(u)du=Y_{-3}^{2}+\begin{cases} O(x^{-1+\varepsilon}), \\ \Omega_{+}(x^{-1}\log\log x).\end{cases}$$

Similarly, for other values of a the order of the error term essetially depends on that of $G_{1-a,2}(x)$.

THEOREM 3. (1)

$$\sum_{n\le x}\frac{\sigma_a(n)}{n^{a+2}}=\zeta(2)\zeta(a+2)-x^{-2}(x^{-a}G_{a,1}(x)+G_{-a,1}(x))+O(x^{-a/2-2})$$

$$+\begin{cases} -x^{-1}\log x-(2\gamma+1)x^{-1}, & a=0, \\ -\zeta(a+1)x^{-1}+\begin{cases} -\dfrac{\zeta(1-a)}{a+1}x^{-1-a}, & 0<a<2, \\ -\dfrac{1}{x^3}G_{1-a,2}(x) & a=2, \\ \left(\dfrac{5}{18}\zeta(a+1)+\dfrac{1}{3}\zeta(a)\right)x^{-3}-\dfrac{1}{x^3}G_{1-a,2}((x), & 2<a\leqq 3,\end{cases}\end{cases}$$

(2) *On defining $E_{-a,2}^{a}(x)$ by*

$$E_{-a-2}^{a}(x)=\sum_{n\le x}\frac{\sigma_a(n)}{n^{a+2}}-\begin{cases} \zeta^2(2)-x^{-1}\log x-(2\gamma+1)x^{-1}, & a=0, \\ \zeta(2)\zeta(a+2)-\zeta(a+1)x^{-1}, & 0<a\leqq 3,\end{cases}$$

we have

$$\sum_{n\leq x} E^{a}_{-a-2}(n)=W^{a}_{-a-2}-\frac{1}{2x^{2}}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x))$$

$$+\begin{cases} -\dfrac{1}{2x}\log x-\left(\gamma+\dfrac{1}{2}\right)x^{-1}, & a=0,\\ \dfrac{\zeta(1-a)}{a(1+a)}\bar{\delta}_{a,2}\bar{\delta}_{a,3}x^{-a}-\dfrac{\zeta(a+1)}{2}x^{-1} \\ +\begin{cases} \dfrac{\zeta(1-a)}{a+1}\left(\bar{B}_{1}(x)-\dfrac{1}{2}\right)x^{-1-a}, & 0<a<1,\\ \left(\bar{B}_{1}(x)-\dfrac{1}{2}\right)G_{-a,1}(x)x^{-2}, & a=1,\\ \left\{\dfrac{\zeta(a+1)}{9}+\dfrac{\zeta(a)}{12}-\dfrac{\zeta(a+1)}{2}\bar{B}_{2}(x)+\left(\bar{B}_{1}(x)-\dfrac{1}{2}\right)G_{-a,1}(x)\right\}x^{-2}, & 1<a\leqq 3,\end{cases}\end{cases}$$

$$+O(x^{-(a+3)/2}),$$

where

$$W^{a}_{-a-2}=\begin{cases}\zeta^{2}(2)+\gamma^{2}-\gamma-\gamma_{1}-1, & a=0,\\ \zeta(2)\zeta(a+2)-\zeta'(a+1)-\zeta(a+1), & \textit{otherwise},\end{cases}$$

(3) $$\int_{1}^{x}E^{a}_{-a-2}(u)du=Y^{a}_{-a-2}-\frac{1}{2x^{2}}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x))$$

$$+\begin{cases}0, & a=0,\\ \dfrac{\zeta(1-a)}{a(1+a)}\bar{\delta}_{a,2}\bar{\delta}_{a,3}x^{-a}+\begin{cases}0, & 0<a\leqq 1,\\ \left(\dfrac{\zeta(a+1)}{9}+\dfrac{\zeta(a)}{12}\right)x^{-2} & 1<a\leqq 3,\end{cases}\end{cases}$$

$$+O(x^{-(a+3)/2}),$$

where,

$$Y^{a}_{-a-2}=\begin{cases}\zeta^{2}(2)-\gamma^{2}-2\gamma-2\gamma_{1}-1, & a=0,\\ \zeta(2)\zeta(a+2)-\zeta(a+1)-\gamma\zeta(a+1)-\zeta'(a+1), & 0<a\leqq 3.\end{cases}$$

THEOREM 4. (1) *For $t\geqq 2$, we have*

$$\sum_{n\leq x}\frac{\sigma_{a}(n)}{n^{a+t}}=\zeta(t)\zeta(a+t)$$

$$+\begin{cases}-\dfrac{1}{t-1}x^{-t+1}\log x-\dfrac{1}{t-1}\left(2\gamma+\dfrac{1}{t-1}\right)x^{-t+1}, & a=0,\\ -\dfrac{\zeta(a+1)}{t-1}x^{-t+1}+\begin{cases}-\dfrac{\zeta(1-a)}{a+t-1}x^{-a-t+1}, & 0<a<2,\\ -\dfrac{t}{2}x^{-t-1}G_{-1,2}(x), & a=2,\\ (\alpha(t)\zeta(a+1)+\beta(t)\zeta(a))x^{-t-1}-\dfrac{t}{2}x^{-t-1}G_{1-a,2}(x), & 2<a\leqq 3,\end{cases}\end{cases}$$

$$-x^{-t}(x^{-a}G_{a,1}(x)+G_{-a,1}(x))+O(x^{-a/2-t}),$$

where,

$$\alpha(t)=\frac{t^2+1}{6(t+1)}, \qquad \beta(t)=\frac{t}{2(t+1)}.$$

(2) *On defining* $E^{a}_{-a-t}(x)$ *by*

$$E^{a}_{-a-t}(x)=\sum_{n\leqq x}\frac{\sigma_a(n)}{n^{a+t}}-g^{a}_{-a-t}(x),$$

where

$$g^{a}_{-a-t}(x)=\zeta(t)\zeta(a+t)+\begin{cases} -\dfrac{1}{t-1}x^{-t+1}\log x-\dfrac{1}{t-1}\left(2\gamma+\dfrac{1}{t-1}\right)x^{-t+1}, & a=0,\\ -\dfrac{\zeta(a+1)}{t-1}x^{-t+1}, & 0<a\leqq 3,\end{cases}$$

we have, for $t\geqq 3$,

$$\sum_{n\leqq x}E^{a}_{-a-t}(n)=W^{a}_{-a-t}$$

$$+\begin{cases} -\dfrac{1}{2(t-1)}x^{-t-1}\log x-\dfrac{1}{2(t-1)}\left(2\gamma+\dfrac{1}{t-1}\right)x^{-t+1}, & a=0,\\ -\dfrac{\zeta(a+1)}{2(t-1)}x^{-t+1}+\dfrac{\zeta(1-a)}{(a+t-1)(a+t-2)}\bar{\delta}_{a,2}\bar{\delta}_{a,3}x^{-a-t+2} & \\ \quad+\begin{cases} \dfrac{\zeta(1-a)}{a+t-1}\left(\bar{B}_1(x)-\dfrac{1}{2}\right)x^{-a-t+1}, & 0<a<1,\\ \left(\bar{B}_1(x)-\dfrac{1}{2}\right)G_{-1,1}(x)x^{-t}, & a=1,\\ \left(\alpha'(t)\zeta(a+1)+\beta'(t)\zeta(a)-\dfrac{\zeta(a+1)}{2}\bar{B}_2(x)\right)x^{-t}+\left(\bar{B}_1(x)-\dfrac{1}{2}\right)G_{-a,1}(x)x^{-t}, & 1<a\leqq 3,\end{cases}\end{cases}$$

$$-\frac{1}{2x^t}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x))+O(x^{-a/2-t+1/2}),$$

where,

$$W^{a}_{-a-t}=\begin{cases} \zeta^2(t)-\zeta^2(t-1)-\dfrac{\zeta(t-1)}{t-1}\left(2\gamma+\dfrac{1}{t-1}\right)-\dfrac{\zeta(t-1)}{t-1}, & a=0,\\ \zeta(t)\zeta(a+t)-\zeta(t-1)\zeta(a+t-1)+\dfrac{\zeta(a+1)}{t-1}\zeta(t-1), & \textit{otherwise}.\end{cases}$$

(3) *For* $t\geqq 3$,

$$\int_1^x E_{-a-t}^{a}(u)du = Y_{-a-t}^{a}$$

$$+\begin{cases} 0, & a=0, \\ \dfrac{\zeta(1-a)}{(a+t-2)(a+t-1)}\bar{\bar{\delta}}_{a,2}\bar{\bar{\delta}}_{a,3}x^{-a-t+2} \\ +\begin{cases} 0, & 0 \leqq a \leqq 1, \\ (\alpha'(t)\zeta(a+1)+\beta'(t)\zeta(a))x^{-t}, & 1 < a \leqq 3, \end{cases} \end{cases}$$

$$-\frac{1}{2x^t}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x))+O(x^{-a/2-t+1/2}),$$

where,

$$Y_{-a-t}^{a}=\begin{cases} \zeta^2(t)-\zeta^2(t-1)+\dfrac{1}{(t-1)(t-2)}\left(2\gamma+\dfrac{1}{t-1}+\dfrac{1}{t-2}\right) & a=0, \\ \zeta(t)\zeta(a+t)-\zeta(t-1)\zeta(a+t-1)+\dfrac{\zeta(a+1)}{(t-1)(t-2)}, & otherwise, \end{cases}$$

$$\alpha'(t)=\alpha(t)-\alpha(t-1), \qquad \beta'(t)=\beta(t)-\beta(t-1).$$

REMARK 2. From (1.1)-(1.5) and [7] the last but one term involving $G_{a,k}$-functions can be estimated as follows:

(i) $-x^{-t}G_{-a,1}(x)+O\ (x^{-a/2-t+\alpha+\varepsilon})$

$$=\begin{cases} O\ (x^{-a/2-t+\alpha+\varepsilon}), & 0 \leqq a \leqq \beta, \\ O\ (x^{\phi(-a)-t+\varepsilon}), & \beta < a < 1, \\ O\ ((\log x)^{2/3}), & a=1, \end{cases}$$

$$=\begin{cases} \Omega_+(x^{-t+1/4}(\log x)^{1/4}(\log\log x)^{(3+2\log 2)/4}\exp(-A\sqrt{\log\log x\ \log x}), & a=0, \\ \Omega_-(x^{-t+1/4}\exp(c(\log\log x)^{1/4}(\log\log\log x)^{-3/4}), & a=0, \\ \Omega_\pm\left(x^{-t}\exp\left(c_{-a}\dfrac{(\log x)^{-a+1}}{\log\log x}\right)\right), & 0<a<1, \\ \Omega_\pm(x^{-t}\log\log x), & a=1, \\ \Omega_\pm(x^{-t}), & a>1. \end{cases}$$

(ii) $-\dfrac{1}{2x^{-t}}G_{1-a,2}(x)+O\ (x^{-a/2-t+3/4})$

$$=\begin{cases} O\ (x^{-a/2-t+3/4}), & 0 \leqq a \leqq 1/2, \\ O\ (x^{\phi(1-a)-t+\varepsilon}), & 1/2 < a \leqq 2, \\ O\ (x^{-t}), & a>2, \end{cases}$$

$$
=\begin{cases}
\Omega_{\pm}(x^{-a/2-t+3/4}), & 0\leqq a<1/2,\\
\Omega_{+}(x^{-a/2-t+3/4}f_a(x)), & 1/2\leqq a\leqq 1,\\
\Omega_{\pm}(x^{-a/2-t+3/4}f_a(x)), & 1<a\leqq 3/2,\\
\Omega_{+}\left(x^{-t}\exp\left(c_{1-a}\dfrac{(\log x)^{2-a}}{\log\log x}\right)\right), & 3/2<a<2,\\
\Omega_{\pm}(x^{-t}\log\log x), & a=2,\\
\Omega_{\pm}(x^{-t}), & a>2.
\end{cases}
$$

3. Preliminaries.

LEMMA 1. (MacLead [8]) *For $t>0$, let*

$$\sum_{n\leqq x} n^{-t}\sigma_a(n)=g^{a}_{-t}(x)+E^{a}_{-t}(x),$$

and suppose the series

$$\sum_{n=1}^{\infty}\frac{E^{a}_{-t}(n)}{n^2},\qquad \sum_{n=1}^{\infty}\frac{E^{a}_{-t}(n)}{n^2(n+1)},\qquad \sum_{n=1}^{\infty}\frac{g^{a}_{-t}(n)}{n^2(n+1)},$$

all converge and that $x^{-2}\sum_{n\leqq x}E^{a}_{-t}(n)\to 0$ as $x\to\infty$. Then we have

(1)
$$\sum_{n\leqq x} n^{-t-1}\sigma_a(n)=\sum_{i=1}^{6}S_i(t)+K^{a}_{-t},$$

where
$$S_1(t)=\sum_{n\leqq x}\frac{g^{a}_{-t}(n)}{n^2}+\sum_{n>x}\frac{g^{a}_{-t}(n)}{n^2(n+1)},$$

$$S_2(t)=-2\sum_{n>x}\frac{G^{a}_{-t}(n)}{n(n+1)^2}-\sum_{n>x}\frac{G^{a}_{-t}(n)}{n^2(n+1)^2},$$

$$S_3(t)=G^{a}_{-t}(x)([x]+1)^{-2},$$

$$S_4(t)=\sum_{n\leqq x}\frac{E^{a}_{-t}(n)}{n^2(n+1)},$$

$$S_5(t)=(\bar{B}_1(x)-1/2)x^{-1}([x]+1)^{-1}\sum_{n\leqq x}n^{-t}\sigma_a(n),$$

$$S_6(t)=x^{-1}\sum_{n\leqq x}n^{-t}\sigma_a(n),$$

and that

(2)
$$\sum_{n\leqq x}(x-n)n^{-t-1}\sigma_a(n)=x\sum_{i=1}^{5}S_i(t)+xK^{a}_{-t},$$

where K^{a}_{-t} is the sum of the three series, $G^{a}_{-t}(x)$ is defined by

$$G^{a}_{-t}(x)=\sum_{n\leqq x}E^{a}_{-t}(n),$$

and the R. H. S. of (2) *should be interpreted as the sum of first five terms in* (1) *multiplied by x, with an error term better than the error term for the L. H. S. of* (1) *by* $x^{-3/2}$.

(3) $$\sum_{n\leqq x} E^{a}_{-t}(n)=\sum_{n\leqq x}(x-n)n^{-t}\sigma_{a}(n)-\sum_{n\leqq x} g^{a}_{-t}(n)-(\bar{B}_{1}(x)-1/2)\sum_{n\leqq x} n^{-t}\sigma_{a}(n),$$

and (*Segal* [17])

(4) $$\int_{1}^{x} E^{a}_{-t}(u)du=\sum_{n\leqq x}(x-n)n^{-t}\sigma_{a}(n)-\int_{1}^{x} g^{a}_{-t}(u)du\,.$$

LEMMA 2. *For $v>0$, let*

$$L_{-v,-1}(x)=\sum_{n>x} n^{-v}(n+1)^{-1}\,,\qquad M_{-v,=1}(x)=\sum_{n>x} n^{-v}(n+1)^{-1}\log n\,,$$

and for $v>-1$, let

$$L_{-v,-2}(x)=\sum_{n>x} n^{-v}(n+1)^{-2}\,,\qquad M_{-v,-2}(x)=\sum_{n\leqq x} n^{-v}(n+1)^{-2}\log n\,.$$

Then for any $N\in \boldsymbol{N}$,

(1) $$L_{-v,-1}(x)=\sum_{r=0}^{N}\frac{(-1)^{r}}{v+r}\sum_{s=0}^{r}(-1)^{s}\binom{v+r}{s}\bar{B}_{s}(x)x^{-v-r}+O\ (x^{-N-v-1}),$$

(2) $$M_{-v,-1}(x)=\sum_{r=0}^{N}\frac{(-1)^{r}}{v+r}\sum_{s=0}^{r}(-1)^{s}\binom{v+r}{s}\bar{B}_{s}(x)(\log x+j_{s-2}(s-v-r-1))x^{-v-r}$$
$$+O\ (x^{-N-v-1}\log x),$$

(3) $$L_{-v,-2}(x)=\sum_{r=0}^{N}\frac{(-1)^{r}}{v+r+1}\sum_{s=0}^{r}(-1)^{s}(r-s+1)\binom{v+r+1}{s}\bar{B}_{s}(x)x^{-v-r-1}+O\ (x^{-N-v-2}),$$

(4) $$M_{-v,-2}(x)=\sum_{r=0}^{N}\frac{(-1)^{r}}{v+r+1}\sum_{s=0}^{r}(-1)^{s}(r-s+1)\binom{v+r+1}{s}\bar{B}_{s}(x)(\log x$$
$$+j_{s-2}(s-v-r-2))x^{-v-r-1}+O\ (x^{-N-v-2}\log x),$$

where $j_{n-1}(u)=\sum_{k=0}^{n-1}1/(u-k)$ *for* $n\geqq 1$, $j_{-1}(u)=0$, $j_{-2}(u)=-1/(u+1)$.

PROOF. Substituting

$$(n+1)^{-1}=\sum_{r=0}^{N}(-1)^{r}n^{-r-1}+O\ (n^{-N-2}),$$

$$(n+1)^{-2}=\sum_{r=0}^{N}(-1)^{r}(r+1)n^{-2-r}+O\ (n^{-N-3}),$$

in the definitions of L's and M's and applying the asymptotic formulas for $L_{u}(x)$ and $M_{u}(x)$ contained in I-Lemma 3, 8 we conclude the assertion.

LEMMA 3. *Let $S_{r}(x)=(x-1)^{r}$. 7hen we have*

(1) $$S_r(x)=\sum_{u=0}^{r} a_u(r)B_u(x),$$

where $a_u(r)=\frac{(-1)^{r+u}}{r+1}\binom{r+1}{u}$ $(u\leqq r)$, *and* 0 *otherwise.*

(2) $$B_u(x)=\sum_{r=0}^{u} b_r(u)S_r(x),$$

$$\textit{where}\quad b_r(u)=\begin{cases}\binom{u}{r}, & 0\leqq r\geqq u-2,\\ u/2, & r=u-1\quad ,\\ 1\quad, & r=u\quad .\end{cases}$$

PROOF. These follow from the recurrence relations of Bernoulli numbers and Bernoulli polynomials.

LEMMA 4. *For any nonnegative integer N, we have*

(1) $$([x]+1)^{-1}=\sum_{r=0}^{N}\Big(\sum_{u=0}^{r}\frac{(-1)^{r-u}}{r+1}\binom{r+1}{u}\bar{B}_u(x)\Big)x^{-r-1}+O\ (x^{-N-2}),$$

(2) $$([x]+1)^{-2}=\sum_{r=1}^{N}\Big(\sum_{u=0}^{r-1}(-1)^{r-u+1}\binom{r}{u}\bar{B}_u(x)\Big)x^{-r-1}+O\ (x^{-N-2}).$$

PROOF. We have

(3.1) $$\begin{aligned}([x]+1)^{-1}&=x^{-1}(1+(1-\{x\})/x)^{-1}\\ &=\sum_{r=0}^{N}S_r(\{x\})x^{-r-1}+O\ (x^{-N-2}),\end{aligned}$$

similarly

(3.2) $$([x]+1)^{-2}=\sum_{r=1}^{N}rS_{r-1}(\{x\})x^{-r-1}+O\ (x^{-N-2}).$$

Now the result follows from Lemma 3.

LEMMA 5. *For any nonnegative integer N, we have*

(1) $$(\bar{B}_1(x)-1/2)x^{-1}([x]+1)^{-1}=\sum_{r=1}^{N}\sum_{u=0}^{r}a_u(r)\bar{B}_u(x)x^{-1-r}+O\ (x^{-N-2}),$$

(2) $$(\bar{B}_1(x)-1/2)([x]+1)^{-2}=\sum_{r=1}^{N}r\sum_{u=0}^{r}a_u(r)\bar{B}_u(x)x^{-1-r}+O\ (x^{-N-2}),$$

(3) $$\begin{aligned}\bar{B}_2(x)([x]+1)^{-2}=&\sum_{r=1}^{N}r(\sum_{u=0}^{r-1}(a_u(r+1)+a_u(r)+a_u(r-1)/6)\bar{B}_u(x)\\ &+\bar{B}_{r+1}(x)+(1-r)\bar{B}_r(x)/2))+O\ (x^{-N-2}),\end{aligned}$$

where $a_n(r)$ *are defined in Lemma* 3.

PROOF. From (3.1) and (3.2) we can expand the L. H. S. of (1), (2) in powers of x with coefficients $S_{r+1}(\{x\})$, which are expressed in terms of $\bar{B}_u(x)$ by Lemma 3, (1). Similarly, for (3) we first express $\bar{B}_2(x)$ in terms of $S_r(\{x\})$ by Lemma 3, (2), and then we can expand the L. H. S. of (3) in powers of x with coefficients involving $S_r(\{x\})$ and again we apply Lemma 3, (1) to express them via $\bar{B}_u(x)$.

LEMMA 6. *For* $u\in\boldsymbol{N}$, *we have*

$$(B_1(x)-1/2)B_u(x)=\frac{1}{u+1}\sum_{m=0}^{u}\binom{u+1}{m}B_{u-m+1}B_m(x)+B_{u+1}(x).$$

PROOF. We have

$$\text{L. H. S.}=(x-1)B_n(x)=\sum_{r=0}^{u}b_r(u)S_{r+1}(x).$$

Using Lemma 3-(1), and interchanging the order of summation gives the result.

LEMMA 7. (cf. Walfisz [18]) *Suppose* $f(x)$ *is a bounded, Riemann integrable function on* $[a, b]$ *and that it has the bounded, Riemann integrable derivative on* $[a, b]$ *except at integer points, where it has the right derivative. Then*

$$\sum_{a<n\leq b}f(n)=\int_a^b f(y)dy+\bar{B}_1(a)f(a)-\bar{B}_1(b)f(b)+\int_a^b \bar{B}_1(y)f'(y)dy\,.$$

PROOF. The proof goes on the same lines as those of Lemma 1.3.1 in Walfisz. Indeed, by dividing the interval $(a, b]$ into subintervals, we see that it suffices to prove our formula in the case where there is at most one integer in $(a, b]$. We distinguish three cases. First, if there is no integer in $(a, b]$, our formula follows by integration by parts.

Secondly, if there is an integer n such that $a<n<b$, we proceed as in Walfisz to arrive at

$$\int_a^b \bar{B}_1(y)f(y)dy=\int_a^n+\int_n^b=\int_a^{n-0}+\int_n^b.$$

Since f has the right derivative at $x=n$, we may integrate the second as well as the first integral by parts, and so we infer that

$$\int_a^b \bar{B}_1(y)f'(y)dy=\bar{B}_1(n-0)f(n)-\bar{B}_1(a)f(a)-\int_a^n f(y)dy$$

$$+\bar{B}_1(b)f(b)-\bar{B}_1(n)f(n)-\int_n^b f(y)dy$$

$$=f(n)+\bar{B}_1(b)f(b)-\bar{B}_1(a)f(a)-\int_a^b f(y)dy\,.$$

Finnally, if b is an integer, we can repeat the argument of Walfisz.

LEMMA 8. *Let $a(n)$ be a polynomial in n of degree t with leading coefficient positive and let $a<2t-2$. Then*

$$T:=\sum_{n>x}\frac{G_{a,k}(n)}{a(n)}=\begin{cases} O\ (x^{a/2-t+1}), & a>-2,\\ O\ (x^{-t}(\log x)^{\delta_{a,-1}}), & a\leqq -2,\end{cases}$$

PROOF. Changing the order of summation, we have

$$T=\sum_{m\geqq 1}m^a\sum_{n>x,\,n\geqq m^2}a(n)^{-1}\bar{B}_k\Big(\frac{n}{m}\Big).$$

Dividing the range of m into two: $m\leqq\sqrt{x}$, $m>\sqrt{x}$ and estimating the sum over $m>x$, $n=m^2$ trivially, we obtain

$$T=\sum_{m\leqq\sqrt{x}}m^aS(x)+\sum_{m>\sqrt{x}}m^aS(m^2)+O\ (x^{a/2-t+1/2}),$$

where

$$S(Y)=\sum_{n>Y}a(n)^{-1}\bar{B}_k\Big(\frac{n}{m}\Big).$$

Now, by Lemma 7 and the second mean value theorem,

$$S(Y)=\int_Y^\infty a(u)^{-1}\bar{B}_k\Big(\frac{u}{m}\Big)du+a(Y)^{-1}\bar{B}_k\Big(\frac{Y}{m}\Big)$$

$$+\int_Y^\infty \bar{B}_1(u)\Big\{\frac{d}{du}a(u)^{-1}\bar{B}_k\Big(\frac{u}{m}\Big)+a(u)^{-1}\frac{k}{m}\bar{B}_{k-1}\Big(\frac{u}{m}\Big)\Big\}du$$

$$=O\ (m^2Y^{-t-1}+mY^{-t}),$$

since, by integration by parts

$$\int_Y^\infty \bar{B}_1(u)\bar{B}_k\Big(\frac{u}{m}\Big)du=O\ (m^2).$$

Hence

$$T=O\Big(\sum_{m\leqq\sqrt{x}} m^{a}(mx^{-t}+m^{2}x^{-t-1})\Big)+O\Big(\sum_{m>\sqrt{x}} m^{a-2t+1}\Big)+O\left(x^{a/2-t+1/2}\right)$$

$$=\begin{cases} O\left(x^{a/2-t+1}\right), & \text{if } a>-2, \\ O\left(x^{-t}(\log x)^{\delta_{a,-2}}\right), & \text{if } a\leqq-2, \end{cases}$$

as asserted.

LEMMA 9. (Richert [16, Satz 2]). *Let* $Z(s)=\sum_{n=1}^{\infty} f(n)n^{-s}$ *be a convergent Dirichlet series whose analytic continuation is meromorphic in every subdomain of* $\boldsymbol{C}$ *and satisfies* $|Z(s)|=O(|t|^{c})$ *uniformly in* σ *with a constant* c. *Suppose there exist real numbers* α, $\bar{\alpha}$ *such that* $f(n)=O(n^{\alpha+\varepsilon})$ *and* $Z(s)$ *is absolutely convergent for* $\sigma>\bar{\alpha}$. *Suppose also that we can express* $Z(s)$ *as*

$$Z(s)=C^{\theta s+\theta_1}G(s)Z_1(r-s)$$

with r, θ *real*, $C>0$, θ_1 *complex, and*

$$G(s)=\sum_{j=1}^{M}\frac{\Gamma(\beta_j-b_js)}{\Gamma(\delta_j+d_js)},$$

where $M\in\boldsymbol{N}$, β_j, $\delta_j\in\boldsymbol{R}$, b_j, $d_j>0$, $1\leqq j\leqq M$, *and where* $Z_1(s)$ *is an analytic function defined by a convergent Dirichlet series* $Z_1(s)=\sum_{n=1}^{\infty} f_1(n)n^{-s}$. *Suppose further that there are two real numbers* α_1, $\bar{\alpha}_1$ *such that* $f_1(n)=O(n^{\alpha_1+\varepsilon})$, *and* $\sum_{n=1}^{\infty} f_1(n)n^{-s}$ *is absolutely convergent for* $\sigma>\bar{\alpha}_1$. *Then, if* $\alpha\geqq-1$ *and* $\kappa\geqq0$, *we have, as* $x\to\infty$,

$$\Gamma(\kappa+1)^{-1}\sum_{n\geqq x} f(n)(x-n)^{\kappa}$$

$$=F_{\kappa}(x)+O\left(x^{\alpha+(\kappa+1)\left(1-\frac{\bar{\alpha}_1+\alpha+1-r}{q(\bar{\alpha}_1-r)+\lambda+1/2}\right)+\varepsilon}\right)+O\left(x^{\alpha+\varepsilon}\right)+O\left(x^{\kappa+\frac{\lambda-1/2-\kappa}{q}+s}\right),$$

where

$$F_{\kappa}(x)=\sum_{\min\left(r-\bar{\alpha}_1,\frac{\lambda-\kappa-1}{q}\right)\leqq\sigma\leqq\alpha+1}\operatorname{Res} x^{s+\kappa}\frac{\Gamma(s)Z(s)}{\Gamma(s+\kappa+1)},$$

$$q=2\sum_{j=1}^{M} d_j,\qquad \lambda=\sum_{j=1}^{M}(\beta_j-\bar{\delta}_j).$$

4. Sketch of proofs

The proof of Theorem 1 goes along similar lines to those of proofs of Theorem 3 in [6]. In order to apply I-Lemma 3 without error term we have to restrict ourselves to the case $a-t\in\boldsymbol{Z}$. In what follows, by (Case: $t=b$) we refer to formulas for

$$(1)\ \sum_{n\leq x} n^{-a-b}\sigma_a(n),\qquad (2)\ \sum_{n\leq x} E^{a}_{-a-b}(n),\qquad (3)\ \int_1^x E^{a}_{-a-b}(u)du\,.$$

Then, using Lemma 1-6, 8 we deduce Theorem 2 (Case: $t=1$) from Theorem 1 with $t=a$, which however, differs from the corresponding formulas in Theorem 4 (Case: $t\geqq 2$). We therefore go on to deduce Theorem 3 (Case: $t=2$). Formula (1) in Theorem 3 coincides with Formula (1) in Theorem 4 with $t=2$, while neither Formulas (2) nor (3) coincide with those in Theorem 4. We then proceed to deduce Formulas (2) and (3) in (Case: $t=3$), which are seen to coincide with Formulas (2) and (3) in Theorem 4 with $t=3$.

To prove Theorem 4-(1) we apply induction on t. The starting point for Formula (1) is $t=2$ and, for Formulas (2) and (3), it is $t=3$. Applying the back-track method to Formulas for some t (which we assume to hold), we can check that the Formulas for $t+1$ are precisely those given in Theorem 4 with $t+1$ in place of t.

In order to derive Formulas (2) and (3), we need Riesz sums, which we calculate with the aid of Lemma 1-(2), Lemma 9. They read

PROPOSITION. *For $3\leqq t\leqq \boldsymbol{Z}$, we have*

$$\sum_{n\leq x}(x-n)\frac{\sigma_a(n)}{n^{a+t}}=\zeta(t)\zeta(a+t)x-\zeta(t-1)\zeta(a+t-1)$$

$$+\begin{cases}\dfrac{1}{(t-1)(t-2)}x^{-t+2}\log x+\dfrac{1}{(t-1)(t-2)}\Big(2\gamma+\dfrac{1}{t-2}+\dfrac{1}{t-1}\Big)x^{-t+2}, & a=0,\\[2mm] \dfrac{\zeta(a+1)}{(t-1)(t-2)}x^{-t+2}+\dfrac{\zeta(1-a)}{(a+t-1)(a+t-2)}\bar{\delta}_{a,2}\bar{\delta}_{a,3}x^{-a-t+2} & \\ \quad+\begin{cases}0, & 0<a\leqq 1,\\ (\alpha'(t)\zeta(a+1)+\beta'(t)\zeta(a))x^{-t} & 1<a\leqq 3,\end{cases}\end{cases}$$

$$-\frac{1}{2x^t}(x^{-a}G_{1+a,2}(x)+G_{1-a,2}(x))+O\,(x^{-a/2-t+1/2}),$$

where

$$\alpha'(t)=\alpha(t)-\alpha(t-1),\qquad \beta'(t)=\beta(t)-\beta(t-1).$$

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Kagoshima National College of Technology
1460-1 Shinko Hayato-cho
Kagoshima 899-51
Japan